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On maps preserving isosceles orthogonality in normed linear spaces

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Abstract. We show that a linear map from a normed linear space X to another normed linear space Y preserves isosceles orthogonality if and only if it is a scalar multiple of a linear isometry.

Keywords: Birkhoff orthogonality, inner product space, isometry, isosceles orthogonality, normed linear space, Singer orthogonality

MSC 2000 classification: primary 46B04, secondary 46B20, 52A21

1 Introduction

The study of distance-preserving maps in normed linear spaces is based on the Mazur-Ulam theorem (see [12, p. 76]), and regarding such isometries also maps preserving orthogonality types are interesting. In the present paper a result in this direction is proved.

Let X be a real normed linear space with *origin* o, *norm* $\|\cdot\|$, and *unit sphere* S_X. We say that $x \in X$ is *Birkhoff orthogonal* to $y \in X$, denoted by $x \perp_B y$, if $\|x + ty\| \geq \|x\|$ holds for any number $t \in \mathbb{R}$; x is said to be *isosceles orthogonal* to y if $\|x + y\| = \|x - y\|$ holds, and in this case we write $x \perp_I y$. Clearly, Birkhoff orthogonality is homogeneous and not symmetric, while isosceles orthogonality is symmetric but not homogeneous, which shows (besides further properties) that these two types of orthogonalities are different in general normed linear spaces. We refer to [2], [6], and [5] for basic properties of Birkhoff orthogonality and isosceles orthogonality.

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One can easily verify that in inner product spaces both types of orthogonality yield usual orthogonality. More precisely, they coincide if and only if the underlying space is an inner product space; cf. [1, (4.1), (4.4)]. Therefore they can be considered as natural extensions of usual (Euclidean) orthogonality to normed linear spaces. It is common to ask what properties of Euclidean orthogonality can be extended to normed linear spaces. For example, one can check whether the following result on inner product spaces (cf. [4, Theorem 1]) can be extended to normed linear spaces, in view of generalized orthogonality types:

> An orthogonality preserving linear map between two inner product spaces is necessarily a scalar multiple of a linear isometry, where a map T is said to be orthogonality preserving if the property that x is orthogonal to y implies that $T(x)$ is orthogonal to $T(y)$.

In [3] this result has been extended to (real or complex) normed linear spaces for the case of Birkhoff orthogonality, namely by the following

1 Theorem. *Let X and Y be two normed linear spaces. A linear map $T : X \mapsto Y$ preserves Birkhoff orthogonality if and only if it is a scalar multiple of a linear isometry.*

A special case of Theorem 1, namely when $X = Y$ and X is real, was obtained in [7].

The aim of the present paper is to prove a similar result for isosceles orthogonality. We only consider the case when X is real and non-trivial, i.e., the dimension of X is at least 2. Our result is based on Theorem 1 and the following lemmas concerning the geometric structure of the *bisector* $B(p,q)$ of the segment between two points p and q, which is defined by

$$B(p,q) := \{z \in X : \ \|z-p\| = \|z-q\|\}.$$

The geometry of bisectors in normed linear spaces is widely discussed in Section 4 of the survey [8]. For any distinct points p and q, we denote by $[p,q]$, $\langle p,q\rangle$, and $[p,q\rangle$ the *segment* between p and q, the *line* passing through p and q, and the *ray* starting from p and passing through q, respectively. For any two-dimensional subspace X_0 of X and $x \in S_{X_0}$ (with S_{X_0} as *the unit circle* of X_0) we denote by H_x^+ and H_x^- the two open half-planes of X_0 bounded by the line $\langle -x, x\rangle$; by $l(x)$ and $r(x)$ we denote the two points such that $[r(x), l(x)]$ is a maximal segment (in the sense that it is not properly contained in any other segment on S_{X_0}) parallel to $\langle -x, x\rangle$ on $S_{X_0} \cap H_x^+$, and that $r(x) - l(x)$ is a positive multiple of x. When there is no non-trivial segment on S_{X_0} parallel to $\langle -x, x\rangle$, the points $l(x)$ and $r(x)$ are chosen so that $r(x) = l(x) \in S_{X_0} \cap H_x^+$, and $l(x) \perp_B x$.

2 Lemma. *[cf. [9, Corollary 16]] For any $x \in X_0\backslash\{o\}$, every line contained in X_0 parallel to $\langle -x, x\rangle$ intersects $B(-x,x)\cap X_0$ in exactly one point if and only if S_{X_0} does not contain a non-trivial segment parallel to $\langle -x, x\rangle$.*

3 Lemma. *[cf. [8, Proposition 22]] For any $x \in S_{X_0}$, $B(-x,x)\cap X_0$ is fully contained in the bent strip contained in X_0 and bounded by the rays $[x, x+r(x)\rangle$, $[x, x-l(x)\rangle$, $[-x, -x+l(x)\rangle$, and $[-x, -x-r(x)\rangle$.*

2 Results and proofs

4 Lemma. *Let X and Y be two real normed linear spaces. If a linear map $T : X \mapsto Y$ preserves isosceles orthogonality, then it also preserves Birkhoff orthogonality.*

Proof. Let x and y be two points such that $x \perp_B y$. We show that $T(x) \perp_B T(y)$. The case that one of the points x and y is the origin is trivial, and since Birkhoff orthogonality is homogeneous, we can assume, without loss of generality, that $x, y \in S_X$. Then it is clear that x and y are linearly independent. Let X_0 be the two-dimensional subspace of X spanned by x and y. We consider the following three cases:

Case I: There exists a non-trivial segment $[a, b]$ parallel to $\langle -y, y\rangle$ and contained in S_{X_0} such that x is a relative interior point of $[a, b]$.

Let $\alpha = \min\{\|x-a\|, \|x-b\|\}$. Then $\alpha > 0$ and, for any $t \in (0, \alpha)$,

$$\|x+ty\| = \|x-ty\| = 1,$$

which means that $x \perp_I ty$ holds for any $t \in (0, \alpha)$. It follows from our assumption on T that

$$T(x) \perp_I tT(y) \quad \forall t \in (0, \alpha).$$

From the convexity of the function $f(t) = \|T(x) + tT(y)\|$ (which is implied by the convexity of the norm) it follows that, for any $t \in (0, \alpha)$, there exists a number $t_0 \in [-t, t]$ such that $f(t)$ attains its minimum at t_0. Hence $f(t)$ attains its minimum at 0, which implies that $T(x) \perp_B T(y)$.

Case II: There exists a non-trivial maximal segment $[a, b]$ parallel to $\langle -y, y\rangle$ and contained in S_{X_0} such that $x = a$.

Let $\{x_n\} \subset [a, b]\backslash\{a, b\}$ be a sequence such that $\lim_{n\to\infty} x_n = x$. It is clear that, for each n, $x_n \perp_B y$. Then it follows from what we have proved in Case I that $T(x_n) \perp_B T(y)$. Since T is a continuous map from X_0 to $T(X_0)$, we have $T(x) = \lim_{n\to\infty} T(x_n)$ and therefore $T(x) \perp_B T(y)$.

Case III: There is no non-trivial segment contained in S_{X_0} and parallel to $\langle -y, y\rangle$. Then, by Lemma 3, $B(-y, y) \cap X_0$ is bounded between the lines

$\langle y, y+x\rangle$ and $\langle -y, -y+x\rangle$. On the other hand, Lemma 2 implies that for any integer $n > 0$ there exists a unique number $\lambda_n \in [0,1]$ such that $\lambda_n(y+nx) + (1-\lambda_n)(-y+nx) \in B(-y, y)$, which implies that

$$(2\lambda_n - 1)T(y) + nT(x) \perp_I T(y).$$

Thus

$$\left\| T(x) + \frac{2\lambda_n}{n} T(y) \right\| = \left\| T(x) - \frac{2(1-\lambda_n)}{n} T(y) \right\| .$$

From the convexity of the function $g(t) = \|T(x) + tT(y)\|$ it follows that, for any integer $n > 0$, $g(t)$ attains its minimum at some number $t_0 \in [-\frac{2(1-\lambda_n)}{n}, \frac{2\lambda_n}{n}]$, which implies that $T(x) \perp_B T(y)$. The proof is complete. QED

5 Theorem. *Let X and Y be two real normed linear spaces. A linear map $T : X \mapsto Y$ preserves isosceles orthogonality if and only if T is a scalar multiple of a linear isometry.*

Proof. If T is linear and preserves isosceles orthogonality, Lemma 4 implies that T preserves Birkhoff orthogonality, and therefore T is a scalar multiple of a linear isometry.

Conversely, if T is a scalar multiple of a linear isometry, then there exists a number $t > 0$ such that tT is a linear isometry. For any $x, y \in X$ with $x \perp_I y$ we have

$$t\,\|T(x) - T(y)\| = \|tT(x) - tT(y)\| = \|x - y\|$$

and

$$t\,\|T(x) + T(y)\| = \|tT(x) + tT(y)\| = \|x + y\| .$$

Hence $T(x) \perp_I T(y)$. QED

6 Remark. In Theorem 5, isosceles orthogonality cannot be replaced by Singer orthogonality (cf. [11]), which is defined as follows: a point $x \in X$ is said to be *Singer orthogonal* to $y \in X$ (we write $x \perp_S y$) if either $\|x\| \cdot \|y\| = 0$ or

$$\left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\| = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| .$$

Take, e.g., X and Y as the two normed planes with norm $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively; this means that for any point $x = (x_1, x_2) \in \mathbb{R}^2$

$$\|x\|_1 := |x_1| + |x_2| \quad \text{and} \quad \|x\|_\infty := \max\{|x_1|, |x_2|\}$$

hold. Let I be the identity map from X to Y that maps each point in $\mathbb{R}^2$ onto itself. It is clear that I is linear, and it can be easily verified that I preserves Singer orthogonality. However, I is apparently not a scalar multiple of a linear isometry from X to Y.

7 Remark. We note that Theorem 5 can also be proved by using another approach which was used to obtain Theorem 3 of [10] (the two-dimensional case is not covered in this paper). This approach can be summarized as follows: first one shows that a map preserving isosceles orthogonality also preserves isosceles triangles, which will in turn imply that this map preserves "equality of distance". Then, by the main result of [13], the map has to be a scalar multiple of a linear isometry.

It is clear that, in contrast to this, our approach is based on the recent result obtained in [3] and the relation between Birkhoff orthogonality and isosceles orthogonality.

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