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On the Generalized Bertrand Curves in Euclidean N-spaces

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Abstract. In this article, we give a necessary condition for a C^∞-special Frenet curve in $\mathbb{R}^N$ being a generalized Bertrand curve.

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1 Introduction

A curve $C : I \to \mathbb{R}^N$ is called a C^∞-*special Frenet curve in* $\mathbb{R}^N$ if it is a smooth and regular curve with well-defined Frenet frames $\{t, n_1, \cdots, n_{N-1}\}$ and *non-zero* curvatures (i.e., curvatures never vanish at any point of curves) along the curve. On the other hand, in literature, a C^∞-special Frenet curve in $\mathbb{R}^3$ is called a Bertrand curve if there exists a distinct curve $\bar{C}(s) = C(s) + r(s) \cdot n_1(s)$, where n_1 is well-defined along C, such that the 1-normal lines of $C(s)$ and $\bar{C}(s)$ are equal for all $s \in I$. Furthermore, $\bar{C}$ is called the Bertrand mate of C.

Bertrand curves in $\mathbb{R}^3$ have many interesting geometric properties (e.g., see p.26 in [1] for more details). These types of curves have also been applied in computer-aided geometric design (CAGD) (e.g., see [5], [7]). In [3] Hayden has suggested to extend the definition of Bertrand curves in $\mathbb{R}^3$ to those in $\mathbb{R}^N$ or Riemannian manifolds. However, Pears in [6] showed that a Bertrand curve in $\mathbb{R}^N$ must belong to a three-dimensional subspace in $\mathbb{R}^N$, since its curvatures of higher order must be identically equal to zero, i.e., $k_j = 0$ for all $j \geq 3$. This implies that a Bertrand curve in $\mathbb{R}^N$ can't be a C^∞-special Frenet curve in $\mathbb{R}^N$. Notice that a C^∞-special Frenet curve in $\mathbb{R}^N$ can not be confined in

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a lower dimensional subspace of $\mathbb{R}^N$ because all of its curvatures are nonzero. Thus the classical definition of Bertrand curves is not suitable for C^∞-special Frenet curves in $\mathbb{R}^N$. This phenomenon motivates us to extend the notion of Bertrand curves to C^∞-special Frenet curves in $\mathbb{R}^N$ when $N \geq 4$. To the authors' knowledge, [4] is the only article extending the notion of Bertrand curves to Euclidean spaces of higher dimensions. The reader is referred to [4] for explicit examples of the so-called $(1,3)$-Bertrand curves in $\mathbb{R}^4$ (which is a type of generalized Bertrand curves).

In this article, we generalize the definition of classical Bertrand curves in $\mathbb{R}^3$ and $(1,3)$-Bertrand curves in $\mathbb{R}^4$ (see [4]) by defining the so-called generalized Bertrand curves for the class of C^∞-special Frenet curves in $\mathbb{R}^N$, where $N \geq 4$ (see Definition 1). Our main result gives a necessary condition for existence of the generalized Bertrand curve in $\mathbb{R}^N$. We found that only a particular type of generalized Bertrand curves exists in $\mathbb{R}^N$.

1 Definition (The Generalized Bertrand Curves)**.** Assume $C : I \to \mathbb{R}^N$ is a C^∞-special Frenet curve. Let $i_p \in \{1, 2, \ldots, N-1\}$, where $p \in \{1, \ldots, m\}$ and $m \in \{1, 2, \ldots, N-1\}$. Denote by n_{i_p} the i_p-th unit normal vector field of the curve C. Then, the curve C is called a $(i_1, \ldots, i_m)$-Bertrand curve if there exists a distinct C^∞-special Frenet curve,

$$\bar{C}(s) = C(s) + \sum_{p=1}^{m} \alpha_{i_p}(s) \cdot n_{i_p}(s), \tag{1}$$

such that the Frenet $(i_1, \ldots, i_m)$-normal planes at $C(s)$ and $\bar{C}(s)$ coincide for all $s \in I$.

For our convenience, we call a $(i_1, \ldots, i_m)$-Bertrand curve the generalized Bertrand curves, and we always let $1 \leq i_1 < i_2, < \ldots\ldots < i_m \leq N-1$.

2 Theorem. *If a C^∞-special Frenet curve in $\mathbb{R}^N$ is a generalized Bertrand curve, then it must be the type of $(1, i_2, \ldots, i_m)$-Bertrand curve.*

3 Remark. The generalized Bertrand curves still keep certain geometric properties. For example, by a straightforward computation, one can verify that the distance between a generalized Bertrand curve and its mate (offset) along the curve remains constant. For geometric properties of $(1,3)$-Bertrand curves in $\mathbb{R}^4$, the reader is referred to [4]. For generalized Bertrand curves in $\mathbb{R}^N$, we leave the discussion to our future work.

2 Proofs

We will argue by contradiction. Namely, we assume that the C^∞-special Frenet curve C in $\mathbb{R}^N$ is a $(i_1, \ldots, i_m)$-Bertrand curve with $i_1 \geq 2$, then a contradiction would happen.

Denote by $\bar{s} = \varphi(s)$ the arc-length parameter of $\bar{C}$. Let

$$K_{j,i}(s) = -k_i(s)\delta_j^{i-1} + k_{i+1}(s)\delta_j^{i+1}, \tag{2}$$

where δ_j^i is the Kronecker's delta and k_i are higher curvatures of $\mathbb{R}^N$ if $i \in \{1, \ldots, N-1\}$; otherwise $k_i = 0$ (see [2]). Then the Frenet equations can be written as

$$n_i'(s) = \sum_{j=0}^{N-1} K_{j,i}(s) n_j(s), \tag{3}$$

where $n_0(s) = t(s)$ and $i \in \{0, \ldots, N-1\}$. By differentiating (1) with respect to s, we obtain

$$\begin{aligned} \varphi'(s) \cdot \bar{n}_0(s) =& \bar{C}'(s) \\ =& n_0(s) + \sum_{p=1}^{m} \alpha_{i_p}'(s) \cdot n_{i_p}(s) + \sum_{j=0}^{N-1} \sum_{p=1}^{m} K_{j,i_p}(s) \cdot \alpha_{i_p}(s) \cdot n_j(s) \\ =& \sum_{j=0}^{N-1} \beta_j(s) n_j(s), \end{aligned} \tag{4}$$

where n_0 and $\bar{n}_0$ denote the unit tangent vectors of C and $\bar{C}$ respectively. Since by assumption the normal plane spanned by $\bar{n}_{i_1}(s), \ldots, \bar{n}_{i_m}(s)$ coincides with the one spanned by $n_{i_1}(s), \ldots, n_{i_m}(s)$, there exists a matrix $T(s) \in O(m)$ such that

$$(\bar{n}_{i_1}(s), \ldots, \bar{n}_{i_m}(s))^t = T(s)(n_{i_1}(s), \ldots, n_{i_m}(s))^t,$$

for all $s \in I$. In other words,

$$\bar{n}_{i_q}(s) = \sum_{p=1}^{m} T_{qp}(s) n_{i_p}(s), \tag{5}$$

where T_{qp} is the (q,p)-th entry of the matrix T. Thus by (4) and (5), we have

$$\begin{aligned} 0 =& \langle \varphi'(s) \cdot \bar{n}_0(s), \bar{n}_{i_q}(s) \rangle = \sum_{p=1}^{m} T_{qp}(s) \langle \sum_{j=0}^{N-1} \beta_j(s) \cdot n_j(s), n_{i_p}(s) \rangle \\ =& \sum_{p=1}^{m} T_{qp}(s) \beta_{i_p}(s), \end{aligned} \tag{6}$$

for each fixed $q \in \{1, 2, \ldots, m\}$. Since $\det T(s) = \pm 1 \neq 0$, it follows from (4) and (6) that

$$0 = \beta_{i_q}(s) = \alpha'_{i_q}(s) + \sum_{p=1}^{m} K_{i_q,i_p}(s) \cdot \alpha_{i_p}(s), \tag{7}$$

for each $q \in \{1, 2, \ldots, m\}$. By (4),

$$\bar{n}_0(s) = \frac{1}{\varphi'(s)} n_0(s) + \sum_{p=1}^{m} \frac{1}{\varphi'(s)} \alpha'_{i_p}(s) \cdot n_{i_p}(s) \\ + \sum_{j=0}^{N-1} \sum_{p=1}^{m} \frac{1}{\varphi'(s)} K_{j,i_p}(s) \cdot \alpha_{i_p}(s) \cdot n_j(s). \tag{8}$$

By differentiating (8) with respect to s, we obtain

$$\begin{aligned}
&\varphi'(s) \cdot \bar{k}_1(s) \cdot \bar{n}_1(s) \\
&\quad = \left(\frac{1}{\varphi'(s)}\right)' n_0(s) + \frac{1}{\varphi'(s)} k_1(s) n_1(s) + \sum_{p=1}^{m} \left(\frac{1}{\varphi'(s)} \alpha'_{i_p}(s)\right)' n_{i_p}(s) \\
&\qquad + \sum_{j=1}^{N-1} \left(\sum_{p=1}^{m} \frac{1}{\varphi'(s)} K_{j,i_p}(s) \cdot \alpha'_{i_p}(s) \right) \cdot n_j(s) \\
&\qquad + \sum_{j=1}^{N-1} \left(\sum_{p=1}^{m} \frac{1}{\varphi'(s)} K_{j,i_p}(s) \cdot \alpha_{i_p}(s) \right)' \cdot n_j(s) \\
&\qquad + \sum_{j=0}^{N-1} \left(\sum_{p=1}^{m} \frac{1}{\varphi'(s)} [K_{j-1,i_p}(s) \cdot k_j(s) - K_{j+1,i_p}(s) \cdot k_{j+1}(s)] \cdot \alpha_{i_p}(s) \right) \cdot n_j(s) \\
&\quad = \sum_{j=0}^{N-1} \gamma_j(s) \cdot n_j(s),
\end{aligned} \tag{9}$$

where $k_N = 0$. By (9), (5) and assuming $i_q \geq 2$ for all $q \in \{1, 2, \ldots, m\}$, we have

$$0 = \langle \varphi'(s) \bar{k}_1(\varphi(s)) \bar{n}_1(\varphi(s)), \bar{n}_{i_q}(\varphi(s)) \rangle = \sum_{p=1}^{m} T_{qp}(s) \gamma_{i_p}(s). \tag{10}$$

Since $\det T(s) = \pm 1 \neq 0$, it follows from (9) and (10) that

$$\begin{aligned}0 =&\gamma_{i_q}(s)\\ =&\Big(\frac{1}{\varphi'(s)}\alpha'_{i_q}(s)\Big)' + \sum_{p=1}^{m}\frac{1}{\varphi'(s)}K_{i_q,i_p}(s)\alpha'_{i_p}(s)\\ &+\sum_{p=1}^{m}\Big(\frac{1}{\varphi'(s)}K_{i_q,i_p}(s)\alpha_{i_p}(s)\Big)'\\ &+\sum_{p=1}^{m}\frac{1}{\varphi'(s)}\big(K_{-1+i_q,i_p}(s)k_{i_q} - K_{1+i_q,i_p}(s)k_{1+i_q}\big)\alpha_{i_p}(s),\end{aligned} \tag{11}$$

for all $q \in \{1, 2, \dots, m\}$. Below we omit the arc-length parameter s of C without confusion. Denote by $A = (\alpha_{i_1}, \dots, \alpha_{i_m})^t$, and let $B = (B_{lp})$, and $R = (R_{lp})$ to be

$$B_{lp} = K_{i_l,i_p}, \tag{12}$$
$$R_{lp} = K_{-1+i_l,i_p}k_{i_l} - K_{1+i_l,i_p}k_{1+i_l}. \tag{13}$$

Then (7) and (11) can be written respectively as

$$A' + BA = 0, \tag{14}$$
$$\Big(\frac{1}{\varphi'}A'\Big)' + \frac{1}{\varphi'}BA' + \Big(\frac{1}{\varphi'}BA\Big)' + \frac{1}{\varphi'}RA = 0. \tag{15}$$

Substituting A' by $-BA$ in (15), we can simplify (15) as

$$(R - B^2)A = 0. \tag{16}$$

4 Lemma. *The $m \times m$ matrix $R - B^2$ is symmetric and can be written as*

$$\begin{pmatrix} D_1+F_1 & N_1 & 0 & \cdots & \cdots & 0\\ N_1 & D_2+F_2 & N_2 & \ddots & \ddots & \vdots\\ 0 & N_2 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & N_{m-2} & 0\\ \vdots & \ddots & \ddots & N_{m-2} & D_{m-1}+F_{m-1} & N_{m-1}\\ 0 & \cdots & \cdots & 0 & N_{m-1} & D_m+F_m \end{pmatrix},$$

where

$$\begin{cases} D_q = K^2_{i_{q-1},i_q} - k^2_{i_q},\\ F_q = K^2_{i_q,i_{q+1}} - k^2_{1+i_q},\\ N_q = K_{-1+i_{q+1},i_q}k_{i_{q+1}}, \end{cases} \tag{17}$$

and we let $K_{i_p,i_q} = 0$, if i_p or i_q is not defined.

Proof. From (12), it is obvious that $B^t = -B$, thus B^2 is symmetric. By applying (13), (12) and (2), it is easy to verify that the matrix R is symmetric and to compute all entries of $R - B^2$. We leave it to the reader. QED

We can decompose the matrix $R - B^2$ into a sum of matrices. Namely,

$$R - B^2 = \sum_{q=1}^{m+1} E_q$$
$$= \begin{pmatrix} D_1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & 0 & F_m \end{pmatrix}$$
$$+ \sum_{q=2}^{m} \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \ddots & \vdots \\ \vdots & \vdots & F_{q-1} & N_{q-1} & \vdots & \vdots \\ \vdots & \vdots & N_{q-1} & D_q & \vdots & \vdots \\ \vdots & \ddots & \cdots & \cdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}.$$

5 Lemma. *For each fixed $q \in \{1, \ldots, m+1\}$ and $X = (x_1, \ldots, x_m)^t$,*

$$\langle E_q X, X \rangle \leq 0. \tag{18}$$

Proof. It is easy to verify (18) by using (17) and (2). We leave it to the reader. QED

Observe that Lemma 5 and (16) imply

$$\langle E_q A, A \rangle = 0,$$

for each fixed q.

6 Lemma. *(i) Assume $i_{q+1} - i_q \geq 3$. Then, $\alpha_{i_q} = 0 = \alpha_{i_{q+1}}$.*

(ii) Assume $i_{q+1} - i_q = 2$. Then, $\alpha_{i_q} = 0$ if and only if $\alpha_{i_{q+1}} = 0$.

(iii) Assume $i_{q+1} - i_q = 1$. Then, $\alpha_{i_{q-1}} = 0 = \alpha_{i_q}$ implies $\alpha_{i_{q+1}} = 0$, where we set $\alpha_{i_0} = 0$.

Proof. **Case (i):** $i_{q+1} - i_q \geq 3$. From

$$0 = \langle E_{q+1}A, A\rangle = -[(k_{1+i_q}\alpha_{i_q})^2 + (k_{i_{q+1}}\alpha_{i_{q+1}})^2],$$

it follows that

$$k_{1+i_q}\alpha_{i_q} = 0 = k_{i_{q+1}}\alpha_{i_{q+1}}.$$

Thus $\alpha_{i_q} = 0 = \alpha_{i_{q+1}}$.

Case (ii): $i_{q+1} - i_q = 2$. From

$$0 = \langle E_{q+1}A, A\rangle = -(k_{1+i_q}\alpha_{i_q} - k_{i_{q+1}}\alpha_{i_{q+1}})^2,$$

it follows that

$$k_{1+i_q}\alpha_{i_q} = k_{i_{q+1}}\alpha_{i_{q+1}}.$$

Thus $\alpha_{i_q} = 0$ if and only if $\alpha_{i_{q+1}} = 0$.

Case (iii): $i_{q+1} - i_q = 1$. By (7), we have

$$-\alpha'_{i_q} = K_{i_q, i_{q-1}}\alpha_{i_{q-1}} - k_{i_{q+1}}\alpha_{i_{q+1}}.$$

By assuming $\alpha_{i_{q-1}} = 0 = \alpha_{i_q}$, it follows that $\alpha_{i_{q+1}} = 0$.

QED

Proof of Theorem 2. By $\langle E_1A, A\rangle = 0$, we obtain $k_{i_1}^2\alpha_{i_1}^2 = 0$. Hence, $\alpha_{i_1} = 0$. Then, by applying Lemma 6 inductively, we obtain $\alpha_{i_2} = \cdots = \alpha_{i_m} = 0$. This implies that $\bar{C}$ coincides with C, which is a contradiction. QED

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