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# $k$–sets of type $(1,h)$ in finite planar spaces

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**Abstract.** A set of type $(m,n)$ is a set $\mathcal{K}$ of points of a planar space with the property that each plane of the space meets $\mathcal{K}$ either in $m$ or $n$ points, and there are both planes intersecting $\mathcal{K}$ in $m$ points and in $n$ points. In this paper, sets of type $(1,h)$ in a planar space whose planes pairwise intersect either in the empty–set or in a line, are studied.

**Keywords:** Linear spaces, projective planes, semiaffine planes, maximal arcs

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## 1 Introduction

A (*finite*) *linear space* is a pair $\mathbb{S} = (\mathcal{P}, \mathcal{L})$ consisting of a (finite) set $\mathcal{P}$ of elements called *points* and a set $\mathcal{L}$ of distinguished subsets of points, called *lines*, such that *any two distinct points are contained in exactly one line, any line has at least two points, and there are at least two lines.*

A *subspace* of a linear space is a subset of points $X$ such that for every pair of distinct points of $X$ the line joining them is entirely contained in $X$.

A (*finite*) *planar space* is a (finite) linear space endowed with a family of subspaces, called *planes*, such that any three non–collinear points are contained in a unique plane, every plane contains at least three non–collinear points, and there are at least two planes.

Clearly, projective and affine spaces of dimension at least three are planar spaces.

A planar space is *non–degenerate* if every line contains at least 3 three points.

Let $\mathbb{S}$ be a finite planar space. We use $v$, $b$ and $c$ to denote respectively the number of points, of lines and of planes of $\mathbb{S}$. For any point $p$, the *degree* of $p$ is the number $r_p$ of lines on $p$, and for any line $L$, the *length* of $L$ is the number $k_L$ of its points.

A $k$–subset $\mathcal{K}$ of points of $\mathbb{S}$ is of *class* $[m_1, \dots, m_s]$ if a plane of $\mathbb{S}$ meets $\mathcal{K}$ in $m_1, m_2, \dots,$ or $m_s$ points. Let $t_{m_j}$ be the number of planes meeting $\mathcal{K}$ into

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exactly $m_j$ points. A $k$–set $\mathcal{K}$ is of type $(m_1, \ldots, m_s)$ if it is of class $[m_1, \ldots, m_s]$ and $t_{m_j} \neq 0$ for every $j = 1, \ldots, s$. The $m_j$'s are the *characters* of $\mathbb{S}$.

A *tangent* plane is a plane meeting $\mathcal{K}$ in exactly one point. A *secant* plane is a plane meeting $\mathcal{K}$ in $h$ points $(h > 1)$.

A line which meets $\mathcal{K}$ in $i$ points is called an *i–secant*. A 0–secant line is an *external* line, and a 1–secant line is a *tangent* line.

In the literature, one can find a number of papers devoted to the study of sets of a finite projective (or affine) space with respect to their intersection with all the subspaces of a given dimension $d$, see e.g [8–10]. Moreover, some authors have extended such a study to other incidence structures [3, 4, 7].

Finite planar spaces whose planes pairwise intersect either in the empty–set or in a line have their *local parameters* (that is the point–degree, the point–degree in every plane, the number of planes through a point, and the number of planes through a line) equal to those of the desarguesian projective space of dimension three. It is a longstanding conjecture [6] to prove that, if there are no disjoint planes, these planar spaces are obtained from $\mathrm{PG}(3, n)$ by deleting a subset of points.

In particular, given such a finite (non–degenerate) planar space there is an integer $n \geq 2$ (the *order* of the planar space) such that

- *every point has degree* $n^2 + n + 1$
- *through every point there are* $n^2 + n + 1$ *planes*
- *in every plane each point has degree* $n + 1$
- *through every line there are* $n + 1$ *planes*
- *every plane contains at most* $n^2 + n + 1$ *lines*
- *every plane contains at most* $n^2 + n + 1$ *points*
- *the number of points is at most* $n^3 + n^2 + n + 1$.

Moreover, if there are no disjoint planes, the number of planes is $n^3+n^2+n+1$ and every plane has $n^2 + n + 1$ lines.

Recall that every plane of such a planar space is embeddable in a finite projective plane, actually if $\pi$ is a plane and $p$ is a point not in $\pi$, the plane $\pi$ can be embedded, by projection through $p$, in the projective plane whose points are the lines through $p$ and whose lines are the planes through $p$.

A *cap* of a planar space is a set $\mathcal{K}$ of points which meets every line in at most two points.

An *ovoid* of a planar spaces $\mathbb{S} = (\mathcal{P}, \mathcal{L}, \mathcal{H})$ is a cap $\mathcal{K}$ such that for every point $p \in \mathcal{K}$ the union of the tangent lines through $p$ is a plane $\pi_p$, called the *tangent plane* at $p$.

Let $\mathbb{S}$ be a planar space whose planes pairwise intersect either in the empty set or in a line, and let $\mathcal{K}$ be a cap, and $p, q$ be two points of $\mathcal{K}$. A plane $\pi$ through $p$ and $q$ is different from $\pi_p$ and intersects $\pi_p$ in a tangent line $t_p$ at $p$. The $n$ lines of $\pi$ trough $p$ and different from $t_p$ are all secant lines, and so $\pi$ meets $K$ exactly in $n+1$ points.

Thus, $k = 2 + (n+1)(n-1) = n^2 + 1$.

In this paper, we study finite planar spaces whose planes pairwise intersect each other either in the empty–set or in a line, and with a set of type $(1, h)$. In particular, the following result will be proved.

**Theorem I** *Let $\mathbb{S} = (\mathcal{P}, \mathcal{L}, \mathcal{H})$ be a finite (non–degenerate) planar space of order $n$ whose planes pairwise intersect each other either in the empty–set or in a line. If $\mathbb{S}$ contains a set $\mathcal{K}$ of type $(1, h)$, then $c \geq n^3 + n^2 + n + 1$, and equality holds if and only if $\mathcal{K}$ is either a line of length $n+1$ or an ovoid.*

## 1.1 Some preliminary results

In this section we collect some results on both finite linear spaces and on two–character sets of a finite linear spaces, which will be useful in the next sections.

### 1.1.1 Linear spaces

**1 Definition.** Let $\mathbb{S}$ be a finite linear space, and let $H$ be a finite set of non–negative integers. $\mathbb{S}$ is *$H$–semiaffine* if for every point–line pair $(p, \ell)$, with $p \notin \ell$, the number $\pi(p, \ell) := r_p - k_\ell \in H$.

We recall part of a result of Doyen and Hubaut ( [2], 1971), whose statement we have rewritten in terms of planar spaces with planes pairwise intersecting either in a line or in the empty–set.

**2 Theorem.** *[Doyen–Hubaut, [2]1971 ] A finite planar space whose planes pairwise intersect either in the empty–set or in a line and with constant line length $s$, is either a projective space, or an affine space or a space in which each plane is $I$–semiaffine, where $I = \{s^2 - s + 1\}$ or $I = \{s^3 + 1\}$.*

### 1.1.2 Caps, ovoids and planar spaces

**3 Theorem.** *[Tallini, [9] 1986] Let $\mathbb{S}$ be a non–degenerate finite planar space with constant line size $n+1$, and constant plane size. If $\mathbb{S}$ contains an ovoid $\Omega$ then $\mathbb{S}$ is $\mathrm{PG}(3, n)$, and $\Omega$ is one of its ovoids.*

**4 Theorem.** *[Thas, [10], 1973] A proper subset* $\mathcal{K}$ *of the point–set of* $\mathrm{PG}(r,n)$, $r \geq 3$, *meeting every hyperplane in either* 1 *or* $h$ *points is a line or* $r = 3$ *and it is an ovoid.*

**5 Theorem.** *[Biondi, [1], 1998] Let* $\mathbb{S}$ *be a non–degenerate finite planar space of order*[1] $n$ *whose planes pairwise intersect in a line, and let* $\Omega$ *be an ovoid of* $\mathbb{S}$. *The planar space* $\mathbb{S}$ *is embeddable if and only if the inversive plane defined by* $\Omega$ *is embeddable.*

**6 Theorem.** *[Durante–Napolitano–Olanda, [5], 2002] Let* $\mathbb{S}$ *be a non–degenerate finite planar space of order* $n$ *whose planes pairwise intersect in a line, and* $\mathcal{K}$ *be a set of type* $(1,h)$ *of* $\mathbb{S}$. *Then* $\mathcal{K}$ *is either a line* ($\textit{of length } n+1$) *or an ovoid of* $\mathbb{S}$.

Thus, Theorem I generalizes Theorem 6 and Theorem 4 when $r = 3$.

## 2 Sets of class $[1,h]$ in $(\mathcal{P},\mathcal{L},\mathcal{H})$

Let $\mathbb{S} = (\mathcal{P},\mathcal{L},\mathcal{H})$ be a non–degenerate finite planar space of order $n$ whose planes pairwise intersect either in the empty set or in a line, and let $\mathcal{K}$ be a subset of $\mathcal{P}$ meeting every plane in either 1 or $h$ ($h \geq 2$) points. A *tangent* plane is a plane meeting $\mathcal{K}$ in exactly one point, a *secant* plane is a plane meeting $\mathcal{K}$ in $h$ points. An *external* line is a line missing $\mathcal{K}$, a *tangent* line is a line meeting $\mathcal{K}$ in just one point, and a *secant* line is a line meeting $\mathcal{K}$ in more than one point.

**7 Proposition.** $h > 2$.

PROOF. Assume by way of contradiction that $h = 2$. Thus, every plane meets $\mathcal{K}$ either 1 or 2 points. Let $p$ and $p'$ be two points of $\mathcal{K}$, and let $\ell$ be the line $pp'$. Since, every plane through $\ell$ meets $\mathcal{K}$ in exactly two points, there is no other point of $\mathcal{K}$ either on $\ell$ or outside $\ell$. Thus, $\mathcal{K} = \{p,p'\}$. Let $\pi$ be a plane trough $\ell$, and $x$ be a point of $\pi$ outside $\ell$. Since $|\ell| \geq 3$, then through $x$ there is an external line $t$, and all the planes through $t$, but $\pi$ does not meet $\mathcal{K}$, a contradiction. QED

**8 Proposition.** *If every plane of* $\mathbb{S}$ *is secant to* $\mathcal{K}$, *then* $\mathcal{K} = \mathcal{P}$ *and* $\mathbb{S}$ *is either* $PG(3,n)$, *or* $AG(3,n)$ *or a space with constant line length* $s$ *in which every plane is* $I$*–affine, where either* $I = \{s^2 - s + 1\}$ *or* $I = \{s^3 + 1\}$.

PROOF. Let $x$ and $y$ be two points of $\mathcal{K}$ and let $t$ be the line joining them. Let $s = |t \cap \mathcal{K}|$. Since every plane through $t$ is a secant plane and the planes through $t$ partition the set $\mathcal{K} \setminus t$, it follows that:

[1] The *order* of a finite planar space with planes pairwise intersecting in a line, is the integer $n$ such that through any line $\ell$ there pass exactly $n + 1$ planes.

$$k = s + (n+1)(h-s). \tag{1}$$

Equation (1) shows that every secant line to $\mathcal{K}$ meets $\mathcal{K}$ in a constant number $s = h - \dfrac{k-h}{n}$ of points. If there is a line $\ell$ missing $\mathcal{K}$, then counting $k$ via the planes through $\ell$ gives

$$k = (n+1)h. \tag{2}$$

Comparing equations (1) and (2) it follows that $n = 0$ or $s = 0$, a contradiction.

If there is a tangent line to $\mathcal{K}$, then

$$k = (n+1)(h-1) + 1. \tag{3}$$

Comparing Equation (1) and Equation (2) we get $s = 1$, a contradiction, since $s \geq 2$. It follows that every line is secant. If $\mathcal{K}$ is a proper subset of $\mathcal{P}$, then there is a point $p$ of $\mathcal{P}$ not in $\mathcal{K}$.

Computing the size of $\mathcal{K}$ via the lines through $p$ we get

$$k = (n^2 + n + 1)s, \tag{4}$$

while computing the size of $\mathcal{K}$ via the lines through a point $p'$ in $\mathcal{K}$, we get

$$k = (n^2 + n + 1)(s-1) + 1. \tag{5}$$

Comparing equations (4) and (5) we get a contradiction. Hence, $\mathcal{K} = \mathcal{P}$. Every plane is contained in $\mathcal{K}$, and so the planes have constant size $h$. Moreover, each line is contained in $\mathcal{K}$, and so all the lines are secant and have constant length $s$.

Thus, $\mathbb{S}$ is a 3–dimensional locally projective planar space with constant line size, by Theorem 2 the assertion follows. *QED*

From now on we may assume that there is at least a tangent plane $\pi_0$ to $\mathcal{K}$ and hence that $\mathcal{K}$ is a proper subset of $\mathcal{P}$. Let $p_0 = \pi_0 \cap \mathcal{K}$. Every line in $\pi_0$ not through $p_0$ is an external line, while a line in $\pi_0$ through $p_0$ is a tangent line.

Let $r$ be a secant line and let $s$ be the number of points in which $r$ intersects $\mathcal{K}$. Now, the same argument involving Equation 1 can be used to compute the number of planes through $r$. Hence, such a number is $s$, it is constant and independent from the choice of the line $r$. So, every line of $\mathbb{S}$ is a $i$–secant, where $i = 0, 1, s$.

Let $t$ be a tangent line to $\mathcal{K}$. Let $\mu$ be the number of secant planes through $t$, then

$$k = 1 + \mu(h-1). \tag{6}$$

Thus $\mu$ is independent from the line $t$ and

$$(h-1)|(k-1). \tag{7}$$

Let $E$ be an external line. Let $\gamma$ be the number of secant planes through $E$. Then

$$k = \gamma h + n + 1 - \gamma = \gamma(h-1) + n + 1. \tag{8}$$

There follows that $\gamma = \dfrac{k-1}{h-1} - \dfrac{n}{h-1}$ and since $h-1$ divides $k-1$ then

$$(h-1)|n \tag{9}$$

hence

$$h \leq n+1. \tag{10}$$

Since for every point $p$ of $\mathcal{K}$ there is at least a tangent line and since through every tangent line there is at least a tangent plane, through every point of $\mathcal{K}$ there is at least a tangent plane. Let $p$ be a point of $\mathcal{K} \setminus \{p_0\}$, and let $\pi_p$ be a tangent plane at $p$. Then $\pi_p \neq \pi_0$, and through the common line $E_0$ of these two planes there are at least two tangent planes. Since $h \leq n+1$ it follows, computing $k$ via the planes through $E_0$

$$n+1 \leq k \leq n^2+1. \tag{11}$$

Let $\pi$ be a plane of $\mathbb{S}$, denote with $i_\pi$ the number of planes intersecting $\pi$, then $i_\pi \leq (n^2+n+1)n+1 = (n+1)(n^2+1)$. Let $i_\pi = (n+1)(n^2+1) - u_\pi$, and let $\delta_\pi$ be the number of planes disjoint from $\pi$, then $c = n^3+n^2+n+1-u_\pi+\delta_\pi$. So the number of planes $c$ of $\mathbb{S}$ may be written as the sum of $n^3+n^2+n+1$ and an integer $z$.

From now on put $c = n^3+n^2+n+1+z$, with $z$ an integer.

Let $\alpha$ and $\beta$ be the number of planes tangent and secant respectively. It follows

$$n^3+n^2+n+1+z = \alpha + \beta. \tag{12}$$

Counting in two ways the pairs $(p,\pi)$ with $p \in \mathcal{K}$ and $p \in \pi$ gives

$$k(n^2+n+1) = \alpha + \beta h. \tag{13}$$

Counting in two ways the pairs $(\{p,p'\},\pi)$ with $p,p' \in \mathcal{K}\cap\pi$ gives

$$k(k-1)(n+1) = \beta h(h-1). \tag{14}$$

From equations (12) and (13) there follows:

$$k(n^2+n+1) - (n^3+n^2+n+1+z) = \beta(h-1). \tag{15}$$

Comparing equations (14) and (15) we get

$$k(k-1)(n+1) = h[k(n^2+n+1) - (n^3+n^2+n+1+z)]. \tag{16}$$

Hence we have the following equation in $k$:

$$k^2(n+1) - k[(n+1)+h(n^2+n+1)] + h(n^3+n^2+n+1+z) = 0. \tag{17}$$

Since $k = s+(n+1)(h-s)$, Equation (17) becomes

$$n(n+1)h^2 - n(sn^2-n^2+s+3sn+1)h + s^2n^3+s^2n^2+sn^2+sn = -hz. \tag{18}$$

Even if part of the proof of the next Lemma (the case $s \geq 3$) is similar to that of the main theorem of [5], we give it since our argument uses only $h$ for both cases $s=2$ and $s \geq 3$, and also to make the reading of the paper independent from that of [5].

**9 Lemma.** *If $z=0$, and $\mathcal{K}$ is a proper subset of $\mathcal{P}$, then $\mathcal{K}$ is a line of length $n+1$ or an ovoid of $\mathbb{S}$.*

Proof. Let $z=0$, then Equation (18) becomes

$$(n+1)h^2 - (sn^2-n^2+s+3sn+1)h + s^2n^2+s^2n+sn+s = 0. \tag{19}$$

The discriminant of Equation (19) is

$$\begin{aligned}\Delta =&(s-1)^2n^4+2s(s-3)n^3+(3s^2-4s-2)n^2+2s(s-1)n+(s-1)^2 = \\ =&[(s-1)n^2-sn-(s-1)]^2+4s(s-2)n^3+4s(s-2)n^2\end{aligned}$$

which is non–negative for $s \geq 2$.

For $s=2$, $\Delta = (n^2-2n-1)^2$, and $h_1 = n+1$, $h_2 = 4 - \dfrac{2}{n+1}$. Since $h_2$ is an integer, we have $n=1$, a contradiction.

Hence $h=n+1$, $k = 2+(n+1)(n-1)n = n^2+1$ and $\mathcal{K}$ is an ovoid.

Now, let $s \geq 3$.

We have

$$h \in \Big\{\frac{(n^2+3n+1)s-(n^2-1)-\sqrt{\Delta}}{2(n+1)},\ \ \frac{(n^2+3n+1)s-(n^2-1)+\sqrt{\Delta}}{2(n+1)}\Big\}.$$

Since $\sqrt{\Delta} > [(s-1)n^2 - sn - (s-1)] + 1$, we have that
$h_2 = \dfrac{(n^2+3n+1)s-(n^2-1)+\sqrt{\Delta}}{2(n+1)} > n+1$, a contradiction.
Let $h_1 = \dfrac{(n^2+3n+1)s-(n^2-1)-\sqrt{\Delta}}{2(n+1)}$. From $\sqrt{\Delta} > [(s-1)n^2 - sn - (s-1)] + 1$ it follows that $h < 2s - \dfrac{2s+1}{2(n+1)} < 2s$.

Thus, there is no plane with two secant lines. It follows that if $r$ is a secant line, then there is no point of $\mathcal{K}$ outside $r$. Since every plane meets $\mathcal{K}$, and there are $(n^2+n)(n+1)$ planes it follows that $sn^2+n+1 = (n^2+1)(n+1)$, that is $s = n+1$. So, $\mathcal{K}$ is a line of length $n+1$ of $\mathbb{S}$. QED

**10 Lemma.** $z \geq 0$.

Proof. Assume $z < 0$. Let $z' = \dfrac{z}{n}$, then Equation (18) becomes

$$(n+1)h^2 - (sn^2 - n^2 + s + 3sn + 1 - z')h + (n+1)s(sn+1) = 0. \qquad (20)$$

Let $\Delta$ be the discriminant of Equation (19) and let $\Delta'$ be the discriminant of Equation (20), and let $h_1', h_2'$ be the roots of equation (20), and $h_1, h_2$ be the roots of Equation (19), as above. Then $\Delta' > \Delta$.

Thus,

$$\begin{aligned} h_2' =& \frac{(sn^2 - n^2 + s + 3sn + 1 - z') + \sqrt{\Delta'}}{2(n+1)} \\ >& \frac{(sn^2 - n^2 + s + 3sn + 1) + \sqrt{\Delta}}{2(n+1)} - \frac{z'}{2(n+1)} \\ =& \frac{(n^2+3n+1)s-(n^2-1)+\sqrt{\Delta}}{2(n+1)} - \frac{z'}{2(n+1)} \\ =& h_2 - \frac{z'}{2(n+1)} > n+1 - \frac{z'}{2(n+1)} > n+1, \end{aligned}$$

which cannot occur since $h \leq n+1$.

Moreover, since $h_1'h_2' = h_1h_2 = s(sn+1)$, from $h_2' > h_2$ it follows that $h_1' < h_1 < 2s$.

Hence, there is no plane with two secant lines. It follows that if $r$ is a secant line, then there is no point of $\mathcal{K}$ outside $r$. Since every plane meets $\mathcal{K}$, and there are $(n^2+n)(n+1)$ planes it follows that $sn^2+n+1=c\leq n^3+n^2+n$, that is $s\leq n$. Let $\alpha$ be a plane through $r$, and let $\ell$ be a line of $\alpha$ disjoint from $r$. Any plane through $r$, different from $\alpha$, is disjoint with $r$, that is there are planes disjoint with $\mathcal{K}$, a contradiction. QED

**11 Lemma.** *If $\mathcal{K}$ is either a line of length $n+1$ or an ovoid then $z=0$.*

Proof. If $\mathcal{K}$ is a line of length $n+1$, then it intersects all the planes of the planar space, and so $c=(n+1)n^2+n+1=n^3+n^2+n+1$.

Now, let $\mathcal{K}$ be an ovoid of $\mathbb{S}$. Then, $k=n^2+1$, and every secant line is 2–secant.

From $k=s+(n+1)(h-s)$, it follows that $h=n+1$. Then by Equation (18) it follows that $z=0$. QED

The previous Lemmata proves Theorem I.

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