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Surjective partial differential operators on real analytic functions defined on a halfspace

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Abstract. Let $P(D)$ be a partial differential operator with constant coefficients and let $A(\Omega)$ denote the real analytic functions defined on an open set $\Omega \subset \mathbb{R}^n$. Let H be an open halfspace. We show that $P(D)$ is surjective on $A(H)$ if and only if $P(D)$ is surjective on $A(\mathbb{R}^n)$ and $P(D)$ has a hyperfunction elementary solution which is real analytic on H.

Keywords: partial differential equations, elementary solutions, surjectivity on real analytic functions

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Dedicated to the memory of Prof. Dr. Klaus Floret

1 Introduction

Since the pioneering papers of Kawai [12] and Hörmander [8], the basic question if

$$P(D) \text{ is surjective on } A(\Omega) \tag{1}$$

has been studied by many authors. Here $P(D)$ is a partial differential operators with constant coefficients, $\Omega \subset \mathbb{R}^n$ is open and $A(\Omega)$ denotes the space of real analytic functions on Ω. A by no means complete list of the corresponding papers is contained in the references (see Andreotti and Nacinovich [1], Kaneko [10,11], Zampieri [23], Braun [3], Braun, Meise and Taylor [4,5] and Langenbruch [13–16], see also the references given in Langenbruch [15]).

For convex $\Omega \subset \mathbb{R}^n$, a characterization of (1) was obtained by Hörmander [8] using a Phragmen-Lindelöf type condition valid on the complex characteristic variety of the principal part P_m of P. For general open sets Ω, a different characterization by means of locally regular elementary solutions was given in Langenbruch [15].

In the present paper, we will concentrate on the case of half spaces

$$\Omega := H_N := \{\, x \in \mathbb{R}^n \mid \langle x, N\rangle > 0 \,\},\ 0 \neq N \in \mathbb{R}^n.$$

Our main result is the following (see Theorem 1 below):
$P(D)$ is surjective on $A(H_N)$ if and only if $P(D)$ is surjective on $A(\mathbb{R}^n)$ and $P(D)$ has a hyperfunction elementary solution E defined on $\mathbb{R}^n$ such that E is real analytic on H_N.

This improves the corresponding results of Langenbruch [15] and Zampieri [23] considerably.

Besides the paper [8] of Hörmander, the present paper relies on the results of Langenbruch [15,16], and the main part of our proof consists in showing that $P(D)$ has an elementary solution as above if $P(D)$ is surjective on $A(H_N)$.

The paper is organized as follows: In the first section, our main result is stated in Theorem 1 and its proof is reduced to showing that

$$P(D)C_\Delta(Z) = C_\Delta(Z) \tag{2}$$

if $P(D)$ is surjective on $A(H_N)$. Here

$$Z := (\mathbb{R}^n \times\,]0,\infty[\,) \cup (H_N \times \{0\})$$

and $C_\Delta(Z)$ are the harmonic germs defined near Z.

Since $C_\Delta(Z)$ in a natural way is the projective limit of a projective spectrum of (DFS)-spaces, the proof of (2) relies on the theory of projective spectra of linear spaces and the corresponding Proj^k-functors which were developed by Palamodov [18,19] (see also Vogt [21] and the recent book of Wengenroth [22]).

The corresponding notions and the key result from Langenbruch [15, Theorem 1.4] (see Theorem 3) are recalled in section 2.

In the last section, the proof of our main theorem is completed using a precise result of Langenbruch [16] on the solvability of partial differential equations for harmonic germs defined near non convex sets (see Theorem 6).

2 The main result

In this section, we will introduce some useful notation and formulate the main result of this paper in Theorem 1. Using the results of Hörmander [8] and Langenbruch [15,16], the proof of the main theorem is then reduced to the proof of the surjectivity of $P(D)$ on a certain space of harmonic germs (see (4) below).

In the present paper, $n \in \mathbb{N}$ always is at least 2 and Ω is an open set in $\mathbb{R}^n$. The real analytic functions on Ω are denoted by $A(\Omega)$. $P(D)$ is always a partial differential operator in n variables with constant coefficients. The degree of P is m and P_m denotes the principal part of P.

Our proofs will be based on harmonic germs in $(n+1)$ variables. Correspondingly, we will use the following notations: A point in $\mathbb{R}^{n+1}$ is written as

$(x,y) \in \mathbb{R}^n \times \mathbb{R}$. $\Delta = \sum_{k\le n}(\partial/\partial x_k)^2 + (\partial/\partial y)^2$ denotes the Laplace operator on $\mathbb{R}^{n+1}$. The harmonic germs near a set $S \subset \mathbb{R}^{n+1}$ are denoted by $C_\Delta(S)$. Of course, $P(D) = P(D_x)$ also operates on the harmonic germs, and in fact we will solve the equation $P(D_x)f = g$ for harmonic germs f and g rather than for hyperfunctions f and g, that is, we will use the following well known representation of hyperfunctions on Ω

$$\mathfrak{B}(\Omega) := \widetilde{C}_\Delta(\Omega \times (\mathbb{R} \setminus \{0\}))/\widetilde{C}_\Delta(\Omega \times \mathbb{R}) \tag{3}$$

(see Bengel [2] and Hörmander [9, Chapter IX]). Here $\widetilde{C}_\Delta(V)$ is the space of harmonic functions on V which are even w.r.t. y.

Let S^n denote the unit sphere in $\mathbb{R}^n$. The half space defined by $N \in S^n$ is denoted by

$$H_N := \{\, x \in \mathbb{R}^n \mid \langle x, N\rangle > 0 \,\}.$$

For $\xi \in \mathbb{R}^n$ let

$$U_k(\xi) := \{\, x \in \mathbb{R}^n \mid \|x - \xi\| < k \,\}, U_k := U_k(0)$$

and

$$U_{k,+} := U_k \cap \{\, x \in \mathbb{R}^n \mid \langle x, N\rangle > 1/k \,\}.$$

The main result of this paper is the following

1 Theorem. *The following statements are equivalent:*

(a) $P(D)$ is surjective on $A(H_N)$.

(b) $P(D)$ is surjective on $A(\mathbb{R}^n)$ and for any $j \in \mathbb{N}$ there are $\delta < 0$ and a hyperfunction F defined on $\{\, x \in \mathbb{R}^n \mid \langle x, N\rangle > \delta \,\}$ such that

$$P(D)F = \delta \text{ on } \{\, x \in \mathbb{R}^n \mid \langle x, N\rangle > \delta \,\} \text{ and } F\,|_{U_{j,+}} \in A(U_{j,+}).$$

(c) $P(D)$ is surjective on $A(\mathbb{R}^n)$ and $P(D)$ has an elementary solution $E \in \mathfrak{B}(\mathbb{R}^n)$ such that $E\,|_{H_N} \in A(H_N)$.

(d) $P(D)$ is surjective on $A(\mathbb{R}^n)$ and for any $g \in \mathfrak{B}(\mathbb{R}^n)$ with $g\,|_{H_N} \in A(H_N)$ there is $f \in \mathfrak{B}(\mathbb{R}^n)$ with $f\,|_{H_N} \in A(H_N)$ such that $P(D)f = g$ on $\mathbb{R}^n$.

The first characterization of surjective partial differential operators on $A(\Omega)$ for general open sets $\Omega \subset \mathbb{R}^n$ has been given in Langenbruch [15]. For convex Ω, a different characterization has been given in the pioneering work of Hörmander [8] by means of a suitable Phragmen-Lindelöf type condition valid on the complex zero variety of the principal part P_m of P. Hence, the statements in Theorem 1 are also equivalent to the corresponding statements for P_m instead of P, and also to the statements for $-N$ instead of N, respectively.

The main feature of Theorem 1 is the implication "$(a) \Longrightarrow (d)$". In fact, the implications "$(d) \Longrightarrow (c) \Longrightarrow (b)$" are obvious, and the equivalence of (a) and (b) easily follows from the results of Hörmander [8] and Langenbruch [15].

Thus, Theorem 1 will be proved if we can show that (a) implies (d). Taking into account the definition of hyperfunctions in (3) it is sufficient to show that

$$P(D)C_\Delta(Z) = C_\Delta(Z) \tag{4}$$

if $P(D)$ is surjective on $A(H_N)$, where

$$Z := (\mathbb{R}^n \times]0,\infty[\,) \cup (H_N \times \{0\})\,.$$

Indeed, a hyperfunction g on $\mathbb{R}^n$ is defined by a harmonic function g_+ defined on $\mathbb{R}^n \times]0,\infty[$. Since $g_{|H_N}$ is real analytic, g_+ can be extended to a harmonic germ near Z. If $P(D)f_+ = g_+$ for some harmonic germ f_+ defined near Z then f_+ defines a hyperfunction f which is analytic on H_N and which solves $P(D)f = g$.

3 Surjectivity via the Proj^1-functor

As was noticed in (4), we have to prove that $P(D)$ is surjective on $C_\Delta(Z)$ for $Z := (\mathbb{R}^n \times]0,\infty[\,) \cup (H_N \times \{0\})$. The natural topology of this space is rather complicated and can be defined as follows: Using a strictly decreasing zero sequence $A_K > 0$ (to be chosen later, see the remarks before Theorem 8 below) we set

$$Z_K := (V_K \times [A_K, K]) \cup (V_{K,+} \times [0,K])$$

where V_k and $V_{k,+}$ denote the closure of U_k and $U_{k,+}$, respectively. Then

$$C_\Delta(Z) = \lim_{\leftarrow K} C_\Delta(Z_K),$$

that is, $C_\Delta(Z)$ is the projective limit of the projective spectrum

$$C^Z_\Delta := \{\, C_\Delta(Z_K), R^K_J \,\}$$

of (DFS)-spaces where the linking maps

$$R^K_J : C_\Delta(Z_J) \to C_\Delta(Z_K) \text{ for } J \geq K$$

are defined by restriction. Notice that the topology of $C_\Delta(Z)$ is independent of the sequence A_K, while the proper choice of A_K is important for the proof of the needed properties of the projective spectrum C^Z_Δ (see Theorem 3 below).

Since the topology of $C_\Delta(Z)$ is so complicated the proof of (4) will rely on the theory of projective spectra of linear spaces and the corresponding Proj^k-functors which were developed by Palamodov [18,19] (see also Vogt [21] and the recent book of Wengenroth [22]). We will shortly introduce the corresponding notions and facts which we need. The reader is referred to these papers for further information.

For $S \subset \mathbb{R}^{n+1}$ let

$$N_P(S) := \{ C_\Delta(S) \mid P(D_x)f = 0 \}$$

and let

$$N_P^Z := \{ N_P(Z_K), R_J^K \}$$

be the projective spectrum of the kernels of $P(D_x)$ in $C_\Delta(Z_K)$. We thus have the short sequence of projective spectra

$$0 \longrightarrow N_P^Z \longrightarrow C_\Delta^Z \xrightarrow{P(D)} C_\Delta^Z \longrightarrow 0. \tag{5}$$

The sequence (5) of projective spectra is called exact if for any $K \in \mathbb{N}$ there is $J \geq K$ such that

$$P(D)C_\Delta(Z_K) \supset R_J^K(C_\Delta(Z_J)). \tag{6}$$

We now have the following key result which is essentially Theorem 5.1 of Vogt [21] in our concrete situation (see also Langenbruch [15, Proposition 1.1] for a proof which can easily be transferred to the present situation).

2 Proposition. *Let the sequence of projective spectra* (5) *be exact. Then*

$$P(D)C_\Delta(Z) = C_\Delta(Z)$$

if (and only if) $Proj^1(N_P^Z) = 0$.

The reader is referred to Palamodov [18, 19], Vogt [21] or Wengenroth [22] for the definition of the Proj^1-functor. We do not need the definition here since we will only use explicit criteria from Langenbruch [15] for the vanishing of the Proj^1-functor of projective (DFS)-spectra (see Theorem 3 below). We shortly introduce the corresponding notions:

Let $\mathfrak{X} = \{X_K, R_J^K\}$ be a projective (DFS)-spectrum, that is, a projective spectrum of (DFS)-spaces $X_K = \lim_{k\longrightarrow} X_{K,k}$ with Banach spaces $X_{K,k}$ and compact inclusion mappings from $X_{K,k}$ into $X_{K,k+1}$. Let $B_{K,k}$ be the unit ball in $X_{K,k}$. For $X := \lim_{\longleftarrow K} X_K$ let

$$R^K : X \longrightarrow X_K$$

be the canonical mapping.

To state our sufficient condition for $Proj^1(\mathfrak{X}) = 0$ from Langenbruch [15] we need two further notions: Firstly, we will use condition (P_3) defined for the spectrum $\mathfrak{X}$ as follows (see Langenbruch [15, section 1]):

$$\forall K\ \exists L\ \forall M\ \exists k\ \forall l\ \exists m, C : R_L^K(B_{L,l}) \subset C\big(R_M^K(B_{M,m}) + B_{K,k}\big). \tag{7}$$

Secondly, we will need, that $\mathfrak{X}$ is reduced in the sense of Braun and Vogt [6, p. 150], that is,

$$\forall K\ \exists L\ \forall M \geq L : \text{ the closure of } R_M^K(X_M) \text{ in } X_K \text{ contains } R_L^K(X_L). \tag{8}$$

In many concrete situations the following theorem allows to check if $Proj^1(\mathfrak{X}) = 0$:

3 Theorem (Langenbruch [15, Theorem 1.4])**.** *$Proj^1(\mathfrak{X}) = 0$ if $\mathfrak{X}$ is a reduced projective (DFS)-spectrum satisfying property (P_3).*

4 The proofs

In this section the proof of our main result Theorem 1 is completed. From the discussion at the end of section 1, Proposition 2 and Theorem 3 we know that we have to show that the sequence of projective spectra (5) is exact (which roughly means that the equation $P(D)f = g$ can be solved semiglobally in $C_\Delta(Z)$) and that the kernel spectrum is reduced (which is a density property) and satisfies condition (P_3) (which is a decomposition with bounds in the kernel spectrum). For this, we need the following two basic Lemmata (see Lemmata 1.1 and 1.2 in Langenbruch [16]). For compact sets $Q \subset S \subset \mathbb{R}^{n+1}$ let

$$R_S^Q : C_\Delta(S) \longrightarrow C_\Delta(Q)$$

be the canonical mapping defined by restriction.

4 Lemma. *Let $Q \subset S \subset \mathbb{R}^{n+1}$ be compact sets such that*

$$\mathbb{R}^{n+1} \backslash Q \textit{ does not have a bounded component.} \tag{9}$$

(and the same for S). Then

$$P(D)C_\Delta(Q) \supset R_S^Q(C_\Delta(S))$$

if for any bounded set B in $C_\Delta(Q)'_b$ the set

$$\widetilde{B} := \{\, \mu \in C_\Delta(Q)' \mid P(-D)\mu \in B \,\}$$

is bounded in $C_\Delta(S)'_b$.

5 Lemma. *Let $Q \subset \mathbb{R}^{n+1}$ be compact with* (9)*. Then for any bounded set B in $C_\Delta(Q)'_b$ the set*

$$\widetilde{B} := \{\, \mu \in C_\Delta(Q)' \mid P(-D)\mu \in B \,\}$$

is bounded in $C_\Delta(\mathrm{conv}(Q))'_b$.

To apply Lemma 4 we need an appropriate representation for $C_\Delta(Q)'_b$. This is provided by the Grothendieck-Tillmann duality: Let

$$G(x,y) := -|(x,y)|^{1-n}/((n-1)c_{n+1}) \tag{10}$$

be the canonical even elementary solution of the Laplacian (see Hörmander [9], and recall that $(n+1) \geq 3$). For $Q \subset \mathbb{R}^{n+1}$ compact let

$$C_{\Delta,0}(\mathbb{R}^{n+1}\backslash Q) := \{\, f \in C_\Delta(\mathbb{R}^{n+1}\backslash Q) \mid \lim_{\xi\to\infty} f(\xi) = 0 \,\}$$

endowed with the topology of $C(\mathbb{R}^{n+1}\backslash Q)$. $C_{\Delta,0}(\mathbb{R}^{n+1}\backslash Q)$ is a Fréchet space. Let

$$\varkappa(\mu)(x,y) := u_\mu(x,y) := \langle \mu_{(s,t)},\, G(s-x,\, t-y)\rangle \text{ for } \mu \in C_\Delta(Q)'_b\,.$$

Then we have the topological isomorphisms

$$\varkappa : C_\Delta(Q)'_b \longrightarrow C_{\Delta,0}(\mathbb{R}^{n+1}\backslash Q) \cong C_\Delta(\mathbb{R}^{n+1}\backslash Q)/C_\Delta(\mathbb{R}^{n+1}) \tag{11}$$

by the Grothendieck-Tillmann duality (Grothendieck [7, Theorem 4], Mantovani, Spagnolo [17], Tillmann [20, Satz 6]).

We will also use the precise surjectivity results for partial differential operators on harmonic germs from Langenbruch [16], so we have to recall some notions introduced in that paper: For a compact $X \subset \Omega$ let

$$S(X,\Omega) := \{\, \xi \in \mathbb{R}^n \mid \xi + X \subset \Omega \,\}$$

and let $S_0(X,\Omega)$ be the component of 0 in $S(X,\Omega)$. The Ω-hull X_Ω of a compact $X \subset \Omega$ is defined by

$$X_\Omega := \{\, x \in \mathbb{R}^n \mid x + S_0(X,\Omega) \subset \Omega \,\} = \bigcap_{\xi \in S_0(X,\Omega)} (\Omega - \xi).$$

Let

$$J(c) := [\,-c, c\,] \text{ for } c > 0.$$

6 Theorem. *Let $P(D)$ be surjective on $A(\Omega)$. Then for any compact $X \subset \Omega$ there is $C > 0$ such that for any $\varepsilon > 0$ there is $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, any compact convex $Y \subset \Omega$ with $Y \supset \widetilde{X}_\varepsilon := (X_\Omega + V_\varepsilon) \cap V_C$ and any $0 < \gamma < \delta$ there is $0 < \beta < \gamma$ such that*

$$P(D)C_\Delta\big((X \times J(\delta)) \cup (Y \times J(\beta))\big) \supset C_\Delta\big((\tilde{X}_\varepsilon \times J(\delta)) \cup (Y \times J(\gamma))\big) \,|_{(X \times J(\delta)) \cup (Y \times J(\beta))} \,.$$

Proof. This is Langenbruch [16, Theorem 2.3.a and d] in the special case where $\eta = 0$ and Y is convex. QED

We first apply the preceding result for $\Omega := H_N$ and for $\Omega := \mathbb{R}^n$, respectively.

7 Corollary. *Let $P(D)$ be surjective on $A(H_N)$.*

(a) For any $L \in \mathbb{N}$ there is $\delta_L > 0$ such that for any $\eta > 0$ there is $E \in C_\Delta\big((V_{L,+} \times [-\delta_L, \infty[\,) \cup (V_L \times [\eta, \infty[\,)\big)$ such that

$$P(D)E = G \text{ near } (V_{L,+} \times [-\delta_L, \infty\,[) \cup (V_L \times [\eta, \infty[\,).$$

(b) For any $L \in \mathbb{N}$ there are $L_0 \in \mathbb{N}$ and $d_L > 0$ such that for any $M \in \mathbb{N}$, any $\xi \in \mathbb{R}^n$ with $M \geq |\xi| \geq L_0$ and any $\eta > 0$ there is $E_\xi \in C_\Delta\big((V_L \times [-d_L, \infty[) \cup (V_M \times [\eta, \infty[)\big)$ such that

$$P(D)E_\xi = G(\cdot - \xi, \cdot) \text{ near } (V_L \times [-d_L, \infty[) \cup (V_M \times [\eta, \infty[\,).$$

Proof. (a) (I) Let $\Omega := H_N$ and

$$X := CN + \{\, x \in \mathbb{R}^n \mid \langle x, N \rangle \geq A, |x| \leq B \,\}$$

for $A, B, C > 0$. Then

$$S_0(X, H_N) = S(X, H_N) = \{\, x \in \mathbb{R}^n \mid \langle x, N \rangle > -A - C \,\}$$

and

$$X_{H_N} = \{\, x \in \mathbb{R}^n \mid \langle x, N \rangle \geq A + C \,\} \tag{12}$$

if $B \geq A$, since $N \in S^n$.

(II) We now fix $L \in \mathbb{N}$ and apply Theorem 6 for $\Omega := H_N$ and

$$X := 2LN + V_{L,+} = 2LN + \{\, x \in \mathbb{R}^n \mid \langle x, N \rangle \geq 1/L, |x| \leq L \,\}$$

and get $C > 0$ from Theorem 6. Using Theorem 6 for $\varepsilon = 1/(2L)$ and (12) we get

$$\begin{aligned}\widetilde{X}_{1/(2L)} &= (X_{H_N} + V_{1/(2L)}) \cap V_C \\ &= \big(\{\, x \in \mathbb{R}^n \mid \langle x, N\rangle \geq 2L + 1/L \,\} + V_{1/(2L)}\big) \cap V_C \subset 2LN + V_{J_0,+}\end{aligned}$$

for some $J_0 \in \mathbb{N}$. From Theorem 6 we thus get $\delta_0 > 0$ such that (with $Y := 2LN + W$ for $W := \operatorname{conv}(V_L, V_{J_0,+})$ and $0 < \gamma := \eta/2 \leq \delta_0/4$)

$$\begin{aligned}P(D)C_\Delta\big(2LN + [(V_{L,+} \times J(\delta_0)) \cup (W \times J(\beta))]\big) \\ \supset C_\Delta\big(2LN + [(V_{J_0,+} \times J(\delta_0)) \\ \cup (W \times J(\eta/2))]\big) \,|_{2LN+[(V_{L,+}\times J(\delta_0))\cup(W\times J(\beta))]}\end{aligned}$$

for some $\beta > 0$. Since

$$G(\cdot - 2LN, \cdot + \eta) \in C_\Delta\big(2LN + [(V_{J_0,+} \times J(\delta_0)) \cup (W \times J(\eta/2))]\big)$$

we may thus find

$$E_1 \in C_\Delta\big(2LN + [(V_{L,+} \times J(\delta_0)) \cup (W \times J(\beta))]\big)$$

such that

$$P(D)E_1 = G(\cdot - 2LN, \cdot + \eta) \text{ near } 2LN + [(V_{L,+} \times J(\delta_0)) \cup (W \times J(\beta))].$$

We now shift the sets and the functions by $(-2LN, \eta)$ and restrict the functions to get

$$E_2 \in C_\Delta\big((V_{L,+} \times (\eta + J(\delta_0))) \cup (V_L \times (\eta + J(\beta)))\big)$$

such that

$$P(D)E_2 = G \text{ near } (V_{L,+} \times (\eta + J(\delta_0))) \cup (V_L \times (\eta + J(\beta))). \tag{13}$$

(III) Choose $\varphi \in C^\infty(\mathbb{R})$ such that $\varphi = 1$ near $]-\infty, \eta]$ and $\varphi = 0$ near $[\eta + \beta/2, \infty[$. The function $\Delta(\varphi(y)E_2(x, y))$ may be trivially extended (i.e. by the value 0) to an infinitely differentiable function $\tilde{h}$ defined on $U_L \times \mathbb{R}$. By the fundamental principle of Ehrenpreis-Palamodov we can find an infinitely differentiable function h such that

$$P(D)h = (1 - \varphi)G \text{ and } \Delta h = -\tilde{h} \text{ on } U_L \times \mathbb{R}. \tag{14}$$

Indeed, $U_L \times \mathbb{R}$ is convex, and the relation

$$P(D)(-\tilde{h}) = \Delta\big((1 - \varphi)G\big) \tag{15}$$

is satisfied. This is trivial on $U_L\times\,]-\infty,\eta[$ and $U_L\times\,]\eta+3\beta/4,\infty[$ while on $U_L\times\,]\eta-\beta,\eta+\beta[$ we get by (13)

$$\begin{aligned} P(D)(-\tilde{h}) &= P(D_x)\Delta(-\varphi E_2) = \Delta(-\varphi P(D_x)E_2) \\ &= \Delta(-\varphi G) = \Delta((1-\varphi)G). \end{aligned} \tag{16}$$

Set

$$E := \varphi E_2 + h.$$

By trivial extension of φE_2, E is then defined and harmonic on $(U_{L,+}\times\,]-2\delta_L,\infty[) \cup (U_L\times\,]\eta,\infty[)$ for $\delta_L := \delta_0/4$ since $\eta \le \delta_0/2$. Moreover, $P(D)E = G$ by (14). This shows the claim in (a) for $L-1$ and 2η instead of L and η, respectively.

(b) Since $P(D)$ is surjective on $A(H_N)$, $P(D)$ is also surjective on $A(\mathbb{R}^n)$ by Hörmander [8]. We may therefore apply Theorem 6 for $\Omega = \mathbb{R}^n$, $X = V_L$, $\varepsilon = 1$ and $Y := V_M$ for $M \ge L_0 := C+1$. For $\xi \in \mathbb{R}^n$ with $M \ge |\xi| \ge L_0$ and $\eta > 0$ we thus obtain (with $\gamma := \eta/2$)

$$E_1 \in C_\Delta\big((V_L \times J(\delta_0)) \cup (V_M \times J(\beta))\big)$$

such that

$$P(D)E_1 = G(\cdot - \xi, \cdot + \eta) \text{ near } (V_L \times J(\delta_0)) \cup (V_M \times J(\beta)).$$

For $E_2 := E_1(\cdot, \cdot - \eta)$ we thus get

$$P(D)E_2 = G(\cdot - \xi, \cdot) \text{ near } (V_L \times (\eta + J(\delta_0))) \cup (V_M \times (\eta + J(\beta))).$$

The proof of b) is now completed as in (a.III) above. *QED*

Let $A_K > 0$ be a strictly decreasing zero sequence such that

$$A_K \le \delta_{2K+1}/2 \tag{17}$$

for δ_{2K+1} from Corollary 7(a) and let

$$Z_{K,\delta} := (V_K \times [A_K - \delta, K + \delta]) \cup (V_{K,+} \times [-\delta, K + \delta]).$$

8 Theorem. *Let $P(D)$ be surjective on $A(H_N)$. Then for any $K \in \mathbb{N}$ there is $J_0 > K$ such that for any $J \ge J_0$ there is $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ and any $0 < \gamma < \delta$*

$$P(D)C_\Delta\big(Z_{K,\delta} \cup Z_{J,\gamma}\big) \supset C_\Delta\big(Z_{J_0,\delta} \cup Z_{2J,\gamma}\big)\,|_{Z_{K,\delta}\cup Z_{J,\gamma}}\,.$$

Proof. (a) We will use Lemma 4 for $S := Z_{J_0,\delta} \cup Z_{2J,\gamma}$ and $Q := Z_{K,\delta} \cup Z_{J,\gamma}$. Let B be bounded in $C_\Delta(Q)'_b$ and let

$$\widetilde{B} := \{ \mu \in C_\Delta(Q)' \mid P(-D)\mu \in B \}.$$

By Lemma 5, $\widetilde{B}$ is bounded in

$$C_\Delta\big(V_J \times [-\delta, J+\delta]\big)'_b. \tag{18}$$

(b) Let $J \geq J_0 \geq 2K$. Fix $\eta > 0$ and let

$$(x,y) \in M_1 := \{ (x,y) \in \mathbb{R}^{n+1} \mid y \in [-1, A_{2J} - \gamma - \eta], \langle x, N \rangle \leq \frac{1}{2J}, |x| \leq J+1 \}.$$

Let $0 < \delta_0 \leq \min(A_J/2, A_K - A_J)$ and choose

$$E \in C_\Delta\big((V_{2J+1,+} \times \, [-2A_J, \infty[\,) \cup (V_{2J+1} \times \, [\eta, \infty[\,)\big)$$

by Corollary 7(a) for $L := 2J+1$ (recall that $A_J \leq \delta_{2J+1}/2$ by (17)). Since

$$Q - M_1 = (Z_{K,\delta} \cup Z_{J,\gamma}) - M_1 \subset (V_{2J+1,+} \times [-2A_J, \infty[) \cup (V_{2J+1} \times [\eta, \infty[))$$

we get by Corollary 7(a)

$$\begin{aligned} u_\mu(x,y) = \langle \mu_{(s,t)}, G(s-x, t-y) \rangle &= \langle \mu_{(s,t)}, P(D)E(s-x, t-y) \rangle \\ &= \langle P(-D)\mu_{(s,t)}, E(s-x, t-y) \rangle \text{ for } (x,y) \in M_1. \end{aligned}$$

Since

$$\{ E(\cdot - x, \cdot - y) \mid (x,y) \in M_1 \}$$

is bounded in $C_\Delta(Q)$ and B is bounded in $C_\Delta(Q)'_b$, this implies that

$$\{ u_\mu \mid \mu \in \widetilde{B} \} \text{ is uniformly bounded on } M_1. \tag{19}$$

Let

$$M_2 := \{ (x,y) \in \mathbb{R}^{n+1} \mid y \in [-1, -\gamma - \eta], \frac{1}{2J} \leq \langle x, N \rangle \leq \frac{1}{J_0}, |x| \leq J+1 \}.$$

Then

$$Q - M_2 \subset (V_{2J+1,+} \times \, [-2A_J, \infty[\,) \cup (V_{2J+1} \times \, [\eta, \infty[\,))$$

and we conclude as above that

$$\{ u_\mu \mid \mu \in \widetilde{B} \} \text{ is uniformly bounded on } M_2. \tag{20}$$

(c) Let $0 < \delta_0 \leq d_{K+1}$ for d_{K+1} from Corollary 7(b). Let $J_0 \geq L_0 + 1$, where L_0 is chosen for $L := K + 1$ by Corollary 7(b) and set

$$R := \{ \xi \in \mathbb{R}^n \mid J_0 \leq |\xi| \leq J + 1 \}.$$

Since R is compact we may choose $\xi_1, \ldots, \xi_r \in R$ such that

$$\bigcup_{j=1}^{r} U_1(\xi_j) \supset R.$$

Fix $\eta > 0$ and choose

$$E_{\xi_j} \in C_\Delta\big((V_{K+1} \times [-d_{K+1}, \infty[\,) \cup (V_{J+1} \times [\eta, \infty[\,)\big)$$

by Corollary 7(b). Since

$$Q - (V_1 \times [-1, -\gamma - \eta]) \subset (V_{K+1} \times [-d_{K+1}, \infty[\,) \cup (V_{J+1} \times [\eta, \infty[\,),$$

we get for $x \in V_1$, $y \in [-1, -\gamma - \eta]$ and $j \leq r$

$$\begin{aligned} u_\mu(\xi_j + x, y) &= \langle \mu_{(s,t)}, G(s - \xi_j - x, t - y) \rangle \\ &= \langle \mu_{(s,t)}, P(D) E_{\xi_j}(s - x, t - y) \rangle = \langle P(-D) \mu_{(s,t)}, E_{\xi_j}(s - x, t - y) \rangle. \end{aligned}$$

Since

$$\{ E_{\xi_j}(\cdot - x, \cdot - y) \mid j \leq r, x \in V_1, y \in [-1, -\gamma - \eta] \}$$

is bounded in $C_\Delta(Q)$, this implies as above that

$$\{ u_\mu \mid \mu \in \widetilde{B} \} \text{ is uniformly bounded on } M_3 \tag{21}$$

for $M_3 := (V_{J+1} \setminus U_{J_0}) \times [-1, -\gamma - \eta]$.

(d) The claim follows from Lemma 4 by (18)-(21) and (11). QED

9 Corollary. *Let $P(D)$ be surjective on $A(H_N)$. Then the sequence of projective spectra* (5) *is exact.*

Proof. To check (6) we fix $K \in \mathbb{N}$ and apply Theorem 8 for $K + 1$ instead of K and $J := J_0$ and set $\widetilde{J} := 2J_0$. If $f \in C_\Delta(Z_{\widetilde{J}})$, then $f \in C_\Delta(Z_{2J_0,\delta})$ for some $0 < \delta < \delta_0$ and by Theorem 8 there is $g \in C_\Delta(Z_{K+1,\delta})$ such that

$$P(D) g = f \mid_{Z_{K+1,\delta}} .$$

Since $Z_{K+1,\delta}$ is a neighborhood of Z_K we can identify g with an element $g_{Z_K} \in C_\Delta(Z_K)$ and $P(D) g_{Z_K} = R^K_{\widetilde{J}}(f)$. QED

10 Corollary. *Let $P(D)$ be surjective on $A(H_N)$. Then the projective spectrum N_P^Z is reduced.*

PROOF. (a) To check (8), we fix $\nu \in C_\Delta(Z_K)'$ such that $\nu \mid_{N_P(Z_M)}= 0$ for some $M \geq 2K$, and we will show that $\nu \mid_{N_P(Z_{2K})}= 0$. In part (a.II.i) of the proof of Langenbruch [15, Proposition 4.3] we already showed that

$$P(-D)\mu = \nu \text{ for some } \mu \in C_\Delta(\mathbb{R}^{n+1})'. \tag{22}$$

By Lemma 5 we have

$$\mu \in C_\Delta(V_K \times [0, K])' \tag{23}$$

since $\operatorname{conv}(Z_K) \subset V_K \times [0, K]$.

Since $C_\Delta(\mathbb{R}^{n+1})$ is dense in $C_\Delta(V_K \times [0, K])$, (22) implies that

$$\langle \nu, f\rangle = \langle P(-D)\mu, f\rangle = \langle \mu, P(D)f\rangle \text{ if } f \in C_\Delta(V_K \times [0, K]) \tag{24}$$

hence

$$\langle \nu, f\rangle = 0 \text{ if } f \in N_P\,(V_K \times [0, K]) \tag{25}$$

(b) By Corollary 7(a) applied to $L := 2K + 1$ there are $\delta_{2K+1} > 0$ and $E \in C_\Delta\big((V_{2K+1,+} \times [-\delta_{2K+1}, \infty[) \cup (V_{2K+1} \times [\eta, \infty[)\big)$ such that

$$P(D)E = G \text{ near } (V_{2K+1,+} \times\ [-\delta_{2K+1}, \infty[\,) \cup (V_{2K+1} \times\ [\eta, \infty[\,)\,. \tag{26}$$

On the other hand, by the fundamental principle of Ehrenpreis-Palamodov there is $F \in C_\Delta(\mathbb{R}^n \times]0, \infty[\,)$ such that

$$P(D)F = G \text{ on } \mathbb{R}^n \times]0, \infty[\,. \tag{27}$$

For $x \in U_{K+1}$ and $y \in] - 2, -1[$ we therefore have

$$P(D)F(\cdot - x, \cdot - y) = G(\cdot - x, \cdot - y) = P(D)E(\cdot - x, \cdot - y) \text{ near } V_K \times [0, K].$$

Hence,

$$F(\cdot - x, \cdot - y) = E(\cdot - x, \cdot - y) + h_{x,y} = 0$$

for some $h_{x,y} \in N_P\big(V_K \times [0, K]\big)$. Since therefore $\nu(h_{x,y}) = 0$ by (25), we get for $x \in U_{K+1}$ and $y \in] - 2, -1[$

$$\begin{aligned} u_\mu(x, y) &= \langle \mu_{(s,t)}, G(s - x, t - y)\rangle = \langle \mu_{(s,t)}, P(D_s)F(s - x, t - y)\rangle \\ &= \langle P(-D)\mu_{(s,t)}, F(s - x, t - y)\rangle = \langle \nu_{(s,t)}, F(s - x, t - y)\rangle \\ &= \langle \nu_{(s,t)}, E(s - x, t - y)\rangle =: v(x, y), \end{aligned} \tag{28}$$

where (28) follows from (24). v is harmonic on

$$M_1 := \{\, (x, y) \in \mathbb{R}^{n+1} \mid y \in] - 2, A_{2K}[,\ |x| < K + 1, \langle x, N\rangle < 1/(2K)\,\}$$

since by (17)

$$Z_K - M_1 \subset (V_{2K+1,+} \times \,[-\delta_{2K+1}, \infty[\,) \cup (V_{2K+1} \times \,[\eta, \infty[\,)$$

if $0 < \eta < A_K - A_{2K}$. Using also (23) we have thus shown that $\mu \in C_\Delta(Z_{2K})'$. Since $C_\Delta(\mathbb{R}^{n+1})$ is dense in $C_\Delta(Z_{2K})$ we have $P(-D)\mu = \nu$ also in $C_\Delta(Z_{2K})'$. Thus

$$\langle \nu, f \rangle = \langle P(-D)\mu, f \rangle = \langle \mu, P(D)f \rangle = 0 \text{ for } f \in N_P(Z_{2K}).$$

QED

We finally must check that the projective spectrum N_P^Z satisfies the property (P_3) (see (7)). For this we need to specify the (DFS)-structure of the step spaces $N_P(Z_K)$: For $K, k, c > 0$ let

$$\widetilde{U}_c := \{\, \xi \in \mathbb{R}^{n+1} \mid |\xi| < c \,\} \text{ and } Z_K(k) := Z_K + \widetilde{U}_{1/k}.$$

For an open set $W \subset \mathbb{R}^{n+1}$ let

$$CB_\Delta(W) := \{\, f \in C_\Delta(W) \mid f \text{ is bounded on } W \,\}$$

and

$$NB_P(W) := N_P(W) \cap CB_\Delta(W).$$

Then the (DFS)-structure of $N_P(Z_K)$ is given by

$$N_P(Z_K) = \lim_{k \to \infty} NB_P(Z_K(k)).$$

11 Theorem. *Let $P(D)$ be surjective on $A(H_N)$. Then the projective spectrum N_P^Z satisfies property (P_3).*

Proof. The proof is similar as for Langenbruch [15, Theorem 4.5]. It is based on Theorem 8: We will first decompose functions in $N_P(Z_{L+1}(l))$ as harmonic functions (see (a) below) and then use Theorem 8 (see (c)) to obtain a decomposition as harmonic zero solutions of $P(D)$ (in (d)).

In the proof below we will often use the notation from section 2.

(a) For any $L, l \in \mathbb{N}$ there is a continuous linear operator

$$R = (R_1, R_2) : CB_\Delta(Z_{L+1}(l)) \longrightarrow CB_\Delta(Y_{1,1}) \times CB_\Delta(Y_{1,2})$$

such that $R_1(f) + R_2(f) = f$ on $\operatorname{int}(Z_{L+1,2l})$. Here

$$Y_{1,1} := U_{L+1,+} \times \,]-\infty, 0[\, \cup \operatorname{int}(Z_{L+1,2l})$$

and

$$Y_{1,2} := \mathbb{R}^n \times \,]-1/(2l), \infty[\,.$$

PROOF. Choose $\varphi \in D(Z_{L+1}(l))$ with $\varphi = 1$ near $Z_{L+1,2l}$.
For $f \in CB_\Delta(Z_{L+1}(l))$, φf can be considered as a function on $Y_{1,2}$, and

$$\tilde{f} := \Delta(\varphi f)\mid_{Y_{1,2}}$$

defines a function f_1 on $\mathbb{R}^{n+1}$ by trivial extension (i.e. by setting $f_1 \equiv 0$ outside $Y_{1,2}$). f_1 is bounded and has compact support. Thus,

$$R_1(f) := G * f_1 \mid_{Y_{1,1}}$$

and

$$R_2(f) := (\varphi f - G * f_1)\mid_{Y_{1,2}}$$

have the required properties. QED

(b) For $f \in NB_P(Z_{L+1}(l))$ we have by (a)

$$P(D)R_1(f) = -P(D)R_2(f) \text{ on } Y_{1,1} \cap Y_{1,2} = \operatorname{int}(Z_{L+1,2l}).$$

Thus, a continuous and linear operator

$$\widetilde{R} : NB_P(Z_{L+1}(l)) \longrightarrow C_\Delta(Y_1),$$
$$Y_1 := Y_{1,1} \cup Y_{1,2} = (\mathbb{R}^n \times]-1(2l), \infty[) \cup (U_{L+1,+} \times \mathbb{R}),$$

is defined by

$$\widetilde{R}(f) := P(D)R_1(f) \text{ on } Y_{1,1}$$

and

$$\widetilde{R}(f) := -P(D)R_2(f) \text{ on } Y_{1,2}.$$

(c) Fix $K \in \mathbb{N}$ and choose $J_0 =: L$ for $K+2$ instead of K by Theorem 8. Let $M \geq L$ and fix $k \in \mathbb{N}$ with

$$k \geq \max\bigl(1/\delta_0, (K+2)^2,\, 1/(A_K - A_L)\bigr),$$

where δ_0 is chosen for $J := M+2$ by Theorem 8. Then for $l \geq k + (M+2)^2$ there is a continuous linear operator

$$S : C_\Delta\bigl(Y_1\bigr) \longrightarrow CB_\Delta(Y), Y := Z_K(k+2) \cup Z_M(5l),$$

such that

$$P(D)S(f) = f \text{ on } Y \text{ for } f \in C_\Delta(Y_1) \tag{29}$$

Proof. Let $W := Z_{K+2,1/k} \cup Z_{M+2,1/(3l)}$. Since

$$Y_1 \supset Z_{L,1/k} \cup Z_{2(M+2),1/(3l)},$$

the mapping

$$P(D)^{-1} : C_\Delta(Y_1) \longrightarrow C_\Delta(W)/N_P(W)$$

is defined, linear and continuous by Theorem 8 and the closed graph theorem.

For an open set Y in $\mathbb{R}^{n+1}$ let

$$(L_2)_\Delta(Y) := L_2(Y) \cap C_\Delta(Y).$$

$(L_2)_\Delta(Y)$ is a Hilbert space and

$$(L_2)_P(Y) := (L_2)_\Delta(Y) \cap \ker(P(D))$$

is a closed subspace. By the choice of k and l,

$$Y_2 := Z_{K+1}(k+1) \cup Z_{M+1}(4l) \subset\subset \operatorname{int}(W),$$

and the restriction defines a continuous linear mapping

$$J_1 : C_\Delta(W)/N_P(W) \longrightarrow (L_2)_\Delta(Y_2)/(L_2)_P(Y_2).$$

Let

$$\Pi : (L_2)_\Delta(Y_2)/(L_2)_P(Y_2) \longrightarrow \big((L_2)_P(Y_2)\big)^\perp$$

be the canonical topological isomorphism. Since

$$Y = Z_K(k+2) \cup Z_M(5l) \subset\subset Y_2,$$

the restriction

$$J_2 : \big((L_2)_P(Y_2)\big)^\perp \longrightarrow CB_\Delta(Y)$$

is defined and continuous. Then

$$S := J_2 \circ \Pi \circ J_1 \circ P(D)^{-1} : C_\Delta(Y_1) \longrightarrow CB_\Delta(Y)$$

is defined, linear and continuous and satisfies (29). QED

(d) Since

$$Z_K(k+2) \subset Y_{1,1} \text{ and } Z_M(5l) \subset Y_{1,2}$$

by the choice of k, we may use the operators constructed in b) and (c) to define

$$T = (T_1, T_2) : NB_P\big(Z_{L+1}(l)\big) \longrightarrow NB_P\big(Z_K(k)\big) \times NB_P\big(Z_M(5l)\big)$$

by

$$T_1(f) := \big(R_1(f) - S \circ \widetilde{R}(f)\big)|_{Z_K(k+2)}$$

and

$$T_2(f) := \big(R_2(f) + S \circ \widetilde{R}(f)\big)|_{Z_M(5l)}$$

for $f \in NB_P(Z_{L+1}(l))$. Notice that $P(D)T = 0$ by the definition of $\widetilde{R}$ in b) and by (29). By (a) it is clear that

$$T_1(f) + T_2(f) = f \text{ on } Z_K(k+2) \cap Z_M(5l)$$

if $f \in NB_P(Z_{L+1}(l))$. This proves (P_3) since T is continuous. QED

Using the remarks at the end of section 1, the proof of Theorem 1 is now completed by the following

12 Theorem. *Let $P(D)$ be surjective on $A(H_N)$. Then*

$$P(D)C_\Delta(Z) = C_\Delta(Z).$$

PROOF. The sequence of projective spectra (5) is exact by Corollary 9. N_P^Z is a reduced projective spectrum satisfying property (P_3) by Corollary 10 and Theorem 11. The claim thus follows from Proposition 2 and Theorem 3. QED

References

[1] A. Andreotti, M. Nacinovich: Analytic Convexity and the Principle of Phragmen-Lindelöf, Scuola Norm. Sup., Pisa 1980.

[2] G. Bengel: *Das Weylsche Lemma in der Theorie der Hyperfunktionen*, Math. Z., **96**, (1967), 373–392.

[3] R.W. Braun: *The surjectivity of a constant coefficient homogeneous differential operator on the real analytic functions and the geometry of its symbol*, Ann. Inst. Fourier, **45**, (1995), 223–249.

[4] R.W. Braun, R. Meise, B.A. Taylor: *The geometry of analytic varieties satisfying the local Phragmen-Lindelöf condition and a geometric characterization of the partial differential operators that are surjective on $A(\mathbb{R}^4)$*, Transactions AMS, to appear.

[5] R.W. Braun, R. Meise, B.A. Taylor: *Surjectivity of constant coefficient partial differential operators on $A(\mathbb{R}^4)$ and Whitney's C_4-cone*, Bull. Soc. Roy. Liege, **70**, (2001), 195–206.

[6] R.W. Braun and D. Vogt: *A sufficient condition for $Proj^1\mathfrak{X} = 0$*, Michigan Math. J., **44**, (1997), 149–156.

[7] A. Grothendieck: *Sur les espaces des solutions d'une classe générale d'équations aux derivées partielles*, J. Analyse Math., **2**, (1952/53), 243–280.

[8] L. Hörmander: *On the existence of real analytic solutions of partial differential equations with constant coefficients*, Invent. Math., **21**, (1973), 151–183.

[9] L. Hörmander: The Analysis of Linear Partial Differential Operators I+II, Grundlehren Math. Wiss. Bde. 256+257, Springer, Berlin/Heidelberg/New York 1983.

[10] A. Kaneko: *On the global existence of real analytic solutions of linear partial differential equations on unbounded domain*, J. Fac. Sci. Tokyo, **32**, (1985), 319–372.

[11] A. Kaneko: *A sharp sufficient geometric condition for the existence of global real analytic solutions on a bounded domain*, J. Math. Soc. Japan, **39**, (1987), 163–170.

[12] T. Kawai: *On the global existence of real analytic solutions of linear differential equations I*, J. Math. Soc. Japan, **24**, (1972), 481–517.

[13] M. Langenbruch: *Localizations of partial differential operators and surjectivity on spaces of real analytic functions*, Studia Math., **140**, n. 1, (2000), 15–40.

[14] M. Langenbruch: *Surjective partial differential operators on real analytic functions defined on convex open sets*, Manuscr. Math., **103**, (2000), 241–263.

[15] M. Langenbruch: *Characterization of surjective partial differential operators on spaces of real analytic functions*, Studia Math., **162(1)**, (2004), 53–96.

[16] M. Langenbruch: *Inheritance of surjectivity for partial differential operators on spaces of real analytic functions*, J. Math. Anal. Appl., **297**, (2004), 696–719.

[17] F. Mantovi and S. Spagnolo: *Funcionali analitici reali e funzioni armoniche*, Ann. Sc. Norm. Sup. Pisa, Ser. III, (1964), 475–513.

[18] V.P. Palamodov: *The projective limit functor in the category of linear topological spaces*, Math. USSR-Sb., **4**, (1968), 529–558.

[19] V.P. Palamodov: *Homological methods in the theory of locally convex spaces*, Russian Math. Surveys, **26**, (1971), 1–64.

[20] H.G. Tillmann: *Dualität in der Potentialtheorie*, Port. Math., **13**, (1954), 55–86.

[21] D. Vogt: *Topics on projective spectra of* (LB)*-spaces*, In: T. Terzioglu, editor, Advances in the Theory of Frèchet-spaces, of *NATO ASI Series C*, **287**, 11–27, Kluwer, 1989.

[22] J. Wengenroth: Derived Functors in Functional Analysis, LN in Math. 1810, Springer, Berlin/Heidelberg/New York 2003.

[23] G. Zampieri: *Propagation of singularity and existence of real analytic solutions of locally hyperbolic equations*, J. Fac. Sci. Univ. Tokyo Ser. IA Math., **31**, (1984), 373–390.