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On a Theorem of Burnside

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Abstract. We introduce an algebraic integer related to the irreducible complex characters of finite groups and use it to obtain a generalization of a theorem of Burnside.

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1 Introduction

Let G be a finite group, $\mathcal{C}$ a conjugacy class of G and χ an irreducible (complex) character of G. Let $|\mathcal{C}|$ denote the length of $\mathcal{C}$. It is well known [see, e. g., [3] , Corollary 3.5, and Theorem 3.6] that both $\chi(\mathcal{C})$ and $\frac{|\mathcal{C}|}{\chi(1)}\chi(\mathcal{C})$ belong to the set $\mathcal{I}$ of the (complex) algebraic integers (over $\mathbb{Z}$).

For $a,\ b \in \mathbb{Z}$, let $(a,\ b)$ denote the greatest common divisor of a and b. In this note we observe (in Section 2) that $\frac{(|\mathcal{C}|,\chi(1))}{\chi(1)}\chi(\mathcal{C})$ and $\frac{(|\mathcal{C}|,\chi(1))}{\chi(1)}\,|\chi(\mathcal{C})|$ are algebraic integers, and use this fact (in Section 3) to prove a generalization (Corollary 5) of

1 Theorem. *[Burnside] Let G be a finite group, $\mathcal{C}$ a conjugacy class of G and χ an irreducible character of G of degree coprime to the length of $\mathcal{C}$. Then either $\chi(\mathcal{C}) = 0$ or $|\chi(\mathcal{C})| = \chi(1)$.*

Throughout this paper, "group" will mean "finite group" and "character" will mean "complex character"; moreover, an "integer" will be an element of $\mathbb{Z}$.

We also recall that $\mathcal{I}$ is a Dedekind ring and that $\mathcal{I} \cap \mathbb{Q} = \mathbb{Z}$ [[4], Theorem 5.3]: thus, a rational algebraic integer is, tout court, an integer.

Last, we observe that $z \in \mathcal{I}$ if and only if $\overline{z} \in \mathcal{I}$ ($\overline{z}$ being the complex conjugate of z); therefore, if $z \in \mathcal{I}$ then $|z| \left(= \sqrt{z \cdot \overline{z}}\right) \in \mathcal{I}$. Hence, in addition to $\chi(\mathcal{C})$ and $\frac{|\mathcal{C}|}{\chi(1)}\chi(\mathcal{C})$, also $|\chi(\mathcal{C})|$ and $\frac{|\mathcal{C}|}{\chi(1)}\,|\chi(\mathcal{C})|$ belong to $\mathcal{I}$.

2 Some algebraic integers related to group theory

2 Proposition. *The following hold:*

(a) Let $z \in \mathcal{I}$, and k, $m \in \mathbb{Z}$, $m \neq 0$. Then $\frac{k}{m} z \in \mathcal{I}$ if and only if $\frac{(k,m)}{m} z \in \mathcal{I}$.

(b) Let G be a group, χ an irreducible character of G and $\mathcal{C}$ a conjugacy class of G. The complex numbers

$$\frac{(\,|\mathcal{C}|\,,\chi(1))}{\chi(1)}\chi(\mathcal{C}) \quad \textit{and} \quad \frac{(\,|\mathcal{C}|\,,\chi(1))}{\chi(1)}\,|\chi(\mathcal{C})|$$

belong to $\mathcal{I}$.

Proof. If $w \in \mathcal{I}$ then also $hw \in \mathcal{I}$ for any $h \in \mathbb{Z}$. So, to prove (a) we only have to show that if $\frac{k}{m} z \in \mathcal{I}$ then $\frac{(k,m)}{m} z \in \mathcal{I}$. Writing

$$(k, m) = rk + sm \qquad \text{for suitable} \quad r,\, s \in \mathbb{Z}$$

we obtain that $(k,\, m)\frac{z}{m}$ $\big($being equal to $(rk + sm)\,\frac{z}{m} = r\frac{k}{m}z + sz\big)$ is a $\mathbb{Z}$-linear combination of algebraic integers, hence is itself an algebraic integer.

Now (b) is obvious, because $\frac{|\mathcal{C}|}{\chi(1)}\chi(\mathcal{C})$ and $\frac{|\mathcal{C}|}{\chi(1)}\,|\chi(\mathcal{C})|$ are algebraic integers. $\boxed{QED}$

In the following, we shall refer to the algebraic integer $\frac{(|\mathcal{C}|,\chi(1))}{\chi(1)}\chi(\mathcal{C})$ as *the standard algebraic integer of the pair* $(\chi\,,\mathcal{C})$. Note that the "classical" algebraic integers $\chi(\mathcal{C})$ and $\frac{|\mathcal{C}|}{\chi(1)}\chi(\mathcal{C})$ are integer multiples of it.

3 The Forbidden Annulus Theorem

We now obtain a generalization of Burnside's Theorem 1 by exploiting the standard algebraic integer of the pair $(\chi\,,\mathcal{C})$.

Let u be an algebraic integer and let

$$x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$$

be the minimal (monic) polynomial of u over $\mathbb{Q}$. By Gauss' lemma its coefficients are rational integers; we call $|a_0|$ the *pseudo-norm* of u.

3 Theorem. *[Forbidden Annulus Theorem] Let d be a positive integer, z a sum of d complex roots of unity and q a positive rational such that*

$$q\frac{z}{d} \quad \textit{is an algebraic integer.}$$

Let r be the algebraic degree of $q\frac{z}{d}$ over $\mathbb{Q}$, and t its pseudo-norm. Then the real number $|z|$ does not belong to the open interval

$$\left(0,\ \frac{dt}{q^r}\right).$$

PROOF. We may assume that $z \neq 0$. We have to show that $|z| \geq dt/q^r$.

Let $\lambda_1 := q\frac{z}{d}$ and let $p(x) := x^r + \cdots + a_0$ be the minimal (monic) polynomial of λ_1 over $\mathbb{Q}$ (so that $t = |a_0|$).

If $r = 1$, then $p(x) = x + a_0$ ($a_0 \neq 0$) so that $0 = p(\lambda_1) = \lambda_1 + a_0$ i. e. $|z| = \frac{dt}{q}$; hence we may suppose that $r > 1$.

Let us assume by contradiction that $|z| < \frac{dt}{q^r}$; from the definition of λ_1 we obtain that

$$|\lambda_1| < \frac{t}{q^{r-1}}. \tag{1}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the algebraic conjugates of λ_1. Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are exactly the roots of the polynomial $p(x)$,

$$t = |\lambda_1| \cdot |\lambda_2| \cdot \cdots \cdot |\lambda_r|.$$

By hypothesis,

$$z = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d \qquad \text{for some roots of unity} \quad \varepsilon_i.$$

Then

$$z = \varepsilon^{m_1} + \varepsilon^{m_2} + \cdots + \varepsilon^{m_d} \qquad \text{for certain} \quad m_i \in \mathbb{N}$$

where ε is a suitable root of unity. Thus we can rewrite λ_1 as

$$\lambda_1 := q\frac{z}{d} = \frac{q}{d}\left(\varepsilon^{m_1} + \varepsilon^{m_2} + \cdots + \varepsilon^{m_d}\right)$$

and its algebraic conjugates as

$$\lambda_i = \frac{q}{d}\left(\varepsilon^{s_1^{(i)}} + \varepsilon^{s_2^{(i)}} + \cdots + \varepsilon^{s_d^{(i)}}\right) \qquad i := 2, 3, \ldots, r$$

for certain $s_1^{(i)}, s_2^{(i)}, \ldots, s_d^{(i)} \in \mathbb{N}$ [see, e.g., [4], Proposition 5.2].

Thus for $i := 2, 3, \ldots, r$ we have

$$\begin{aligned} |\lambda_i| = \frac{q}{d} \cdot \left|\varepsilon^{s_1^{(i)}} + \varepsilon^{s_2^{(i)}} + \cdots + \varepsilon^{s_d^{(i)}}\right| \leq \\ \leq \frac{q}{d} \cdot \left(\left|\varepsilon^{s_1^{(i)}}\right| + \left|\varepsilon^{s_2^{(i)}}\right| + \cdots + \left|\varepsilon^{s_d^{(i)}}\right|\right) = \frac{q}{d}d = q. \end{aligned} \tag{2}$$

From (1) and (2) we finally obtain

$$t = |\lambda_1| \cdot |\lambda_2| \cdot \cdots \cdot |\lambda_r| < \frac{t}{q^{r-1}} \cdot q^{r-1} = t$$

a contradiction. $\boxed{QED}$

In the above theorem, any appropriate choice of the positive rational q provides a "forbidden annulus" (and the smaller q is, the larger the annulus we get).

Let $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary part of the complex number z. Since $2\,\mathrm{Re}(z)$ and $2\,\mathrm{Im}(z)$ are in $\mathcal{I}$ whenever $z \in \mathcal{I}$, analogous bounds on $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ can be obtained by mimicking the above proof.

We now specialize to groups.

4 Theorem. *Let G be a group, χ an irreducible character of G and $\mathcal{C}$ a conjugacy class of G. Take $q \in \mathbb{Q}^+$ such that*

$$q\frac{\chi(\mathcal{C})}{\chi(1)} \quad \textit{is an algebraic integer}$$

and let r be its algebraic degree over $\mathbb{Q}$, and t its pseudo-norm. Then the real number $|\chi(\mathcal{C})|$ does not belong to the open interval

$$\left(0,\ \frac{\chi(1)t}{q^r}\right).$$

Proof. Let $d := \chi(1)$. Let k be the order of any $g \in \mathcal{C}$; then [see e. g. [4], pag. 59]

$$\chi(\mathcal{C}) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d \qquad \text{for certain } k\text{-th roots of } 1.$$

Now apply Theorem 3 (with $z := \chi(\mathcal{C})$) to obtain the result. $\boxed{QED}$

Finally we observe that the standard algebraic integer of the pair $(\chi,\ \mathcal{C})$ gives $(|\mathcal{C}|\,,\ \chi(1))$ as a possible value of q in Theorem 4, and we obtain

5 Corollary. *Let G be a group, χ an irreducible character of G, $\mathcal{C}$ a conjugacy class of G, r the algebraic degree of $\chi(\mathcal{C})$ over $\mathbb{Q}$ and t the pseudo-norm of $(|\mathcal{C}|\,,\ \chi(1))\frac{\chi(\mathcal{C})}{\chi(1)}$. Then the real number $|\chi(\mathcal{C})|$ does not belong to the open interval*

$$\left(0,\ \frac{\chi(1)t}{((|\mathcal{C}|\,,\ \chi(1)))^r}\right).$$

4 Final remarks

(1) When $(|\mathcal{C}|\,,\ \chi(1) = 1)$, Corollary 5 yields Burnside's Theorem 1.

(2) It is intuitive that, for a given positive integer k, the modulus of a sum of $k-$th roots of unity (if non-zero) cannot be arbitrarily small (think of them as unit vectors in the complex plane). The problem of how small this sum can be has been already addressed by several authors [see e. g. [2] and [5]], but no general result seems to have emerged which can be useful in our context.

(3) It is natural to ask whether there exist any other forbidden annuli that could be described in similar terms. For example, how close can $\chi(\mathcal{C})$ get to $\chi(1)$? Special cases of this problem are discussed, e.g., in [1].

(4) As the group A_5 shows, the integer r in the statement of Corollary 5 cannot be replaced, in general, by a smaller integer. However, since $|\chi(\mathcal{C})| \in \mathcal{I}$, if it is not integer it is irrational: hence, in such a case, $|\chi(\mathcal{C})|$ is strictly greater than $\dfrac{\chi(1)t}{((|\mathcal{C}|\,,\ \chi(1)))^r}$.

References

[1] D. Gluck, K. Magaard: *Character and fixed point ratios in finite classical groups*, Proc. London Math. Soc. (3) **71** (1995), 547–584.

[2] R. L. Graham, N. J. A. Sloane: *Anti-Hadamard matrices*, Linear Algebra Appl. **62** (1984), 113–137.

[3] I. M. Isaacs: Character Theory of Finite Groups, Academic Press, (1976).

[4] W. Ledermann: Introduction to group characters, Cambridge University Press, (1977).

[5] G. Myerson: *How small can a sum of roots of unity be?*, Amer. Math. Monthly **93** (1986), 457–459.