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On Harmonious Coloring of Middle Graph of $C(C_n)$, $C(K_{1,n})$ and $C(P_n)$

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Abstract. In this paper, we present the structural properties of middle graph of central graph of cycles C_n, star graphs $K_{1,n}$ and paths P_n denoted by $M(C(C_n))$, $M(C(K_{1,n}))$ and $M(C(P_n))$ respectively. We mainly have our discussion on the harmonious chromatic number of $M(C(C_n))$, $M(C(K_{1,n}))$ and $M(C(P_n))$.

Keywords: central graph, middle graph, harmonious coloring

MSC 2000 classification: 05C75, 05C15

Introduction

For a given graph $G = (V, E)$ we do an operation on G, by subdividing each edge exactly once and joining all the non-adjacent vertices of G. The graph obtained by this process is called central graph [1, 29, 31–35] of G denoted by $C(G)$.

The middle graph [6] of G, is defined with the vertex set $V(G) \cup E(G)$ where two vertices are adjacent iff they are either adjacent edges of G or one is a vertex and the other is an edge incident with it and it is denoted by $M(G)$. Additional graph theory terminology used in this paper can be seen in [3, 15].

A harmonious coloring [2, 7, 8, 10–14, 16–28, 36, 37, 39] of a simple graph G is proper vertex coloring such that each pair of colors appears together on at most one edge. Formally, a harmonious coloring [4, 5, 9] is a function c from a color set C to the set $V(G)$ of vertices of G such that for any edge e of G, with end points x, y say $c(x) \neq c(y)$, and for any pair of distinct edges e, e' with end points x, y and x', y' respectively, then $\{c(x), c(y)\} \neq \{c(x'), c(y')\}$. The

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harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring.

The first paper on harmonious graph coloring was published in 1982 by Frank Harary and M. J. Plantholt [16]. However, the proper definition of this notion is due to J. E. Hopcroft and M. S. Krishnamoorthy [18] in 1983. S.Lee and John Mitchum [22], published a paper consisting of the upper bound for the harmonious chromatic number of graphs in 1987.

In 1988, Z. Miller and D. Pritikin, [26] worked on harmonious coloring and gave the harmonious chromatic number of graphs, in the Proceedings of 250th Anniversary Conference on Graph Theory (Fort Wayne, Indiana, 1986) (eds. K. S. Bagga et al.), Congressus Numerantium. D.G. Beane, N.L.Biggs and B.J. Wilson, studied the growth rate of harmonious chromatic number in 1989. Again Z. Miller and D. Pritikin published a paper on the topic the harmonious colouring number of a graph in 1991.

In 1991 C. J. H. McDiarmid and Luo Xinhua [25] gave the Upper bounds for harmonious colorings. Zhikang Lu [38] gave a published work on the harmonious chromatic number of a complete binary and trinary tree, in 1993. He also published a paper on Estimates of the harmonious chromatic numbers of some classes of graphs (Chinese), Journal of Systems Science and Mathematical Sciences, 13 (1993).

A combined work by L. R. Casse, C. M. ÓKeefe and B. J. Wilson [5] gave us the Minimal harmoniously colorable designs in 1994. In the same year, I. Krasikov and Y. Roditty [21] gave a paper on bounds for the harmonious chromatic number of a graph.

Zhikang Lu [40], in 1995, published a paper on the harmonious chromatic number of a complete 4-ary tree. Also K. J. Edwards [7] worked and gave results on the harmonious chromatic number of almost all trees. In the next year (1996) he investigated on the harmonious chromatic number of bounded degree trees [8].

John P. Georges [20] published a paper on the harmonious colorings on collection of graphs in 1995. In 1996, a paper on the harmonious chromatic number of quasistars, was given by I. Havel and J.M. Laborde Manuscript, Prague and Grenoble, 1996.

In 1997, K. J. Edwards, [9] continued his work on the harmonious chromatic number of bounded degree graphs, and also published papers relating harmonious coloring and achromatic number.

Zhikang Lu [41, 42] published a paper on the exact value of the harmonious chromatic number of a complete binary tree in 1997 and trinary tree in 1998.

In 1998, K. J. Edwards [10] published a work emphasizing a new upper bound for the harmonious chromatic number, and in 1999 on the harmonious chromatic number of complete $r-$ary trees.

J. Mitchem and E. Schmeichel, published a paper the harmonious chromatic number of deep and wide complete n-ary trees, in The Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications (Kalamazoo, Michigan, 2000) (eds. Y. Alavi, D. Jones, D. R. Lick and Jiuqiang Liu), Electronic Notes in Discrete Mathematics, 11 (2002).

A work on the harmonious chromatic number of $P(\alpha,\ K_n)$, $P(\alpha,\ K_{1,n})$ and $P(\alpha,\ K_{m,n})$, was published by M. F.Mammana [24] in 2003.

D. Campbell and K. J.Edwards [4] again gave a new lower bound for the harmonious chromatic number in 2004.

In 2007, Vernold Vivin.J in his Ph.D thesis [35] had a detailed study on the harmonious chromatic number of central graph families.

Vernold Vivin.J et al. [32] published a paper on the harmonious coloring of central graph in 2008. For some background on this topic, see [29–35].

1 Structural properties of $M(C(C_n))$

In $M(C(C_n))$, there are n vertices of degree 2, n vertices of degree $(n-1)$, $2n$ vertices of degree $(n+1)$ and $\frac{n^2-3n}{2}$ vertices of degree $2(n-1)$.

Therefore

- The number of vertices ,$p_{M(C(C_n))} = \dfrac{n^2+5n}{2}$.
- The number of edges, $q_{M(C(C_n))} = \dfrac{n^3-n^2+6n}{2}$.
- $\Delta = 2(n-1)$.

2 Harmonious coloring of $M(C(C_n))$

1 Theorem. *The harmonious chromatic number of middle graph of central graph of cycles C_n, $\chi_H(M(C(C_n))) = \left\lceil \dfrac{n^2+5}{2} \right\rceil$.*

Proof. Let $V(C(C_n)) = \{v_1, v_2, \dots, v_n\}$. On the process of centralization of C_n, let u_i be the vertex of subdivision of the edge $v_iv_{i+1}(1 \le i \le n)$. Also let $u_iv_i = e_i(1 \le i \le n)$ and $u_iv_{i+1} = e'_i(1 \le i \le n-1)$ and $u_nv_1 = e'_n$. Also for non-adjacent vertices v_i and v_j of C_n, let $e_{ij} = v_iv_j$. Since we consider only undirected graphs $e_{ij} = e_{ji}$. Middle graph of $C(C_n)$ has the vertex set $V(C(C_n)) \cup E(C(C_n))$ $= \{v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_n, e'_1, e'_2, \dots, e'_n, e_{13}, e_{15}, \dots, e_{24}, e_{25}, \dots\}$. Each v_i is incident with the edges $e_i, e'_{i-1}, e_{ij}(i \ne j)$ and $(2 \le i \le n)$. Also v_1 is incident with $e_1, e'_n, e_{13}, e_{15}, \dots, e_{1(n-1)}$. i.e., Total number of incident edges with

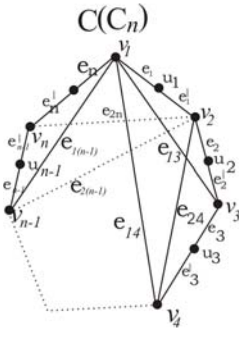

Figure 1. Central Graph of Cycles C_n

v_i is $(n-1)\forall\ (i=1,2,\ldots,n)$. By the definition of middle graph the edges incident with v_i together with the vertex v_i induces a clique of n vertices in $M(C(C_n))$ $(1 \leq i \leq n)$.

Let $K_n^{(i)}$ be the cliques in $M(C(C_n))(i=1,2,\ldots,n.)$ Since $e_{ij}=e_{ji}$, each $K_n^{(i)}$ shares their exactly $(n-3)$ vertices with the remaining cliques. Therefore in each clique, the harmonious coloring is performed by distinct n colors. $K_n^{(1)}$ is assigned n colors, where as since $K_n^{(2)}$ shares one vertex with $K_n^{(1)}$, it needs distinct $(n-1)$ colors which are distinct from the set of colors assigned to $K_n^{(1)}$. Since $K_n^{(3)}$ shares one vertex with $K_n^{(1)}$ and one with $K_n^{(2)}$, it needs only $(n-2)$ colors and so on. Now we turn our proof in the direction of induction.

Case (i)

If n is odd, for $n=3, C(C_3)$ is C_6 and for its middle graph the harmonious chromatic number is $\frac{n^2+5}{2}=7$. Therefore the result is trivial for $n=3$. Now we assume that the result is valid for C_n, when n is odd. i.e.,$\chi_H(M(C(C_n)))=\frac{n^2+5}{2}$. Now consider C_{n+2} by introducing two new vertices v_{n+1} and v_{n+2}. Consider the incident edges of v_{n+1} and v_{n+2} in $C(C_{n+2})$. These edges together with the vertices v_{n+1} and v_{n+2} induces two more cliques of order $n+2$ in $M(C(C_n))$. The vertices $v_{n+1}, v_{n+2},\ e_{n+1}, e_{n+2},\ u_{n+1}, u_{n+2},\ e'_{n+1}, e'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by $(n-3)+(n-3)=(2n-6)$ colors. Therefore $\chi_H(M(C(C_{n+2})))=\frac{n^2+5}{2}+2n-6+8=\frac{n^2+5}{2}+(2n+2)$. Hence $\chi_H(M(C(C_{n+2})))=\frac{(n+2)^2+5}{2}$. Therefore by induction hypothesis $\chi_H(M(C(C_n)))=\frac{n^2+5}{2}$, if n is odd.

Case (ii)

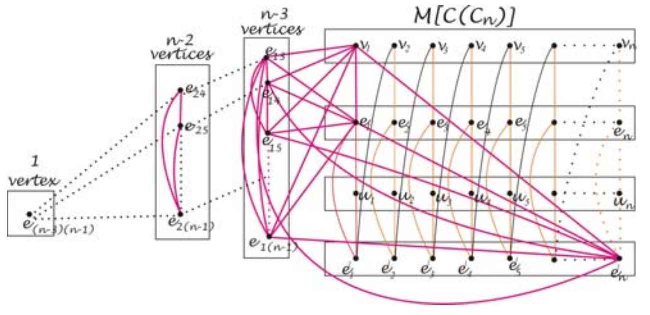

Figure 2. Middle Graph of Central Graph of Cycles C_n

If n is even, we prove that, $\chi_H(M(C(C_n))) = \frac{n^2+6}{2}$. For $n = 4$, the harmonious chromatic number of the middle graph of $C(C_4)$ is $\frac{n^2+6}{2} = 11$. Now we assume that the result is valid for n, when n is even. i.e., $\chi_H(M(C(C_n))) = \frac{n^2+6}{2}$. Now consider C_{n+2} by introducing two new vertices v_{n+1} and v_{n+2}. Consider the incident edges of v_{n+1} and v_{n+2} in $C(C_{n+2})$. These edges together with the vertices v_{n+1} and v_{n+2} induces two more cliques of order $n+2$ in $M(C(C_n))$.

The vertices $v_{n+1}, v_{n+2}, e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}$ are assigned by 8 colors and the cliques $K_{n+2}^{(n+1)}$ and $K_{n+2}^{(n+2)}$ are assigned by $(n-3)+(n-3) = (2n-6)$ colors. Therefore $\chi_H(M(C(C_{n+2}))) = \frac{n^2+6}{2} + 2n - 6 + 8 = \frac{n^2+6}{2} + 2n+2$. Hence $\chi_H(M(C(C_{n+2}))) = \frac{(n+2)^2+6}{2}$. Therefore by induction hypothesis $\chi_H(M(C(C_n))) = \frac{n^2+6}{2}$, if n is even . If n is odd then $\frac{n^2+5}{2} = \left\lceil \frac{n^2+5}{2} \right\rceil$, if n is even then $\frac{n^2+6}{2} = \left\lceil \frac{n^2+5}{2} \right\rceil$. Therefore in both the cases, $\chi_H(M(C(C_n))) = \left\lceil \frac{n^2+5}{2} \right\rceil$. $\boxed{QED}$

3 Structural properties of $M(C(K_{1,n}))$ and $M(C(P_n))$

In $M(C(K_{1,n}))$ and $M(C(P_n))$ there are n vertices of degree 2, $n+1$ vertices of degree n, $2n$ vertices of degree $(n+2)$ and $\frac{n^2-n}{2}$ vertices having degree $2n$. Therefore

- The number of vertices, $p_{M(C(P_n))} = \dfrac{n^2+7n+2}{2}$.
- The number of edges, $q_{M(C(P_n))} = \dfrac{n^3+2n^2+7n}{2}$.

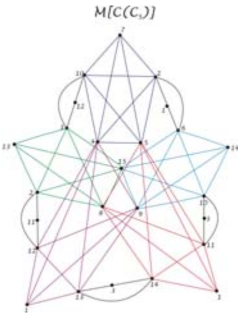

Figure 3. Middle Graph of Central Graph of Cycle C_5 $\chi_H(M(C(C_5))) = \left\lceil \dfrac{5^2+5}{2} \right\rceil = 15$.

- $\Delta = 2n$.
- We infer that $M(C(K_{1,n}))$ and $M(C(P_n))$ are isomorphic graphs.

4 Harmonious coloring of $M(C(K_{1,n}))$ and $M(C(P_n))$

The harmonious chromatic number of $M(C(K_{1,n}))$ and $M(C(P_n))$ are equal since they are isomorphic graphs.

2 Theorem. *The harmonious chromatic number of middle graph of central graph of star graphs $K_{1,n}$, $\chi_H(M(C(K_{1,n}))) = \left\lceil \frac{n^2+2n+5}{2} \right\rceil$.*

Proof. Let $V(K_{1,n}) = \{v, v_1, v_2 \ldots, v_n\}$ where $\deg v = n$. On the process of centralization of $K_{1,n}$, let us denote the vertices of subdivision by $u_1, u_2, \cdots, u_n$. i.e., vv_i is subdivided by $u_i (1 \leq i \leq n)$. Let $e_i = v_iu_i$ and $e'_i = vu_i (1 \leq i \leq n)$. i.e., $V(C(K_{1,n})) = \{v, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots u_n\}$, $E(C(K_{1,n})) = \{e_1, e_2, \ldots, e_n, e'_1, e'_2, \ldots, e'_n, e_{12}, e_{13}, \ldots, e_{1n}, e_{23}, \ldots, e_{2n}, \ldots, e_{(n-1)n}\}$. By the definition of middle graph, $V(M(C(K_{1,n}))) = V(C(K_{1,n})) \cup E(C(K_{1,n}))$. The structure is described below. The vertices $v_1, e_1, e_2, \ldots, e_n$ induces a clique of order $(n+1)$ in its middle graph. The vertices u_i is adjacent to e_i and $e'_i (1 \leq i \leq n)$. Let $S_i = \{e_{ij} : j = 1, 2, \ldots i-1, i+1, \ldots, n\}, (1 \leq i \leq n)$. Clearly $S_i \cap S_j = \{e_{ij}\}$ if $i \neq j$ and let $S^{(n)} = \cup_{i=1}^{n} S_i$. Clearly $|S^{(n)}| = \binom{n}{2}$. Now the vertices v_i and e'_i together with vertices of S_i induces a clique of order $(n+1), (1 \leq i \leq n)$. Therefore $M(C(K_{1,n}))$ contains $n+1$ clique of order $(n+1)$. Now we prove that the harmonious chromatic number of this graph is $\left\lceil \frac{n^2+2n+5}{2} \right\rceil$ by induction

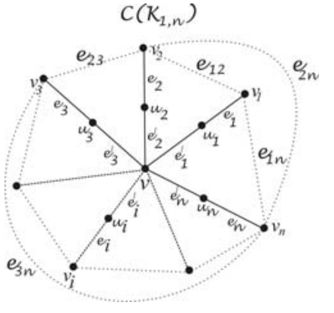

Figure 4. Central Graph of Star Graphs $K_{1,n}$

method.

Case (i)

If n is odd

We prove $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+5}{2}$. If $n=3$, then $C(K_{1,3})$ has 7 vertices and its middle graph is shown in figure 6.

Clearly $\chi_H(M(C(K_{1,3}))) = \frac{n^2+2n+5}{2} = 10$. Therefore the result is true for $n=3$. Assume that the result is true for any integer n and we prove the same for $n+2$. i.e., $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+5}{2}$. Let v_{n+1}, v_{n+2} be two non-adjacent vertices introduced in $K_{1,n}$ which are adjacent to v. Let u_{n+1} and u_{n+2} be the vertices of subdivision in its centralization. Clearly by the structure given in figure 5, the middle graph $C(K_{1,n+2})$ has the following structural property. (i) There are $(n+3)$ cliques $K_{n+3}^{(1)}, K_{n+3}^{(2)}, \ldots, K_{n+3}^{(n+3)}$ of order $(n+3)$. (ii) The vertices u_i is adjacent with e_i and e'_i $(1 \le i \le n+2)$. (iii) Each $K_{n+3}^{(i)}$ has exactly one vertex common with $K_{n+3}^{(j)}$ where $(2 \le i \le n+3), (2 \le j \le n+3)$ and $i \ne j$. By induction hypothesis the minimum number of colors for the harmonious coloring in $C(K_{1,n})$ is $\frac{n^2+2n+5}{2}$. By the above said structure of $M(C(K_{1,n})), |S^{(n)}| = |S_1 \cup S_2 \cup \cdots \cup S_n| = \binom{n}{2} = \frac{n(n-1)}{2}$. Also in $M(C(K_{1,n+2}))$, $|S^{(n+2)}| = |S_1 \cup S_2 \cup \cdots \cup S_{n+2}| = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$, also we have new vertices $e_{n+1}, e_{n+2}, u_{n+1}, u_{n+2}, e'_{n+1}, e'_{n+2}, v_{n+1}, v_{n+2}$ in $M(C(K_{1,n+2}))$. Therefore the total number of added vertices in $M(C(K_{1,n+2})) = \frac{(n+2)(n+1)}{2} - \frac{n(n-1)}{2} + 8 = 2n+1+8 = 2n+9$. Now we find the minimal harmonious coloring in $M(C(K_{1,n+2}))$ as below. By the induction hypothesis $C(K_{1,n})$ has harmonious coloring with the minimum number of $\frac{n^2+2n+5}{2}$ colors. With this same colors assigned to the vertices of $M(C(K_{1,n+2}))$, we assign some new colors to the remaining vertices as below. The vertices u_{n+1} and u_{n+2} are assigned the same

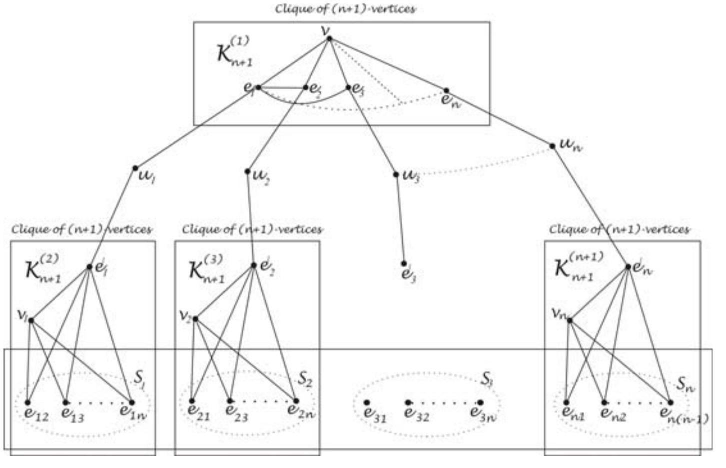

Figure 5. Middle Graph of Central Graph of Star Graph $K_{1,n}$

color as in $u_i(1 \leq i \leq n)$. Then all the e_{n+1} vertices of $S^{(n+2)}$ are assigned $(2n+1)$ new colors. Also, among the remaining vertices $e_{n+1}, e_{n+2}, e'_{n+1}, e'_{n+2}$, v_{n+1} and v_{n+2}, $(e_{n+1}, e_{n+2},)$ (e'_{n+1}, v_{n+1}) $(e'_{n+2}, v_{n+2}) \in E(M(C(K_{1,n+2})))$, we use three distinct colors to color these vertices. Clearly the above said new coloring is a minimal harmonious coloring. Here we use $2n+1+3 = 2n+4$ colors. Therefore $\chi_H(M(C(K_{1,n+2}))) = \frac{n^2+2n+5}{2} + 2n+4 = \frac{(n+2)^2+2(n+2)+5}{2}$. Hence by induction hypothesis the result follows, $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+5}{2}$.

Case (ii)

If n is even, we prove $\chi_H(M(C(K_{1,n}))) = \frac{n^2+2n+6}{2}$ by induction method, following the same procedure as above , $\frac{n^2+2n+5}{2} = \left\lceil \frac{n^2+2n+5}{2} \right\rceil$ and $\frac{n^2+2n+6}{2} = \left\lceil \frac{n^2+2n+5}{2} \right\rceil$ $\forall n$. Therefore $\chi_H(M(C(K_{1,n})))=\left\lceil \frac{n^2+2n+5}{2} \right\rceil$. $\boxed{QED}$

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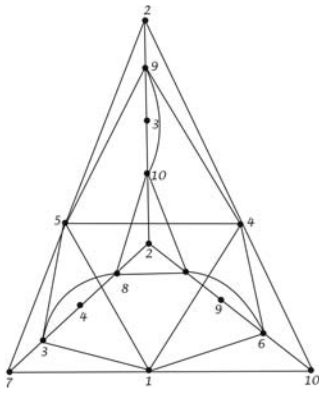

Figure 6. Middle Graph of Central Graph of Star Graphs $K_{1,3}$

$$\chi_H(M(C(K_{1,3}))) = \left\lceil \frac{3^2 + 2(3) + 5}{2} \right\rceil = 10.$$

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