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Generalizing the Kantor-Knuth Spreads

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Abstract. The Kantor-Knuth conical flock spreads are generalized to large dimension. Any such spread is derivable and admits double Baer groups of large order.

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1 Introduction

The Kantor-Knuth semifield spreads are important and unusual in that they are semifield flock spreads in $PG(3,q)$ that are derivable by a non-regulus net. Any such conical flock spread in $PG(3,K)$, where K is a field isomorphic to $GF(q)$ is a union of q reguli that share a common line of $PG(3,K)$. The Kantor-Knuth conical flock spreads have odd order and may be represented by

$$x = 0, y = x \begin{bmatrix} u & \gamma t^{\sigma} \\ t & u \end{bmatrix};\ u, t \in GF(q),$$

where γ is a non-square in $GF(q)$ and σ is a non-trivial automorphism of $GF(q)$, where x and y are considered 2-vectors over $GF(q)$.

Consider the subspread

$$D_{\sigma} : x = 0, y = x \begin{bmatrix} 0 & \gamma t^{\sigma} \\ t & 0 \end{bmatrix};\ t \in GF(q),$$

we may see that this is a derivable net that is not a regulus as follows: Change bases by the mapping $(x, y) \to (x, y \begin{bmatrix} 0 & 1 \\ \gamma^{-1} & 0 \end{bmatrix})$ to represent the subspread in the form

$$x = 0, y = x \begin{bmatrix} t^{\sigma} & 0 \\ 0 & t \end{bmatrix}; t \in GF(q).$$

Since the associated matrices form a field isomorphic to $GF(q)$, it follows that this spread is a derivable partial spread. Let π be a flock spread that admits a derivable net that is not a regulus net. This is an extremely rare situation and the second author has shown that the Kantor-Knuth spreads are precisely the spreads with these properties. A 'derivable flock of a quadratic cone' is a flock whose corresponding conical flock spread admits a derivable partial spread sharing the line shared by the q reguli.

1 Theorem. *[Johnson [5]] If $\mathcal{F}$ is a derivable flock of a quadratic cone in $PG(3,q)$ then q is odd and $\mathcal{F}$ is a Kantor-Knuth flock or the flock is linear.*

The uniqueness of the Kantor-Knuth spreads suggests that certain generalizations of these spreads are of interest. In this article, we give a generalization of the Kantor-Knuth spreads to spreads of larger dimension than 2, that is, whose spreads are not in $PG(3,q)$. (The reader is directly to the Handbook [2] or the Foundations' text [1] for any background not directly given.)

2 Large Dimension Kantor-Knuth Semifield Spreads

We now show how a generalization of the Kantor-Knuth Semifield spreads might be considered using the idea of the companion semifield. The idea arose from an article dealing with a spread-only consideration of the dual of a semifield. This is as follows: Suppose we have a semifield spread of order p^n written over the prime field $GF(p)$, the rows of an associated matrix spread set are given in terms of linear transformations A_i of the n-dimensional $GF(p)$-vector space. That is, it can be shows that a semifield spread may be represented in the form:

$$y = x \begin{bmatrix} w \\ wA_2 \\ wA_2 \\ \vdots \\ wA_t \end{bmatrix}, \text{ for all } t\text{-vectors } x \text{ over } GF(p),$$

where w is an arbitrary t-vector. The semifield corresponding to the dual semifield is then shown to be

$$x = 0, y = x \left[\sum_{i=1}^{n} \alpha_i A_i \right],$$

$$\text{for all } t\text{-vectors } x \text{ over } GF(p), \text{ for all } \alpha_i \in GF(p),\ A_1 = I.$$

This result is given in Jha and Johnson [3]. We call the associated spread the 'companion semifield' and refer to the 'companion semifield construction'.

We shall see how this idea actually generates the manner of generalization of the Kantor-Knuth spreads that we consider here.

Consider $GF(q^2)$, q odd and let $\{1, e\}$, for $e^2 = \theta$, for θ a non-square in $GF(q)$. Then the involutory automorphism mapping $GF(q^2)$ to $GF(q^2)$ and fixing $GF(q)$ pointwise takes $u + te$ to $u - te$. Represent $u + te$ as the matrix $\begin{bmatrix} u & t\theta \\ t & u \end{bmatrix}$. Then $\begin{bmatrix} u & t\theta \\ t & u \end{bmatrix}$ maps to $\begin{bmatrix} u & -t\theta \\ -t & u \end{bmatrix}$. Now let $\gamma = \begin{bmatrix} \gamma_1 & \gamma_2\theta \\ \gamma_2 & \gamma_1 \end{bmatrix}$, be a non-square in $GF(q^2)$, so, $\gamma_2 \neq 0$. Note that

$$\begin{bmatrix} u & -t\theta \\ -t & u \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2\theta \\ \gamma_2 & \gamma_1 \end{bmatrix} = \begin{bmatrix} u\gamma_1 - t\gamma_2\theta & u\gamma_2\theta - t\gamma_1\theta \\ -t\gamma_1 + u\gamma_2 & -t\gamma_2\theta + u\gamma_1 \end{bmatrix}.$$

Now take the Kantor-Knuth spread of order q^4.

$$x = 0, y = x \begin{bmatrix} w & r^q\gamma \\ r & w \end{bmatrix};$$

for all $w, r \in GF(q^2)$. Let $r = \begin{bmatrix} u & t\theta \\ t & u \end{bmatrix}$ and $w = \begin{bmatrix} k & s\theta \\ s & k \end{bmatrix}$. Then $r^q = \begin{bmatrix} u & -t\theta \\ -t & u \end{bmatrix}$. Now represent the Kantor-Knuth spread in its 4-dimensional representation.

$$\begin{bmatrix} k & s\theta & u\gamma_1 - t\gamma_2\theta & u\gamma_2\theta - t\gamma_1\theta \\ s & k & -t\gamma_1 + u\gamma_2 & -t\gamma_2\theta + u\gamma_1 \\ u & t\theta & k & s\theta \\ t & u & s & k \end{bmatrix}$$

We consider now the additive spread obtained by the span of the non-singular linear transformations mapping the 4th row into the 4th, 3rd, 2nd and 1st rows respectively, call these $A_4 = I_4, A_3, A_2, A_1$, respectively. Regarding (t, u, s, k) as $t(1, 0, 0, 0) + u(0, 1, 0, 0) + s(0, 0, 1, 0) + k(0, 0, 0, 1)$, we observe that

$$sA_3 = s \begin{bmatrix} 0 & \theta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ (t, u, s, k) \begin{bmatrix} 0 & \theta & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & 1 & 0 \end{bmatrix} = 3rd \ \text{row}$$

$$uA_2 = u \begin{bmatrix} 0 & 0 & -\gamma_1 & -\gamma_2\theta \\ 0 & 0 & \gamma_2 & \gamma_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, (t, u, s, k) \begin{bmatrix} 0 & 0 & -\gamma_1 & -\gamma_2\theta \\ 0 & 0 & \gamma_2 & \gamma_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \text{2nd row}$$

$$tA_1 = t \begin{bmatrix} 0 & 0 & -\gamma_2\theta & -\gamma_1\theta \\ 0 & 0 & \gamma_1 & \gamma_2\theta \\ 0 & \theta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, (t, u, s, k) \begin{bmatrix} 0 & 0 & -\gamma_2\theta & -\gamma_1\theta \\ 0 & 0 & \gamma_1 & \gamma_2\theta \\ 0 & \theta & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \text{1st row}$$

Then

$$kI_4 + sA_3 + uA_2 + tA_1 = \begin{bmatrix} k & s\theta & -u\gamma_1 - t\gamma_2\theta & -u\gamma_2\theta - t\gamma_1\theta \\ s & k & u\gamma_2 + t\gamma_1 & u\gamma_1 + t\gamma_2\theta \\ u & t\theta & k & s\theta \\ t & u & s & k \end{bmatrix}.$$

Now note that

$$\begin{aligned} \begin{bmatrix} -u\gamma_1 - t\gamma_2\theta & -u\gamma_2\theta - t\gamma_1\theta \\ u\gamma_2 + t\gamma_1 & u\gamma_1 + t\gamma_2\theta \end{bmatrix} &= \begin{bmatrix} -u & -t\theta \\ t & u \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2\theta \\ \gamma_2 & \gamma_1 \end{bmatrix} \\ &= \begin{bmatrix} u & -t\theta \\ -t & u \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2\theta \\ \gamma_2 & \gamma_1 \end{bmatrix} r^q \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma. \end{aligned}$$

Hence, we see that the construction maps

$$\begin{bmatrix} w & r^q\gamma \\ r & w \end{bmatrix} \text{ to } \begin{bmatrix} w & r^q \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma \\ r & w \end{bmatrix}.$$

Since the latter spread does not have $GF(q^2)$ as kernel, the second spread cannot be isomorphic to the first.

We may now generalize Kantor-Knuth spreads as follows:

2 Theorem. *Let the Kantor-Knuth spread of odd order q^4 and kernel $GF(q^2)$ be given by*

$$x = 0, y = x \begin{bmatrix} w & r^q\gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^2),$$

where γ is a non-square in $GF(q^2)$.

(1) Then using the 'companion semifield spread' construction, the following defines a semifield spread of order q^4 and kernel $GF(q)$ (which is the dual of the Kantor-Knuth semifield plane by the main result of [3]).

$$x = 0, y = x \begin{bmatrix} w & r^q \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma \\ r & w \end{bmatrix}$$

(2) Let σ be an automorphism of $GF(q^k)$. Let $y = xM$ be a k-dimensional subspace of a $2k$-dimensional vector space on which there is a Desarguesian spread Σ

$$x = 0, y = xm;\ m \in GF(q^k)/\{0\}.$$

Assume further that $y = xM$ is contained in the partial spread of non-zero squares $S = \{y = xm^2;\ m \in GF(q^k)/\{0\}\}$ and is not a component of Σ. Then the following gives a spread

$$x = 0, y = x \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^k).$$

(3) *If σ is not q or 1,then the kernel of this spread is $GF(q)$,the right nucleus is $GF(q) \cap Fix\sigma$, and the middle nucleus is $Fix\sigma$.*

(4) *This spread is the dual of the corresponding Kantor-Knuth spread if and only if σ is q, or 1.*

Note that for the Kantor-Knuth spread above, the kernel is $GF(q^k)$,the right nucleus is $Fix\sigma$, and the middle nucleus is $Fix\sigma$.

(5) *If σ is not q or 1 then the spread*

$$x = 0, y = x \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^2).$$

is not isomorphic to either the Kantor-Knuth spread, the dual of the Kantor-Knuth spread or to the transpose of the Kantor-Knuth spread.

Proof. We note that we have the subspread

$$x = 0, y = x \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}; \forall w \in GF(q^2).$$

Assume that the kernel of the new spread is isomorphic to $GF(q^2)$. Let $Diag(A, A, A, A)$ be an element of the kernel. The kernel leaves each component invariant, which implies that $AwA^{-1} = w$ and then it follows that A is in the original field F isomorphic to $GF(q^2)$. But, then it follows that $r^\sigma M\gamma$ must commute with F. However, since r^σ and γ are elements of F, it follows that M must commute with F, a contradiction. Hence, the kernel is the subfield of F isomorphic to $GF(q)$.

In order that this spread is the dual of the corresponding Kantor-Knuth spread, it must be that there is a collineation group with elements $(x, y) \to (x, yM)$, where M belongs to a field isomorphic to $GF(q^2)$. It is essentially immediate that $M = \begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix}$ for all $v \in GF(q^2)$. However, an easy calculation shows that this implies that

$$MrM^{-1} = r^\sigma.$$

This implies that σ is either q or q^2. Now since

$$\begin{bmatrix} u & t\theta \\ t & u \end{bmatrix}^q = \begin{bmatrix} u & -t\theta \\ -t & u \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & t\theta \\ t & u \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

it follows that when $\sigma = q$, we obtain this structure is the companion spread to the Kantor-Knuth spread and is therefore the dual Kantor-Knuth spread by the main result of [3]):

When σ is not q or 1, clearly the kernel is then $GF(q)$. Consider,

$$\begin{bmatrix} v & 0 \\ 0 & v \end{bmatrix} \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix} = \begin{bmatrix} vw & vr^\sigma M\gamma \\ vr & vw \end{bmatrix},$$

which clearly implies that $v^\sigma = v$. So the middle nucleus is $Fix\sigma$.

Then

$$\begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix} \begin{bmatrix} [c]ccv & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} wv & r^\sigma M\gamma v \\ rv & wv \end{bmatrix},$$

implies

$$r^\sigma M\gamma v = (rv)^\sigma M\gamma,$$

which implies that

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} v = v^\sigma \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

which implies that the right nucleus is $GF(q) \cap Fix\sigma$.

Part (3) follows easily since there are no $GF(q^k)$'s in the right, middle, or right nuclei. QED

Now consider the spread

$$x = 0, y = x \begin{bmatrix} w & r^\sigma \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^2)$$

and note that, of course, we have a derivable net

$$D_{(w,w)} : x = 0, y = x \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}; \forall w \in GF(q^2)$$

that feels like a regulus net, except that projectively the spread is in $PG(7, q)$ and not in any $PG(3, q^2)$. which is generated by the subkernel group, sub-middle nucleus homology group and the right nucleus homology groups.

Moreover, change bases by $(x,y) \to (x, y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$ to represent the spread in the form:

$$x = 0, y = x \begin{bmatrix} r^{\sigma} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma & u \\ u & r \end{bmatrix} ; \forall r \in GF(q^2)$$

now change bases by $(x,y) \to (x, y \begin{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma^{-1} & 0 \\ 0 & 1 \end{bmatrix})$ to finally represent the spread in the form:

$$x = 0, y = x \begin{bmatrix} r^{\sigma} & u \\ u \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma^{-1} & r \end{bmatrix} ; \forall u, r \in GF(q^2).$$

Consider the matrix $\begin{bmatrix} e & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e \end{bmatrix}$, which maps

$$x = 0, y = x \begin{bmatrix} r^{\sigma} & u \\ u \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma^{-1} & r \end{bmatrix}$$

onto

$$x = 0, y = x \begin{bmatrix} e^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r^{\sigma} & u \\ u \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma^{-1} & r \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix} = \begin{bmatrix} e^{-1} r^{\sigma} & u \\ u \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \gamma^{-1} & er \end{bmatrix}.$$

Now choose $e^{\sigma} = e^{-1}$, if possible. For example, if $q = p^r$ and $\sigma = p^c$, for c properly dividing r, we obtain $e^{p^c} = e^{-1}$ if and only if $e^{p^c+1} = 1$. Therefore, in this setting, we have a left nucleus $GF(q)$, middle nucleus= $GF(p^c)$ =right nucleus and we have a Baer group of order p^c+1. Note that since the left nucleus contains the right/middle nucleus, we see that we have another Baer group of order p^c+1, namely with elements $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Now take the generated group

$$\left\langle \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & f \end{bmatrix} ; f, e \text{ of orders dividing } p^c + 1 \right\rangle.$$

Now in the special case when $q = p^{ce}$, where e is even, there is a subkernel group of order $p^{2c} - 1$. Multiplication of this kernel will produce a double-Baer group of order $p^c + 1$. All of this may be generalized as follows.

3 Theorem. *Representing the spread as*

$$x = 0, y = x \begin{bmatrix} r^\sigma M\gamma & u \\ u & r \end{bmatrix}; \forall r \in GF(q^2),$$

and $\sigma : x \to x^{p^c}$, *for* $q = p^{ce}$, *and* $e > 1$, *we have a double-Baer group of order* $p^e + 1$.

Then we see that have another derivable net

$$D_{(r^\sigma, r)} : x = 0, y = x \begin{bmatrix} r^\sigma & 0 \\ 0 & r \end{bmatrix}; \forall u, r \in GF(q^2).$$

Now consider that we derive either of the derivable nets mentioned. We are now deriving a semifield plane of order q^4. It follows by Johnson [8], that the full collineation of any of these derived spreads is the inherited group.

If we derive $D_{(w,w)}$, we note by Johnson [6], that since the net is a regulus net, the Baer subplanes are $GF(q^k)$-subspaces. Hence, when we derive this spread, the kernel is still $GF(q)$. The right and middle nuclei associated homology groups leave invariant this derivable net, so they are inherited as collineation groups isomorphic to the multiplicative subgroups of $GF(q) \cap Fix\sigma$ and $Fix\sigma$, respectively.

When we derive the $D_{(r^\sigma, r)}$ derivable net, we note that the Baer subplanes are $Fix\sigma$-subspaces, by Johnson [6]. Hence, the kernel of the derived plane now becomes $GF(q) \cap Fix\sigma$, since the remaining components are $GF(q)$-subspaces. Therefore, we have proved the following about the derived spreads.

4 Theorem. *Assume that* σ *is not* q *or* 1. *In the spread*

$$x = 0, y = x \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^k),$$

there are two derivable nets $D_{(w,w)}$ *and, after a basis change,* $D_{(r^\sigma, r)}$.

(1) Derivation of $D_{(w,w)}$ *produces a translation plane with kernel* $GF(q)$ *that admits affine Baer groups isomorphic to the multiplicative subgroups of* $GF(q) \cap Fix\sigma$ *and* $Fix\sigma$, *respectively.*

(2) Derivation of $D_{(r^\sigma, r)}$, *representing the spread as*

$$x = 0, y = x \begin{bmatrix} r^\sigma & u \\ uM\gamma^{-1} & r \end{bmatrix}; \forall u, r \in GF(q^k).$$

produces a translation plane with kernel $GF(q) \cap Fix\sigma$, and also admits Baer groups isomorphic to the multiplicative subgroups of $GF(q) \cap Fix\sigma$ and $Fix\sigma$, respectively.

If $\sigma : x \to x^{p^c}$, for $p^{ce} = q$, and $e > 1$, we admit symmetric affine homology groups of orders $p^c + 1$.

5 Definition. We call any of the spreads

$$x = 0, y = x \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix}; \forall w, r \in GF(q^k),$$

'generalized Kantor-Knuth spreads'.

Of course, the question is, are there any new semifield spreads that may be constructed in this way. Letting Σ be the associated Desarguesian affine plane of order q^k, then we ask what are the various subspaces $y = xM$ that lie within the net of non-zero squares? Of course, if $y = xM$ is $y = x^{q^i}z$, where z is a square does have this property. For this set of subspaces, it is not difficult to verify that these generalized Kantor-Knuth spreads are the Knuth generalized Dickson semifields (see e.g. Handbook of Finite Translation Planes [2]). In general, any such $y = xM$ has the general form $\sum_{i=1}^{kr} f_i x^{p^i}$, where $q^k = p^{rk}$, for p a prime and $f_i \in GF(q^k)$. Then the following defines the corresponding semifield spread:

$$\begin{aligned}(x, z) \circ (r, w) = (x, z) \begin{bmatrix} w & r^\sigma M\gamma \\ r & w \end{bmatrix} &= (xw + zr, xr^\sigma M\gamma + zw) \\ &= (xw + zr, \sum_{i=1}^{kr} f_i (xr^\sigma)^{p^i} + zw).\end{aligned}$$

So if $y = xM = x^{q^i}z$, we see the semifield is a Knuth generalized Dickson semifield.

6 Problem. Show that there exist subspaces $y = xM$ within the subspread of non-zero squares of a Desarguesian affine plane of order q^k that are not of the form $y = x^{q^i}z$.

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