Note di Matematica **27,** n. 2, 2007, 249–255.

How to climb the face of a mountain

Piero Villaggio
Dipartimento di Ingegneria Strutturale, Universitá di Pisa.
Via Diotisalvi, 2, I-56126 Pisa.

Received: 17/10/2006; accepted: 23/10/2006.

Abstract. The shortest line connecting a point at the base of a mountain with the top is a geodesic on the surface of the mountain. But this path is often unpracticable since the inclination of the geodesic with respect the horizontal is often too high to being directly overcome by a climber. We here propose a strategy for reaching the top along curves not exceeding the maximum inclination superable by the climber. We here discuss two particular examples for which an explicit solution is obtainable in terms of elliptic functions.

Keywords: Optimal paths

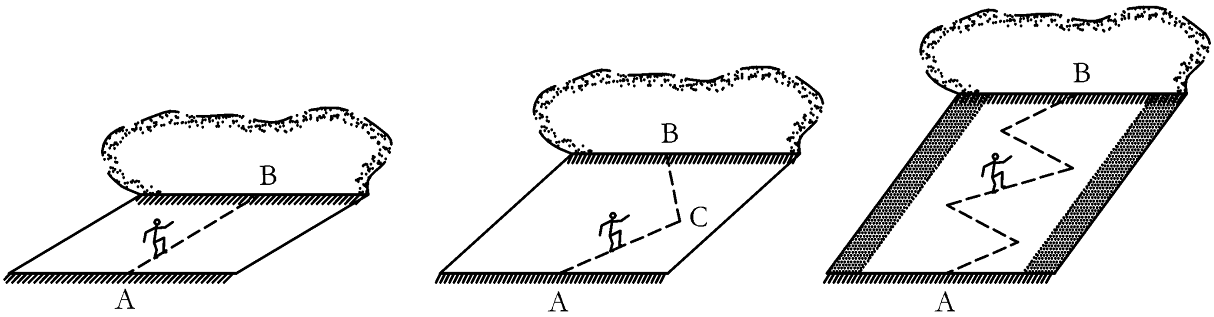
MSC 2000 classification: primary 49J15

To Alan Day for his mechanical taste

1 Introduction

For several millennia human beings have climbed the flanks of mountains either in search of food, for escaping enemies, or even (in the last two centuries) for mere pleasure.

Faces of mountains are surface full of asperities and climbers must find the best path for reaching the top spending minimum effort and avoiding the objective dangers of falling stones and avalanches. Since this problem is mathematically intractable, some simplifications are necessary in order to propose a model. We neglect the asperities and the objective dangers, and assume, for the moment, that the face to be climbed is an inclined plane (Fig. 1(a)). A man, starting from the point A of the base, must reach a point B of the crest placed on the straight line of maximum slope. If the inclination of AB does not exceed a given value $\bar{\alpha}$, less than 45° (according to Sturm and and Zintl [6]), the climber must follow this path (which is the geodesic) in order to reach the top in the shortest time. The segment AB is, in theory, the line of quickest ascent provided that the climber may progress with constant speed. But this route is often unpracticable since, if the inclination exceeds 45°, the climber must deviate from the geodesic and advance along a straight line of inclination $\bar{\alpha}$. If the climber follows this line, he gets the crest but not at the point B and hence, after having reached the point C situated at mid height, he must change



(a) Direct ascent (b) Skew ascent (c) Tortuous ascent

Figure 1.

direction and proceed along the segment CB, also inclined of $\bar{\alpha}$ (Fig. 1(b)). But other routes are possible. For example, instead of going until C, the climber can describe a sequence of zigzags having constant inclination $\bar{\alpha}$ (Fig. 1(c)). This itinerary has the same length as ACB, and hence does not shorten the time of ascent, but has the advantage of being contained in a narrower strip adiacent to the line AB. This choice is necessarily taken by any walker when he must mount a steep staircase confined between two walls, and by any sailor maneuvering a sailboat, equipped with oars, moving over a lake from a point A to B against the wind (Petrov [5, §4]). A similar situation occurs in the ascent of the flank of a mountain, where the terrain is not everywhere uniform but more practicable in a corridor containing the line AB. This explains why the path leading to the high huts in the Alps are so tortuous (see, for example, the diagrams recorded in the guidebook by Collomb and Crew on the Mont Blanc [1]).

In the case of a plane slope the optimum path is either a rectilinear segment (for $\alpha \leq \bar{\alpha}$) or polygonal lines whose sides have the same inclination $\bar{\alpha}$. If, however, the slope is a curved surface the optimum path is no longer linear or piecewise linear, but a curve (or curves) defined by an ordinary differential equation. We discuss the structure of these curves in two cases in which their equation is explicit. For simplicity, we limit ourselves to finding the trajectories with the minimum number of winds, namely those corresponding to the solution sketched in Fig. 1(b) when the slope is plane.

2 The surface is cylindrical (Makalu, SW face)

We assume the surface to be a piece of circular cylinder of radius R, defined by the Cartesian equation (Fig. 2)

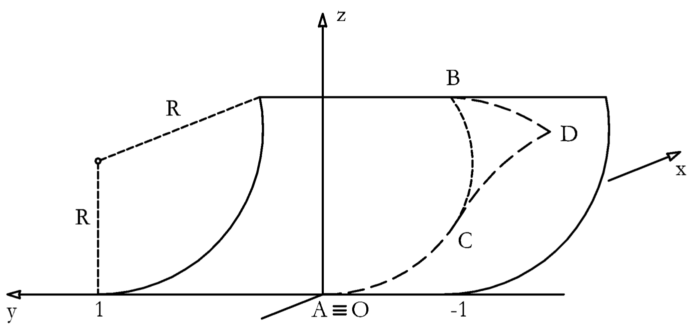


Figure 2. The route of ascent is A-C-D-B

$$z = R - \sqrt{R^2 - x^2}\,, \qquad (1)_a$$

$$(0 \le x \le R\,, \quad -1 \le y \le 1)\,. \qquad (1)_b$$

A climber starting from the point A(0,0) must reach the point B$(-R, R)$. The geodesic is the circular arc connecting A with B, but he cannot follow this path until its end since his velocity of ascent decreases monotonically with the slope α, and, when α reaches a maximum values $\bar{\alpha}$ $(0 < \bar{\alpha} < \pi/2)$, he cannot longer advance. After this point, the climber must change direction and follow the line or the lines in the surface with prescribed inclination $\bar{\alpha}$. In the case of the cylinder of equation $(1)_a$ the inclination of the circular arc $\widehat{AB}$ is zero at A and reaches its maximum allowed value $\bar{\alpha}$ at the point C$(R\sin\bar{\alpha},\ 0,\ R(1-\cos\bar{\alpha}))$ (Fig. 2). Hence the climber can go along the arc $\widehat{AC}$, but after C he must take another route.

In order to find the equation or equations of these possible routes, we assume that they are described by the parametric equation

$$x = x\,, \quad y = y(x)\,, \quad z = R - \sqrt{R^2 - x^2}\,, \qquad (2)_a$$

$$R\sin\bar{\alpha} \le x \le R\,. \qquad (2)_b$$

Clearly $y = y(x)$ is the equation of the vertical projection of the curve (or curves) on the x, y-plane. In addition, $\sin\bar{\alpha}$ is given by the ratio

$$\sin\bar{\alpha} = \frac{dz}{ds} = \frac{z^{'}}{\sqrt{1 + y^{'2} + z^{'2}}} \,, \tag{3}$$

where the prime denotes differentiation with respect to x. Since $z^{'}$ is known from $(2)_a$, we derive $y^{'}$ from (3)

$$y^{'} = \frac{\pm 1}{\sin\bar{\alpha}} \left(\frac{x^2 - R^2 \sin^2\bar{\alpha}}{R^2 - x^2} \right)^{1/2} = \frac{\pm 1}{\sin\bar{\alpha}} \frac{(x^2 - R^2 \sin^2\bar{\alpha})}{\sqrt{(R^2 - x^2)\,(x^2 - R^2 \sin^2\bar{\alpha})}} \,, \quad (4)_a$$

$$R \sin\bar{\alpha} \le x \le R \,. \quad (4)_b$$

Equation $(4)_a$ can be solved explicitly in terms of elliptic functions. Omitting the details and using the tables of Gröbner and Hofreiter [3, Sec 244] we evaluate

$$a_0 = -1,\;\; \kappa^2 = \frac{(1-\sin\bar{\alpha})^2}{(1+\sin\bar{\alpha})^2},\;\; a = -R^2\sin\bar{\alpha}(1-\sin\bar{\alpha}),\;\; b = -R^2\sin\bar{\alpha}(1+\sin\bar{\alpha}),$$

$$c = R\,(1-\sin\bar{\alpha})\,,\; d = R\,(1+\sin\bar{\alpha})\,,\; \delta = 2\,R^3\,\sin\bar{\alpha}\,\cos^2\bar{\alpha}\,,\; \frac{\delta}{\gamma} = \frac{1}{R\,(1+\sin\bar{\alpha})}\,. \tag{5}$$

Therefore the two solution of $(4)_a$ in the interval $(4)_b$ assume the form

$$y = \frac{\pm 1}{\sin\bar{\alpha}} \left(\mathcal{Z} + R^2 \sin^2\bar{\alpha}\,\mathcal{Y}_0\right) + \mathcal{C}\,, \tag{6}$$

where $\mathcal{Y}_0$, $\mathcal{Z}$ are elliptic integrals of first and second kind, respectively, whose expressions in terms of elliptic functions are

$$\mathcal{Y}_0 = \frac{2\,\delta}{\gamma}\,F(\varphi,\,\kappa)\,, \quad (7)_a$$

$$\mathcal{Z} = -\frac{\delta}{c\,d\,\gamma}\left[\left(-2\,a\,b + \frac{b\,\delta}{c+d} - \frac{a\,\delta}{c+\kappa^2\,d}\right) F(\varphi,\,\kappa) + \frac{\delta^2}{(c+d)\,(c+\kappa^2\,d)}\left(E(\varphi,\,\kappa) + \frac{c\,\sin\varphi\,\cos\varphi\,\sqrt{1-\kappa^2\varphi}}{c\,\sin^2\varphi + d}\right)\right]\,, \quad (7)_b$$

where $F(\varphi,\,\kappa)$, $E(\varphi,\,\kappa)$ are Legendre normal integrals of the first an second kind respectively, and φ is related to x by the equation

$$x = \frac{a\sin^2\varphi + b}{c\sin^2\varphi + d}\,. \tag{8}$$

Solving (8) with respect to φ and giving a, b, c, d their expressions recorded in (5) we obtain

$$\varphi = \arcsin\sqrt{\frac{d\,x - b}{a - c\,x}} = \arcsin\sqrt{\frac{(1+\sin\bar\alpha)\,(x - R\sin\bar\alpha)}{(1-\sin\bar\alpha)\,(x + R\sin\bar\alpha)}}\,,$$

which shows that, for $R\,\sin\bar\alpha \le x \le R$, φ ranges in the interval $0 \le \varphi \le \pi/2$. The arbitrary constant $\mathcal{C}$ appearing in (6) is determined by the initial condition that $y = 0$ for $\varphi = 0$ ($x = R\,\sin\bar\alpha$). Since (*cf.* Janke-Emde-Lösch [4, Sec. V]) $F(0,\,\kappa) = E(0,\,\kappa) = 0$, we obtain $\mathcal{C} = 0$. The double sign in (6) means that, after the point C, there are two directions with prescribed inclination $\bar\alpha$ on the surface, one pointed right and one pointed left of the arc $\overset{\frown}{AB}$. For definiteness, we choose the first of them (Fig. 2), corresponding to the minus sign in $(4)_a$ and (6).

However, this branch goes monotonically away from the x, z-plane and hence cannot be followed after a certain point D, to be determined. In order to find D we consider the curve of constant inclination $\bar\alpha$. but pointed at left of $\overset{\frown}{AB}$, starting from the point B $(R\,,\,0\,,\,R)$, for which $\varphi = \frac{\pi}{2}$, $F(\frac{\pi}{2},\,\kappa) = E(\frac{\pi}{2},\,\kappa) = 1$ (*cf.* Janke-Emde-Lösch [4, Sec. V]). The initial condition $y(R) = 0$ determines the value $\mathcal{C} = \bar{\mathcal{C}}$ of the constant. This branch too is monotone and intersect the branch starting from C at a unique point D (Fig. 2). Thus a possible strategy of ascent is the union of the arcs $\overset{\frown}{AC}$, $\overset{\frown}{CD}$, $\overset{\frown}{DB}$. Of course, the symmetric curve with respect to the x, z-plane is also allowed, and these two curves are the only best trajectories with the least number of winds (one at D or at its symmetric point).

3 The mountain is a hemisphere (Everest, Summit)

In his lectures on Calculus of Variations Richard Courant [2, p. 171] solves the problem of the road of quickest ascent to the top of a mountain, under the assumption that it is symmetric about the vertical axis through its apex. Courant's solution is qualitative and does not consider the presence of many zigzags. We here solve Courant's problem with the least number of winds in the particular case in which the surface is hemispherical.

Consider the hemispherical surface defined by the polar equation (Fig. 3)

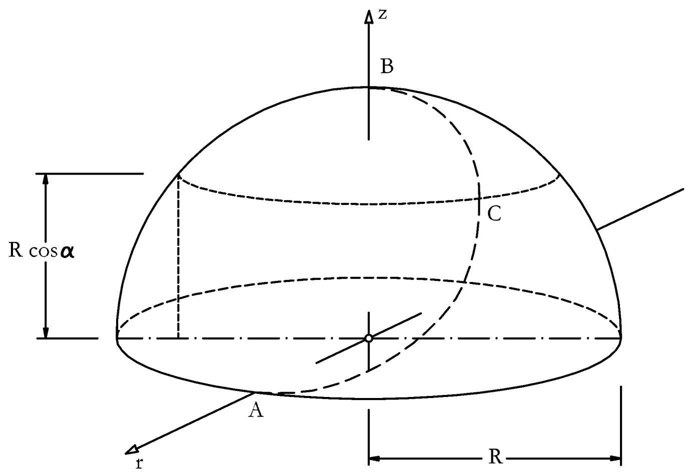


Figure 3. The route of ascent is A-C-B

$$z = \sqrt{R^2 - r^2}\,, \tag{8$_a$}$$

$$(0 \le r \le R\,, \quad 0 \le z \le R\,, \quad 0 \le \theta \le 2\,\pi)\,. \tag{8$_b$}$$

A climber starts from A(R, 0, 0) with the intent of getting the top B(0, 0, R) in the shortest time. The maximum inclination he can overcome is $0 < \bar{\alpha} < \frac{\pi}{2}$ and therefore he cannot proceed along the geodesic, which is the circular arc $\overset{\frown}{AB}$, since its slope is $\frac{\pi}{2}$ at A.

Assume that a possible path has the parametric equation

$$\vartheta = \theta\,, \quad r = r(\vartheta)\,, \quad z = \sqrt{R^2 - r^2}\,, \tag{9$_a$}$$

$$0 \le \vartheta \qquad (\vartheta \text{ may exceed } 2\,\pi)\,. \tag{9$_b$}$$

The lines of prescribed inclination $\bar{\alpha}$ are defined by the equation

$$\sin \bar{\alpha} = \frac{dz}{ds} = \frac{z^{'}}{\sqrt{1 + r^{'2} + z^{'2}}}\,, \tag{10}$$

where the prime denotes differentiation with respect to ϑ. From $(9)_a$ we obtain $z^{'}$ as function of r, and hence from (10) we derive

$$r^{'} = \pm 1 \sin \bar{\alpha} \left(\frac{R^2 - r^2}{r^2 - R^2 \sin^2 \bar{\alpha}} \right)^{1/2}, \quad (0 \leq \vartheta \leq 2\pi),$$

or alternatively,

$$\frac{d\vartheta}{dr} = \frac{\pm 1}{\sin \bar{\alpha}} \left(\frac{r^2 - R^2 \sin^2 \bar{\alpha}}{R^2 - r^2} \right)^{1/2}, \quad (0 \leq \vartheta \leq 2\pi), \tag{11}$$

which is exactly $(4)_a$ except for the replacement of x, y, with r, ϑ respectively. The analytical structure of the solutions is again (6). The constant $\mathcal{C}$ is determined by the initial condition at A

$$r(0) = R, \qquad \text{or} \qquad \vartheta(R) = 0. \tag{12}$$

Let us choose, for definitess, the positive sign in (11) which defines the branch whose projection in the plane $z = 0$ in counter-clockwise directed (Fig. 3). This curve rises monotonically from the bottom until it attains at C the level $z = R \cos \bar{\alpha}$, where the geodesic $\overset{\frown}{CB}$ has inclination $\bar{\alpha}$. After C the shortest way of ascent is the circular arc $\overset{\frown}{CB}$ (Fig. 3). The union of $\overset{\frown}{AC}$ and $\overset{\frown}{CB}$ is an optimum path.

References

[1] R. G. Collomb, P. Crew: Selected Climbs in the Mont Blanc Range, The Alpine Club, London 1967.

[2] R. Courant: Calculus of Variations, New York Univ, 1962.

[3] W. Gröbner, N. Hofreiter: Integraltafel, Springer, Wien 1966.

[4] Janke-Emde-Lösch: Tafeln Höherer Funktionen, Teubner, Stuttgart, 1966.

[5] P. Iu. Petrov: Variational Methods in Optimum Control Theory, Academic Press, New York, 1968.

[6] G. Sturm, F. Zintl: Sicher Klettern in Fels und Eis, BLV, München, 1969.