Note di Matematica **27,** n. 2, 2007, 107–119.

On a continuum-mechanical theory for turbulence

Eliot Fried
Department of Mechanical, Aerospace and Structural Engineering
Washington University in St. Louis
St. Louis, MO 63130-4899, USA
mechanicist@gmail.com

Morton Gurtin
Department of Mathematical Sciences
Carnegie Mellon University
Pittsburgh, PA 15213-3890, USA
mg0c@andrew.cmu.edu

Received: 20/10/06; accepted: 13/11/06.

Abstract. We discuss a continuum-mechanical formulation and generalization of the Navier–Stokes-α equation based on a comprehensive framework for fluid-dynamical theories with gradient dependencies (Fried & Gurtin 2006). Our flow equation entails two additional material length scales: one energetic, the other dissipative. In contrast to Lagrangian averaging, our formulation delivers boundary conditions — involving yet another length scale — and a complete structure based on thermodynamics applied to an isothermal system.

Keywords: Turbulence, hypertraction and hyperstress, virtual power, boundary conditions

MSC 2000 classification: primary 76F05, secondary 76D06, 76A05

1 Background

The Lagrangian averaged Navier–Stokes-α model for (statistically homogeneous and isotropic) turbulent flow yields a governing equation for the fluid velocity $\mathbf{v}$ that can be written in the form

$$\rho\dot{\mathbf{v}} = -\operatorname{Grad} p + \mu(1-\alpha^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2 \operatorname{div}\overset{\circ}{\mathbf{D}}; \tag{1}$$

(1) is commonly referred to as the *Navier–Stokes-α equation*. In this equation: $\mathbf{v}$ is subject to the incompressibility constraint

$$\operatorname{div}\mathbf{v} = 0; \tag{2}$$

$\dot{\mathbf{v}} = \mathbf{v}' + (\operatorname{Grad}\mathbf{v})\mathbf{v}$ is the material time derivative of $\mathbf{v}$; p is the pressure; Δ is the Laplace operator;

$$\mathbf{D} = \frac{1}{2}(\operatorname{Grad}\mathbf{v} + (\operatorname{Grad}\mathbf{v})^{\top})$$

is the stretch-rate;

$$\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{DW} - \mathbf{WD}, \tag{3}$$

with

$$\mathbf{W} = \frac{1}{2}(\mathrm{Grad}\,\mathbf{v} - (\mathrm{Grad}\,\mathbf{v})^{\top})$$

the spin, is the corotational rate of $\mathbf{D}$. The Lagrangian averaged Euler equation, which is (1) with $\mu = 0$, was first derived by Holm, Marsden & Ratiu (1998a, 1998b). Subsequently, Chen, Foias, Holm, Olson, Titi & Wynne (1998, 1999a, 1999b) added the viscous term to the Lagrangian averaged Euler equation, giving (1). See also Shkoller (2000) Marsden & Shkoller (2001).

Aside from the density ρ and the shear viscosity μ, the flow equation (1) involves an additional material parameter $\alpha > 0$ carrying dimensions of length. Within the framework of Lagrangian averaging, α is the statistical correlation length of the excursions taken by a fluid particle away from its phase-averaged trajectory. More intuitively, α can be interpreted as the characteristic linear dimension of the smallest eddy that the model is capable of resolving. Like equations arising from Reynolds averaging, the Navier–Stokes-α equation provides an approximate model that resolves motions only above some critical scale, while relying on filtering to approximate effects at smaller scales. In recognition of this, $\mathbf{v}$ might be best viewed as a *filtered* velocity. A synopsis of properties and advantages of the Navier–Stokes-α equation is provided by Holm, Jeffrey, Kurien, Livescu, Taylor & Wingate (2005).

The structure of (1) is formally suggestive of a conservation law expressing the balance of linear momentum, and one might ask whether there is a complete continuum mechanical framework in which the Navier–Stokes-α equation is embedded along with suitable boundary conditions. Based on experience with theories for plates, shells, and other structured media, the presence of a term involving the fourth-order spatial gradient of the velocity indicates that any such framework should involve a *hyperstress* in addition to the classical stress. Within the context of turbulence theory, a hyperstress might be viewed as providing a means to account for interactions across disparate length scales.

2 Principle of virtual power

To see the need for an additional hyperstress assume an inertial frame, neglect non-inertial body forces, and note first that the weak form of the classical momentum balance

$$\mathrm{div}\,\mathbf{T} + \mathbf{b} = \mathbf{0}, \tag{4}$$

with inertial force

$$\mathbf{b} = -\varrho\dot{\mathbf{v}} \tag{5}$$

treated for convenience as a body force, has the form

$$\underbrace{\int\limits_{\partial R} \mathbf{t}(\mathbf{n}) \cdot \boldsymbol{\phi} \, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \boldsymbol{\phi} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R,\boldsymbol{\phi})} = \underbrace{\int\limits_{R} \mathbf{T} \colon \mathrm{Grad}\, \boldsymbol{\phi} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R,\boldsymbol{\phi})}, \tag{6}$$

with

$$\mathbf{t_n} = \mathbf{Tn} \tag{7}$$

the classical surface-traction of Cauchy. Granted smoothness (6) holds for all *virtual velocities* (i.e., test fields) $\boldsymbol{\phi}$ and all control volumes R if and only if the balance (4) is satisfied at all points in the fluid and the traction condition (7) is satisfied — for any choice of the unit vector $\mathbf{n}$ — at all points in the fluid. Moreover, the requirement of frame-indifference applied to (6) yields the symmetry of the stress $\mathbf{T}$.

When $\boldsymbol{\phi}$ represents the velocity $\mathbf{v}$ of the fluid, the weak balance (6) is a physical balance

$$\underbrace{\int\limits_{\partial R} \mathbf{t_n} \cdot \mathbf{v} \, \mathrm{d}a + \int\limits_{R} \mathbf{b} \cdot \mathbf{v} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R)} = \underbrace{\int\limits_{R} \mathbf{T} \colon \mathrm{Grad}\, \mathbf{v} \, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R)} \tag{8}$$

between:

(i) the external power $\mathcal{W}_{\mathrm{ext}}(R)$, which represents power expended on R by tractions acting on ∂R and power expended by the inertial force $\mathbf{b}$;

(ii) the internal power $\mathcal{W}_{\mathrm{int}}(R)$, the integrand of which represents the classical stress power $\mathbf{T} \colon \mathrm{Grad}\, \mathbf{v}$ expended within R by the stress field $\mathbf{T}$.

Here and in what follows, we write $\mathcal{W}_{\mathrm{ext}}(R)$ for the external power associated with an actual flow and $\mathcal{W}_{\mathrm{ext}}(R, \boldsymbol{\phi})$ for the (virtual) external power associated with a virtual velocity field $\boldsymbol{\phi}$.

The balance (6) represents a nonstandard form of the classical principle of virtual power (Gurtin 2002). This nonstandard form was generalized by Fried & Gurtin (2006) to develop a gradient theory for liquid flows at small length scales and, when combined with suitable constitutive relations, results in a partial differential equation slightly more general than (1) but with the term involving the corotational rate of $\mathbf{D}$ removed. Conventional versions of this principle are formulated for the fluid region as a whole rather than for control volumes and as such generally involve particular boundary conditions. Here the principle of virtual power is used instead as a basic tool in determining the structure of the tractions and of the local force balances. Accordingly, conditions on the external boundary play a role no different from those on the boundary of any control volume. Basic to this view is the premise, central to all of continuum mechanics, that any basic law for the body should hold also for all subregions of the body. On a more pragmatic note, the nonstandard formulation allows for the derivation of the associated angular momentum balance.

To capture the internal power associated with the formation of eddies during turbulent flow we generalize the classical theory by including in the internal power a term linear in the *vorticity* gradient $\operatorname{Grad}\boldsymbol{\omega} = \operatorname{Grad}\operatorname{curl}\mathbf{v}$. Specifically, we introduce a second-order tensor-valued *hyperstress* $\mathbf{G}$ via an internal power expenditure of the form $\mathbf{G}\colon \operatorname{Grad}\boldsymbol{\omega}$ and rewrite the power expended within R in the form

$$\mathcal{W}_{\mathrm{int}}(R) = \int\limits_R \big(\mathbf{T}\colon \operatorname{Grad}\mathbf{v} + \mathbf{G}\colon \operatorname{Grad}\boldsymbol{\omega}\big)\, \mathrm{d}v. \tag{9}$$

In conjunction with the internal power expenditure (9), we introduce a corresponding external power expenditure

$$\mathcal{W}_{\mathrm{ext}}(R) = \int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\, \mathrm{d}a + \int\limits_R \mathbf{b}\cdot\mathbf{v}\, \mathrm{d}v, \tag{10}$$

in which $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ represent tractions on the bounding surface $\mathcal{S} = \partial R$ of R, while $\mathbf{b}$ represents the inertial body force (5). Here the term

$$\mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}, \tag{11}$$

which is not present in classical theories, is needed to balance the effects of the internal-power term $\mathbf{G}\colon \operatorname{Grad}\boldsymbol{\omega}$, which involves the second gradient of $\mathbf{v}$.

The *principle of virtual power* replaces $\mathbf{v}$ by $\boldsymbol{\phi}$ and (hence) $\boldsymbol{\omega}$ by $\operatorname{curl}\boldsymbol{\phi}$ and is based on the requirement that the internal and external power expenditures be equal:

$$\underbrace{\int\limits_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\, \mathrm{d}a + \int\limits_R \mathbf{b}\cdot\mathbf{v}\, \mathrm{d}v}_{\mathcal{W}_{\mathrm{ext}}(R)} = \underbrace{\int\limits_R \big(\mathbf{T}\colon \operatorname{Grad}\mathbf{v} + \mathbf{G}\colon \operatorname{Grad}\boldsymbol{\omega}\big)\, \mathrm{d}v}_{\mathcal{W}_{\mathrm{int}}(R)} \tag{12}$$

for all control volumes R and any choice of the virtual velocity field $\boldsymbol{\phi}$.

3 Local balance law for linear momentum. Traction conditions

Consequences of the virtual power principle and the requirement that the internal power expenditure be frame-indifferent are that:

(i) The classical macroscopic balance $\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T}$ must be replaced by the balance

$$\rho\dot{\mathbf{v}} = \operatorname{div}\mathbf{T} + \operatorname{curl}\operatorname{div}\mathbf{G}, \tag{13}$$

with $\mathbf{T}$ symmetric as in the classical theory.

(ii) Cauchy's classical condition $\mathbf{t_n} = \mathbf{Tn}$ for the traction across a surface $\mathcal{S}$ with unit normal $\mathbf{n}$ must be replaced by the conditions

$$\left.\begin{aligned} \mathbf{t}_{\mathcal{S}} &= \mathbf{Tn} + \operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times) + \mathbf{n}\times(\operatorname{div}\mathbf{G} - 2K\mathbf{Gn}),\\ \mathbf{m}_{\mathcal{S}} &= \mathbf{n}\times\mathbf{Gn}, \end{aligned}\right\} \tag{14}$$

in which $\operatorname{div}_{\mathcal{S}}$ is the divergence operator on $\mathcal{S}$ and K is the mean curvature of $\mathcal{S}$.[1] Thus, interestingly, the traction $\mathbf{t}_{\mathcal{S}}$ depends on the mean curvature; in fact, the term $\operatorname{div}_{\mathcal{S}}(\mathbf{Gn}\times)$ results in a dependence on the curvature tensor $-\operatorname{Grad}_{\mathcal{S}}\mathbf{n}$.

The balance (13) and the traction conditions (14) are special cases of equations (5.11) and (5.12) of Fried & Gurtin (2006), whose theory replaces $\operatorname{curl}\boldsymbol{\omega}$ in the internal power with the full second gradient $\operatorname{Grad}^2\mathbf{v}$ and $\mathbf{G}$ by an analogous third-order hyperstress.[2]

4 Energetics

We restrict attention to a purely mechanical theory based on the requirement that *the temporal increase in free energy of an arbitrary region that convects with the body* $\mathcal{R}(t)$ *be less than or equal to the power expended on that region.* Precisely, letting ψ denote the *specific free energy*, this requirement takes the form of a free energy imbalance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathcal{R}(t)} \rho\psi\,\mathrm{d}v \le \mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)). \tag{15}$$

The imbalance (15) is consistent with standard continuum thermodynamics based on balance of energy and an entropy imbalance (the Clausius–Duhem

[1] Within the framework of finite deformations of an *elastic solid* with couple-stress, the balance (13) was first derived by the Cosserats (1909); see, also, Toupin (1962, 1964), Mindlin & Tiersten (1962), and Green & Naghdi (1968). The traction conditions (14) are special cases of traction conditions derived variationally by Toupin (1962, 1964) for the boundary of the elastic solid.

[2] Cf. Bluestein & Green (1967), who discuss second-gradient fluids based on the multipolar theory of second-gradient materials due to Green & Rivlin (1964). This theory results in redundant boundary conditions, which Bluestein & Green (1967) reduce using ad hoc arguments.

inequality), when that imbalance is restricted appropriately to isothermal processes.

Balance of mass implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{R}(t)} \rho\psi \,\mathrm{d}v = \int_{R(t)} \rho\dot{\psi}\,\mathrm{d}v;$$

since $\mathcal{W}_{\mathrm{ext}}(\mathcal{R}(t)) = \mathcal{W}_{\mathrm{int}}(\mathcal{R}(t))$, we may therefore use the expression (9) for the internal power $\mathcal{W}_{\mathrm{int}}(\mathcal{R}(t))$ in conjunction with the symmetry of $\mathbf{T}$ to localize (15); the result is the *local free energy imbalance*

$$\rho\dot{\psi} - \mathbf{T}\colon\mathbf{D} - \mathbf{G}\colon \mathrm{Grad}\,\boldsymbol{\omega} \le 0, \tag{16}$$

where $\mathbf{D} = \frac{1}{2}(\mathrm{Grad}\,\mathbf{v} + (\mathrm{Grad}\,\mathbf{v})^{\top})$ is the stretching.

5 Simple constitutive equations

We assume that the fluid is incompressible, so that

$$\rho = \text{constant} \qquad \text{and} \qquad \mathrm{div}\,\mathbf{v} = \mathrm{tr}\,\mathbf{D} = 0. \tag{17}$$

Without loss in generality, we may then suppose that

$$\mathbf{T} = \mathbf{S} - p\mathbf{1}, \quad \mathrm{tr}\,\mathbf{S} = 0, \tag{18}$$

where the *pressure* p is a constitutively indeterminate field that does not affect the internal power (9); the field $\mathbf{S}$ represents the *extra stress*. Then, by $(17)_2$,

$$\mathbf{T}\colon\mathbf{D} = \mathbf{S}\colon\mathbf{D}, \tag{19}$$

and the local free-energy imbalance (16) reduces to

$$\mathbf{S}\colon\mathbf{D} + \mathbf{G}\colon \mathrm{Grad}\,\boldsymbol{\omega} - \rho\dot{\psi} \ge 0. \tag{20}$$

Guided by the presence of the term involving the corotational rate $\overset{\circ}{\mathbf{D}} = \dot{\mathbf{D}} + \mathbf{D}\mathbf{W} - \mathbf{W}\mathbf{D}$ of the stretching tensor $\mathbf{D}$ in the Navier–Stokes-α equation (1), we suppose that the specific free energy ψ and the extra stress $\mathbf{S}$ are given by constitutive equations of the form

$$\psi = \alpha^2|\mathbf{D}|^2 \qquad \text{and} \qquad \mathbf{S} = 2\mu\mathbf{D} + 2\rho\alpha^2\overset{\circ}{\mathbf{D}}, \tag{21}$$

with α and μ constant. Further, based on a result of Mindlin & Tiersten (1962) for an elastic solid, we assume that the hyperstress is given by a constitutive equation of the form[3]

$$\mathbf{G} = \mu\beta^2(\operatorname{Grad}\boldsymbol{\omega} + \gamma(\operatorname{Grad}\boldsymbol{\omega})^\top), \tag{22}$$

with $\beta > 0$ and γ constant. With the choices (21) and (22), the dissipation inequality (20) holds if and only if

$$\mu \geq 0 \qquad \text{and} \qquad |\gamma| \leq 1. \tag{23}$$

Whereas μ is the conventional shear viscosity, the constitutive parameters α and β carries dimensions of length. Whereas α is related to the specific free energy and, therefore, nondissipative contribution to the extra stress, β is associated with the wholly dissipative hyperstress. To ensure that the specific free energy has a strict minimum when $\mathbf{D} = \mathbf{0}$ we assume that

$$\alpha > 0; \tag{24}$$

to ensure that the hyperstress is nontrivial when $\operatorname{Grad}\boldsymbol{\omega} \neq \mathbf{0}$, we assume that

$$\beta > 0. \tag{25}$$

When discussing turbulence, it is common to divide the range of eddy scales into mutually disjoint *integral*, *inertial*, and *dissipative subranges* (Richardson 1922; Kolmogorov 1941a-c, 1962; Pope 2000). The integral scales are the largest and are associated with external driving forces. The dissipative scales are the smallest and are associated with the conversion of kinetic energy into heat. The intermediate, inertial, scales are thought to be dissipationless. We expect that the energetic length α should in some way characterize eddy scales within the inertial subrange. Further, we expect that β should characterize eddy scales within the dissipative subrange.

Using (21) and (22) in (13) and bearing in mind that the moduli μ, α, β, and γ are assumed to be constant, we arrive at the flow equation

$$\rho\dot{\mathbf{v}} = -\operatorname{Grad}p + \mu(1 - \beta^2\Delta)\Delta\mathbf{v} + 2\rho\alpha^2\operatorname{div}\mathring{\mathbf{D}}, \tag{26}$$

which, for the particular choice

$$\beta = \alpha \tag{27}$$

[3]These choices are familiar from the theory of Rivlin–Ericksen fluids; cf. Rivlin & Ericksen, 1955; Truesdell & Noll, 1965, §119; Dunn & Fosdick, 1974.

specializes to the Navier–Stokes-α equation (1). In view of the foregoing discussion, the choice (27) embodies a questionable assumption concerning the relationship between the scales of inertial and dissipative eddies.

Interestingly, the material parameter γ, which is dimensionless, does not enter the flow equation (26). However, as is clear from $(14)_2$ and (22), γ would generally be present in any boundary condition in which the hypertraction is prescribed.[4]

6 Boundary conditions

We develop counterparts of the classical notions of a free surface and a fixed surface without slip. Our results hinge on rewriting the external power expenditure (10) for the entire fluid body B and focusing on that portion of this expenditure associated with tractions. In this regard, we derive boundary force and moment balances

$$\mathbf{t}_{\mathcal{S}} = \mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n} \qquad \text{and} \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{28}$$

giving the tractions $\mathbf{t}_{\mathcal{S}}$ and $\mathbf{m}_{\mathcal{S}}$ in terms of their environmental counterparts $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$, and use these balances to express the power expended by tractions in the form

$$\int\limits_{\partial B} \Big((\mathbf{t}^{\mathrm{env}}_{\partial B} + 2\sigma K\mathbf{n}) \cdot \mathbf{v} + \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot \mathbf{P}\frac{\partial \mathbf{v}}{\partial n} \Big)\, \mathrm{d}a, \tag{29}$$

where $\mathbf{P} = \mathbf{1} - \mathbf{n} \otimes \mathbf{n}$. We assume that the mean curvature K of — and the surface tension σ at — the boundary ∂B are known; (29) then suggests that reasonable boundary conditions might, at each point of ∂B, consist of

(i) a prescription of $\mathbf{t}^{\mathrm{env}}_{\partial B}$ or $\mathbf{v}$, or a relation between $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{v}$; and

(ii) a prescription of $\mathbf{m}^{\mathrm{env}}_{\partial B}$ or $\mathbf{P}\partial\mathbf{v}/\partial n$, or a relation between $\mathbf{m}^{\mathrm{env}}_{\partial B}$ and $\mathbf{P}\partial\mathbf{v}/\partial n$.

Consistent with this, we consider specific boundary conditions in which a portion $\mathcal{S}_{\mathrm{free}}$ of ∂B is a free surface and the remainder $\mathcal{S}_{\mathrm{nslp}}$ is a fixed surface without slip. On $\mathcal{S}_{\mathrm{free}}$, the environmental tractions $\mathbf{t}^{\mathrm{env}}_{\partial B}$ and $\mathbf{m}^{\mathrm{env}}_{\partial B}$ vanish and the classical condition $\mathbf{Tn} = \sigma K\mathbf{n}$ is replaced by the conditions

$$\mathbf{Tn} + \mathrm{div}_{\mathcal{S}}(\mathbf{Gn}\times) = \sigma K\mathbf{n} \qquad \text{and} \qquad \mathbf{n} \times \mathbf{Gn} = \mathbf{0}. \tag{30}$$

To describe the conditions on $\mathcal{S}_{\mathrm{nslp}}$, we first note that, if $\mathbf{v} = \mathbf{0}$ on $\mathcal{S}_{\mathrm{nslp}}$, then

$$\mathbf{P}\frac{\partial \mathbf{v}}{\partial n} = \boldsymbol{\omega} \times \mathbf{n} \tag{31}$$

[4] See, in particular, the condition (33).

with $\boldsymbol{\omega} = \operatorname{curl}\mathbf{v}$ the vorticity. Based on this identity, we take, as boundary condition on $\mathcal{S}_{\mathrm{nslp}}$, the classical condition

$$\mathbf{v} = \mathbf{0} \tag{32}$$

supplemented by a condition of the form

$$\mathbf{n} \times \mathbf{G}\mathbf{n} = \mathbf{m}^{\mathrm{env}}_{\partial B} \tag{33}$$

with $\mathbf{m}^{\mathrm{env}}_{\partial B} = \mu\ell\mathbf{P}\partial\mathbf{v}/\partial n = \mu\ell\boldsymbol{\omega}\times\mathbf{n}$, where ℓ carries dimensions of length. Thus we are led to the boundary condition

$$\mathbf{n} \times (\mathbf{G}\mathbf{n} + \mu\ell\boldsymbol{\omega}) = \mathbf{0}. \tag{34}$$

We refer to (34) as the *wall-eddy condition* and to ℓ as the *wall-eddy modulus.*

7 Free energy imbalance, dissipation, and the sign of the wall-eddy modulus

Recently (Fried & Gurtin 2006), we provided a general discussion of the use of an energy imbalance for a boundary pillbox to develop constitutive relations describing the interaction of the fluid and its environment. We here sketch the corresponding analysis, but only as it applies to the boundary conditions (32) and (34). Let $\mathcal{S}$ denote a fixed (i.e. time-independent) subsurface of $\mathcal{S}_{\mathrm{nslp}}$ with $\mathcal{S}$ viewed as a fixed *boundary pillbox* of infinitesimal thickness involving (Figure 1):

- a surface $\mathcal{S}$ with unit normal $\mathbf{n}$; $\mathcal{S}$ is viewed as lying in the *environment* at the interface of the fluid and the environment;
- a surface $-\mathcal{S}$ with unit normal $-\mathbf{n}$; $-\mathcal{S}$ is viewed as lying in the fluid adjacent to the boundary;

Let ψ^{x} denote the *excess free energy* of the fluid at the surface $\mathcal{S}_{\mathrm{nslp}}$, measured per unit area, so that

$$\int_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a$$

represents the net free energy of the pillbox. In the definition (10) of the external power the quantity $\mathcal{W}_{\mathrm{surf}}(\mathcal{S})$ defined by

$$\mathcal{W}_{\mathrm{surf}}(\mathcal{S}) = \int_{\mathcal{S}} \Big(\mathbf{t}_{\mathcal{S}}\cdot\mathbf{v} + \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\Big)\,\mathrm{d}a \tag{35}$$

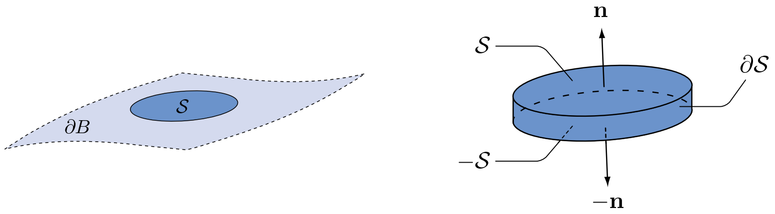

Figure 1. Pillbox corresponding to a subsurface $\mathcal{S}$ of the boundary ∂B of the region B of space occupied by the body. Only a portion of ∂B is depicted. Whereas $\mathbf{n}$ is oriented into the environment, $-\mathbf{n}$ is oriented into the fluid. The outward unit normal on the lateral face $\partial\mathcal{S}$ of the pillbox is denoted by $\boldsymbol{\nu}$.

represents the power expended on the boundary of a control volume. However, because the tractions are local, this definition is also meaningful for an arbitrary surface $\mathcal{S}$ with orientation $\mathbf{n}$. In this instance $\mathcal{W}_{\rm surf}(\mathcal{S})$ represents the power expended *by* the material on the plus side of $\mathcal{S}$ *on* the material on the minus side of $\mathcal{S}$. Then, since $-\mathcal{S}$ has curvature tensor $-\mathbf{K}$, we see that, by (14),[5]

$$\mathbf{t}_{\mathcal{S}} = -\mathbf{t}_{-\mathcal{S}}, \qquad \mathbf{m}_{\mathcal{S}} = \mathbf{m}_{-\mathcal{S}}. \tag{36}$$

Moreover, by (36) (and since $\partial\mathbf{v}/\partial n = \mathbf{n}\cdot\operatorname{Grad}\mathbf{v}$), we have the *power balance*

$$\mathcal{W}_{\rm surf}(\mathcal{S}) = -\mathcal{W}_{\rm surf}(-\mathcal{S}). \tag{37}$$

Since $\mathbf{v} = \mathbf{0}$, it is then clear from (31) that

$$\mathcal{W}_{\rm surf}(-\mathcal{S}) = -\int\limits_{\mathcal{S}} \mathbf{m}_{\mathcal{S}}\cdot\frac{\partial\mathbf{v}}{\partial n}\,{\rm d}a = -\int\limits_{\mathcal{S}} \mathbf{m}_{\mathcal{S}}\cdot(\boldsymbol{\omega}\times\mathbf{n})\,{\rm d}a \tag{38}$$

represents the power expended by the fluid on the pillbox surface $-\mathcal{S}$. We assume that the power expended by the environment on the pillbox surface $\mathcal{S}$ vanishes and hence that the environment is *passive*.[6] The power expended by the fluid on the lateral face of the pillbox by surface tension vanishes, because the boundary curve $\partial\mathcal{S}$ is stationary. Thus since

$$\mathbf{m}^{\rm env}_{\partial B} = \mu\ell\boldsymbol{\omega}\times\mathbf{n} \tag{39}$$

[5]Importantly, (36) represents an *action-reaction principle* for oppositely oriented surfaces that touch and are tangent at a point.

[6]One might wonder how an environment with $\mathbf{m}^{\rm env}_{\partial B} \neq \mathbf{0}$ can be passive. Because $\mathcal{S}_{\rm nslp}$ abuts a motionless, nondeformable environment, the environmental tractions $\mathbf{t}^{\rm env}_{\partial B}$ and $\mathbf{m}^{\rm env}_{\partial B}$ must be indeterminate and hence incapable of expending power.

the net power expended on the pillbox is given by

$$-\int_{\mathcal{S}} \mathbf{m}_{\mathcal{S}} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,\mathrm{d}a = -\int_{\mathcal{S}} \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,\mathrm{d}a. \tag{40}$$

Consider the quantity $D(\mathcal{S})$ defined by

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a}_{\text{free energy rate}} \; - \underbrace{\int_{\mathcal{S}} \big(- \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n})\big)\,\mathrm{d}a}_{\text{power expenditure}} = -D(\mathcal{S}). \tag{41}$$

Were we to parallel the development in bulk with the requirement that the temporal increase in free energy of $\mathcal{S}$ be less than or equal to the power expended on $\mathcal{S}$, then $D(\mathcal{S}) \geq 0$ would represent the *energy dissipated* within the pillbox. Assuming that ψ^{x} is constant and recalling that $\mathcal{S}$ is fixed, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{S}} \psi^{\mathrm{x}}\,\mathrm{d}a = 0,$$

we would find, as a consequence of (41), that

$$D(\mathcal{S}) = -\int_{\mathcal{S}} \mathbf{m}^{\mathrm{env}}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n})\,\mathrm{d}a \geq 0.$$

Thus

$$-\mathbf{m}^{\mathrm{env}}_{\partial B} \cdot (\boldsymbol{\omega} \times \mathbf{n}) \tag{42}$$

would represent the *dissipation per unit area*, so that, by (39),

$$-\int_{\mathcal{S}} \mu\ell|\boldsymbol{\omega} \times \mathbf{n}|^2\,\mathrm{d}a \geq 0. \tag{43}$$

Thus, since $\mathcal{S}$ was arbitrarily chosen, we would conclude that

$$\ell \leq 0. \tag{44}$$

However, as we show elsewhere (Fried & Gurtin 2007), for flow in a channel with the boundary conditions (32) and (34), our theory with $\ell \leq 0$ delivers solutions that agree *neither quantitatively nor qualitatively* when compared to the direct numerical simulations of Kim, Moin & Moser (1987) and Moser, Kim & Mansour (1999) and the experimental results of Wei & Wilmarth (1989); on the other hand there is *excellent agreement* when

$$\ell > 0. \tag{45}$$

Interestingly, such values of ℓ imply that $\mu\ell|\mathbf{n} \times \boldsymbol{\omega}|^2$ — a term which would usually be termed *dissipative* — is *negative*![7]

[7] Cf. the sentence containing (43).

This observation renders the theory with $\ell > 0$ *incompatible with thermodynamics* as embodied in a free energy imbalance. While we know of no successful continuum mechanical theory for which experiments yield moduli of signs opposite to those imposed by thermodynamics, one might argue that continuum thermodynamics is inapplicable to a discussion of turbulence when applied at a fixed boundary without slip. Moreover, turbulent eddies generated at such boundaries might render the state of the fluid there sufficiently far removed from equilibrium that standard continuum thermodynamical laws might no longer be valid. In this regard it is interesting to note that the free energy imbalance applied in bulk delivers moduli of signs consistent with those of the Navier–Stokes–α equation.

Acknowledgments

This work was supported by the U. S. Department of Energy.

References

[1] J. L. Bluestein & A. E. Green: *Dipolar fluids*, Int. J. Eng. Sci. **5**, (1967), 323–340.

[2] S. Chen, C. Foias, E. Olson, E. S. Titi & S. Wynne: *The Camassa–Holm equations as a closure model for turbulent channel and pipe flow*, Phys. Rev. Lett., **81**, (1998), 5338–5341.

[3] S. Chen, C. Foias, E. Olson, E. S. Titi & S. Wynne: *The Camassa–Holm equations and turbulence*, Physica D, **133**, (1999a), 49–65.

[4] S. Chen, C. Foias, E. Olson, E. S. Titi & S. Wynne: *A connection between the Camassa–Holm equations and turbulent flows in channels and pipes*, Phys. Fluids, **11**, (1999b.), 2343–2353.

[5] E. Cosserat & F. Cosserat: Théorie des Corps Déformables, Paris, Hermann, 1909.

[6] J. E. Dunn & R. L. Fosdick: *Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade*, Arch. Rat. Mech. Anal., **56**, (1974), 191–252.

[7] E. Fried & M. E. Gurtin: *Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small length scales*, Arch. Rat. Mech. Anal., **182** (2006), 513–554.

[8] E. Fried & M. E. Gurtin: *Cosserat fluids and the continuum mechanics of turbulence: a generalized Navier–Stokes-α equation with complete boundary conditions*, Theoret. Comput. Fluid Dyn., (2007), submitted.

[9] A. E. Green & R. S. Rivlin: *Simple force and stress multipoles*, Arch. Rat. Mech. Anal., **16**, (1964), 325–353.

[10] A. E. Green & P. M. Naghdi: *A note on simple dipolar stresses*, J. Mécanique, **7**, (1968), 465–474.

[11] M. E. Gurtin: *A gradient theory of single-crystal viscoplasticity that accounts for geometrically necessary dislocations*, J. Mech. Phys. Solids, **84**, (2002), 809–819.

[12] D. D. Holm, C. Jeffrey, S. Kurien, D. Livescu, M. A. Taylor & B. A. Wingate: *The LANS-α model for computing turbulence: Origins, results, and open problems*, Los Alamos Science, **29**, (2005), 152–171.

[13] D. D. Holm, J. E. Marsden, T. Ratiu: *The Euler–Poincaré equations and semidirect products with applications to continuum theories*, Adv. Math. **137**, (1998a), 1–81.

[14] D. D. Holm, J. E. Marsden, T. Ratiu: *Euler–Poincaré models of ideal fluids with nonlinear dispersian*, Phys. Rev. Lett., **80**, (1998b), 4273–4277.

[15] J. Kim, P. Moin & R. D. Moser: *Turbulence statistics in fully developed channel flow at low Reynolds number*, J. Fluid Mech., **177**, (1987), 133–166.

[16] A. N. Kolmogorov: *The local structure of turbulence in an incompressible viscous fluid for very large Reynolds numbers*, Doklady Akad. Nauk S.S.S.R., **30**, (1941a), 301–305.

[17] A. N. Kolmogorov: *On degeneration of isotropic turbulence in an incompressible viscous fluid*, Doklady Akad. Nauk S.S.S.R., **31**, (1941b), 538–540.

[18] A. N. Kolmogorov: *Dissipation of energy in locally isotropic turbulence*, Doklady Akad. Nauk S.S.S.R., **32**, (1941c), 16–18.

[19] A. N. Kolmogorov: *A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number*, J. Fluid Mech., **13**, 82–85.

[20] J. E. Marsden & S. Shkoller: *Global well-posedness for the Lagrangian averaged Navier–Stokes-α (LANS-α) equations on bounded domains*, Phil. Trans. R. Soc. A, **359**, (2001), 1449-1468.

[21] R. D. Mindlin & H. F. Tiersten: Effects of couple-stresses in linear elasticity, Arch. Rat. Mech. Anal., **11**, (1962), 415–448.

[22] R. D. Moser, C. S. Kim & S. Mansour: *Direct numerical simulation of turbulent flow up to* $Re_\tau = 590$, Phys. Fluids, **11**, (1999), 943–945.

[23] S. B. Pope,: Turbulent Flows, Cambridge, UK: Cambridge University Press, 2000.

[24] L. F. Richardson: Weather Prediction by Numerical Process, Cambridge, UK: Cambridge University Press, 1922.

[25] R. S. Rivlin & J. L. Ericksen: *Stress-deformation relations for isotropic materials*, J. Ration. Mech. Anal., **4**, (1955), 323–425.

[26] S. Shkoller: *Analysis on groups of diffeomorphisms of manifolds with boundary and the averaged motion of a fluid*, J. Diff. Geom., **55**, (2000), 145–191.

[27] R. A. Toupin: *Elastic materials with couple stresses*, Arch. Ration. Mech. Anal., **11**, (1962), 385–414.

[28] R. A. Toupin: *Theory of elasticity with couple stresses*, Arch. Ration. Mech. Anal., **17**, (1964), 85–112.

[29] C. A. Truesdell & W. Noll: *The non-linear field theories of mechanics.* In *Handbuch der Physik III/3.* (S. Flügge, ed.), Springer, Berlin, (1965).

[30] T. Wei & W. W. Willmarth: *Reynolds-number effects on the structure of a turbulent channel flow*, J. Fluid. Mech., **204**, (1989), 57–95.