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Ideals of polynomials generated by weakly compact operators

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Abstract. The aim of these notes is to describe the properties and the relationships between the classes of homogeneous polynomials between Banach spaces which are generated by the ideal of all weakly compact operators, namely, the class of weakly compact polynomials and the class of polynomials which the factorization and the linearization methods generate by the ideal of all weakly compact operators. Containment relationships with ideals of absolutely summing polynomials are also investigated.

Keywords: ideals of polynomials, weakly compact operators, weakly compact polynomials.

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Dedicated to the memory of Klaus Floret (1941-2002)

Introduction

The background of the theory of operator ideals can be found in the works of Grothendieck [57], Lindenstrauss-Pełczyński [64] and Pietsch [83], and the core of the theory is the study of classes of linear operators which enjoy some special property, for example properties related to the improvement of convergence of series and properties related to types of compactness. Nowadays operator ideals belong to the mainstream of Banach Space Theory and the books by Defant-Floret [40], Diestel-Jarchow-Tonge [42] and Pietsch [84] are excellent references.

Special classes of multilinear mappings and homogeneous polynomials between Banach spaces have been a permanent topic of investigation, see for example Alencar [1], Aron-Schottenloher [14], Gupta [58], Nachbin [74], Pełczyński [75] and Ryan [88]. But it was only after the 1983 paper by Pietsch [85] that special classes of multilinear mappings and homogeneous polynomials started to be studied under the inspiration of the ideas and techniques derived from the theory of operator ideals. The 1989 paper by Alencar-Matos [5] is another cornerstone in this line of thought. This approach turned out to be very successful and a number of operator ideals have been fruitfully generalized to the

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multilinear and polynomial settings in recent years by several authors (a list of references is omitted because it would grow very large). It must be clear that there is not a unique natural way to generalize a given operator ideal to polynomials and multilinear mappings, a fact that is already present in Pietsch [85]. A remarkable example is the ideal of absolutely summing operators, which has already been generalized in several different ways, each of them enjoying some of the properties of the original ideal (see section 5).

The case of weakly compact operators is quite simpler for two reasons: (i) there is a natural definition of weakly compact polynomials/multilinear mappings, (ii) when applied to the ideal of weakly compact operators, the two methods outlined by Pietsch in [85] generate one single class of polynomials/multilinear mappings (see Theorem 14). So there are only two classes to be studied and compared, and the purpose of this paper is to provide examples, counterexamples and to describe the relationships between these two classes. These multilinear and polynomial generalizations of the ideal of weakly compact operators have been extensively studied, see for example [8,11,12,20,31,55, 60,63,75,87–89,96]. Quite often these two classes have been studied separately, and this paper can be regarded as an attempt to shorten this gap.

For the sake of simplicity we will center attention on the classes of homogeneous polynomials. Multilinear versions of some of the results that are stated for polynomials are omitted. The results presented in sections 1, 2 and 3 can be easily found in the literature, so the proofs are omitted. The most important examples and results appear in sections 4 and 5. Among the results and examples that appear along the notes, some are folklore, some are not-so-well-known and some appear here for the first time.

Throughout these notes n is a positive integer, E, F, G, H, $E_1, \ldots, E_n$, $G_1, \ldots, G_n$ will stand for (real or complex) Banach spaces and B_E denotes the closed unit ball of E.

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1 Operator ideals and weakly compact operators

Weakly compact operators are used in this section to illustrate the abstract notion of operator ideals, an idea that can be traced back to the seminal works of Grothendieck in the 50's, was formally introduced by Pietsch in the 60's and has become an important branch of Banach space theory (see [40,42,84] and references therein).

1 Definition. An *operator ideal* $\mathcal{I}$ is a subclass of the class $\mathcal{L}$ of all continuous linear operators between Banach spaces such that for all Banach spaces E and F its components $\mathcal{I}(E,F) = \mathcal{L}(E,F) \cap \mathcal{I}$ satisfy:
(i) $\mathcal{I}(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$ which contains the finite rank operators.
(ii) The ideal property: if $u \in \mathcal{L}(E,F)$, $v \in \mathcal{I}(F,G)$ and $t \in \mathcal{L}(G,H)$, then the composition tvu is in $\mathcal{I}(E,H)$.

2 Example. According to the standard terminology and to the author's preferences, some of the most outstanding operator ideals are: finite rank operators, compact operators, approximable operators, weakly compact operators, completely continuous operators, absolutely summing operators, nuclear operators, Hilbert-Schmidt operators and integral operators. Next the ideal of weakly compact operators is studied in detail.

Weakly compact operators

A continuous linear operator $u : E \to F$ is said to be *weakly compact*, in symbols $u \in \mathcal{W}(E,F)$, if u maps B_E onto a relatively weakly compact subset of F. The following (elementary-functional-analysis) proposition provides a number of examples.

3 Proposition. *If either E or F is a reflexive Banach space, then every operator $u : E \to F$ is weakly compact.*

The standard example of an operator which fails to be weakly compact is the identity operator on a non-reflexive Banach space. The most important properties enjoyed by weakly compact operators are summarized in the following proposition. The proof can be found, e.g., in Conway [37, Theorems V.13.1, VI.5.4, VI.5.5]. The equivalence (a) $\Leftrightarrow$ (e) below, which is a celebrated result due to Davis-Figiel-Johnson-Pełczyński [39], shows that Proposition 3 is nothing but a clue of the connection between weakly compact operators and reflexive spaces.

4 Proposition. *Let $u : E \to F$ be a continuous linear operator. The following are equivalent:*

(a) u is weakly compact.

(b) For every bounded sequence (x_j) in E, $(u(x_j))$ has a weakly convergent subsequence.

(c) The adjoint operator $u^* : F' \to E'$ is weakly compact.

(d) $u^{**}(E'') \subseteq F$.

(e) *There is a reflexive Banach space G and operators $v : E \to G$ and $t : G \to F$ such that $u = tv$.*

As an operator ideal, $\mathcal{W}$ enjoys the following properties.

5 Proposition.

(a) *$\mathcal{W}$ is a closed operator ideal with respect to the operator norm, that is, when endowed with the usual operator norm, $\mathcal{W}(E;F)$ is a closed subspace of $\mathcal{L}(E;F)$.*

(b) *$\mathcal{W}$ is an injective operator ideal, that is, for every metric injection $I : F \to G$ the following holds: if $T \in \mathcal{L}(E;F)$ and $IT \in \mathcal{W}(E;G)$ then $T \in \mathcal{W}(E;F)$.*

(c) *$\mathcal{W}$ is a surjective operator ideal, that is, for every metric surjection $Q : G \to E$ ($\|Q(x)\| = \inf\{\|y\| : Q(x) = Q(y)\}$) the following holds: if $T \in \mathcal{L}(E;F)$ and $TQ \in \mathcal{W}(E;G)$ then $T \in \mathcal{W}(E;F)$.*

Proof. For (a) see Dunford-Schwartz [46] or Woytaszczyk [97, Theorem II.C.6]. For (b) and (c) see Heinrich [60, Theorem 2.3]. $\boxed{QED}$

6 Remark. A *normed* (resp. *p-normed*) *operator ideal* $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is an operator ideal $\mathcal{I}$ together with a function $\|\cdot\|_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}^+$ such that
(i) $\|\cdot\|_{\mathcal{I}}$ restricted to $\mathcal{I}(E;F)$ is a norm (resp. p-norm) for all Banach spaces E and F.
(ii) $\|id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.
(iii) If $u \in \mathcal{L}(E,F)$, $v \in \mathcal{I}(F,G)$ and $t \in \mathcal{L}(G,H)$, then $\|tvu\|_{\mathcal{I}} \le \|t\|\|v\|_{\mathcal{I}}\|u\|$.

If the components $\mathcal{I}(E;F)$ are complete with respect to $\|\cdot\|_{\mathcal{I}}$ we say that $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is Banach (resp. p-Banach) operator ideal. The notions of closed, injective and surjective operator ideals are defined in the same spirit of Proposition 5. All operator ideals considered along these notes are supposed to be normed or p-normed.

2 Multilinear mappings and homogeneous polynomials

Given $n \ge 2$, the space of all continuous n-linear mappings $A : E_1 \times \cdots \times E_n \to F$ will be denoted by $\mathcal{L}(E_1, \ldots, E_n; F)$. It becomes a Banach space with the natural norm

$$\|A\| = \sup\{\|A(x_1, \ldots, x_n)\| : \|x_j\| \le 1,\ j = 1, \ldots, n\}.$$

If $E_1 = \cdots = E_n = E$ we write $\mathcal{L}(^nE;F)$. Given a continuous n-linear mapping $A \in \mathcal{L}(^nE;F)$, the map

$$P : E \to F : P(x) = A(x, x, \ldots, x) \quad \text{for every } x \in E,$$

is called a *continuous n-homogeneous polynomial.* The space of all continuous n-homogeneous polynomials from E to F will be denoted by $\mathcal{P}(^nE;F)$, and it becomes a Banach space under the norm

$$\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\} = \inf\{C : \|P(x)\| \leq C \cdot \|x\|^n, \forall x \in E\}.$$

If F is the scalar field we shall use the simplified notations $\mathcal{L}(E_1, \ldots, E_n)$, $\mathcal{L}(^nE)$ and $\mathcal{P}(^nE)$. For the sake of simplicity, we will henceforth write 'n- homogeneous polynomial' instead of 'continuous n-homogeneous polynomial'.

The fact that P is the polynomial generated by A, that is $P(x) = A(x, x, \ldots, x)$, will be denoted by $\hat{A} = P$. Given a polynomial $P \in \mathcal{P}(^nE;F)$, there is a unique symmetric continuous n-linear mapping $\check{P} \in \mathcal{L}(^nE;F)$ such that $P(x) = \check{P}(x, x, \ldots, x)$. $\check{P}$ can be obtained from P by the polarization formula

$$\check{P}(x_1, \ldots, x_n) = \frac{1}{n!\,2^n} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n).$$

The correspondence $P \leftrightarrow \check{P}$ establishes an isomorphism between $\mathcal{P}(^nE;F)$ and the space $\mathcal{L}_s(^nE;F)$ of all symmetric continuous n-linear mappings from E to F. Actually it is true that

$$\|P\| \leq \|\check{P}\| \leq \frac{n^n}{n!}\|P\|.$$

The notation $\overset{[i]}{\ldots}$ means that the i-th coordinate is not involved. For $i = 1, \ldots, n$, the operator $I_i : \mathcal{L}(E_1, \ldots, E_n; F) \to \mathcal{L}(E_i; \mathcal{L}(E_1, \overset{[i]}{\ldots} E_n; F))$ given by

$$I_i(A)(x_i)(x_1, \overset{[i]}{\ldots}, x_n) = A(x_1, \ldots, x_n),$$

defines an isometric isomorphism which will be quite useful later. Of course, if A is symmetric, then $I_1(A) = I_2(A) = \cdots = I_n(A)$. In this case we write $I(A)$ instead of $I_i(A)$. For the case $n = 1$ to make sense, we let $\mathcal{L}(^0E;F)$ be the Banach space of constants maps from E to F, or, equivalently, define $\mathcal{L}(^0E;F) = F$ (in this case I is the identity operator on $\mathcal{L}(E;F)$).

Approximable and weakly sequentially continuous polynomials

The subspace of $\mathcal{L}(^nE;F)$ generated by the mappings $A(x_1, \ldots, x_n) = \varphi_1(x_1) \cdots \varphi_n(x_n)\, b$, where $\varphi_j \in E'$ and $b \in F$, is denoted by $\mathcal{L}_f(^nE;F)$ and these are the *n-linear mappings of finite type*. The subspace of $\mathcal{P}(^nE;F)$ generated by the mappings $P(x) = \varphi(x)^n b$, where $\varphi \in E'$ and $b \in F$, is denoted by $\mathcal{P}_f(^nE;F)$ and these are the *polynomials of finite type*. We denote the closure of $\mathcal{P}_f(^nE;F)$ in $\mathcal{P}(^nE;F)$ by $\mathcal{P}_A(^nE;F)$, and these are the *approximable polynomials*.

A polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *weakly sequentially continuous*, in symbols $P \in \mathcal{P}_{wsc}(^nE;F)$, if P sends weakly convergent sequences onto norm convergent sequences. Is is easy to see that $\mathcal{P}_A(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F)$.

7 Theorem. *(Aron-Schottenloher [14, Proposition 5.3]) Let $m, n \in \mathbb{N}$, $m \leq n$. Then $\mathcal{P}(^mE)$ is isomorphic to a complemented subspace of $\mathcal{P}(^nE)$.*

Examples are postponed to sections 4 and 5. For the general theory of multilinear mappings and homogeneous polynomials we refer to Dineen [45] and Mujica [71].

3 Ideals of polynomials and multilinear mappings

The notion of ideals of multilinear mappings (multi-ideals) goes back to Pietsch [85]. Further developments can be found in Braunss [26–28], Braunss-Junek [29,30], Floret-García [47], Floret-Hunfeld [48], Geiss [53] and Junek [61].

8 Definition. (a) An *ideal of multilinear mappings* $\mathcal{M}$ is a subclass of the class of all continuous multilinear mappings between Banach spaces such that for all $n \in \mathbb{N}$ and Banach spaces $E_1, \ldots, E_n$ and F, the components $\mathcal{M}(E_1, \ldots, E_n, F) = \mathcal{L}(E_1, \ldots, E_n, F) \cap \mathcal{M}$ satisfy:
(i) $\mathcal{M}(E_1, \ldots, E_n, F)$ is a linear subspace of $\mathcal{L}(E_1, \ldots, E_n, F)$ which contains the n-linear mappings of finite type.
(ii) The ideal property: if $A \in \mathcal{M}(E_1, \ldots, E_n, F)$, $u_j \in \mathcal{L}(G_j, E_j)$ for $j = 1, \ldots, n$ and $t \in \mathcal{L}(F, H)$, then $tA(u_1, \ldots, u_n)$ is in $\mathcal{M}(G_1, \ldots, G_n, H)$.
If $\| \cdot \|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}^+$ satisfies
(i') $\| \cdot \|_{\mathcal{M}}$ restricted to $\mathcal{M}(E_1, \ldots, E_n; F)$ is a norm (resp. quasi-norm) for all Banach spaces $E_1, \ldots, E_n$ and F and all n,
(ii') $\|A : \mathbb{K}^n \to \mathbb{K} : A(x_1, \ldots, x_n) = x_1 \cdots x_n\|_{\mathcal{M}} = 1$ for all n,
(iii') If $A \in \mathcal{M}(E_1, \ldots, E_n, F)$, $u_j \in \mathcal{L}(G_j, E_j)$ for $j = 1, \ldots, n$ and $t \in \mathcal{L}(F, H)$, then $\|tA(u_1, \ldots, u_n)\|_{\mathcal{M}} \leq \|t\| \|A\|_{\mathcal{M}} \|u_1\| \cdots \|u_n\|$,
then $(\mathcal{M}, \| \cdot \|_{\mathcal{M}})$ is called a *normed* (resp. *quasi-normed*) *ideal of multilinear mappings*.
(b) An *ideal of homogeneous polynomials*, or simply an *ideal of polynomials*, $\wp$ is a subclass of the class of all continuous homogeneous polynomials between Banach spaces such that for all $n \in \mathbb{N}$ and Banach spaces E and F, the components $\wp(^nE, F) = \mathcal{P}(^nE, F) \cap \wp$ satisfy:

(i) $\wp(^nE, F)$ is a linear subspace of $\mathcal{P}(^nE, F)$ which contains the n-homogeneous polynomials of finite type.
(ii) The ideal property: if $u \in \mathcal{L}(G, E)$, $P \in \wp(^nE, F)$ and $t \in \mathcal{L}(F, H)$, then the composition tPu is in $\wp(^nG, H)$.
If $\|\cdot\|_\wp : \wp \to \mathbb{R}^+$ satisfies
(i') $\|\cdot\|_\wp$ restricted to $\wp(^nE; F)$ is a norm (resp. quasi-norm) for all Banach spaces E and F and all n,
(ii') $\|P : \mathbb{K} \to \mathbb{K} : P(x) = x^n\|_\wp = 1$ for all n,
(iii') If $u \in \mathcal{L}(G, E)$, $P \in \wp(^nE, F)$, $t \in \mathcal{L}(F, H)$, then $\|tPu\|_\wp \le \|t\|\|P\|_\wp\|u\|^n$,
then $(\wp, \|\cdot\|_\wp)$ is called a *normed* (resp. *quasi-normed*) *ideal of polynomials.*

Given an ideal of multilinear mappings $\mathcal{M}$, it is easy to check that the classes $\{\hat{A} : A \in \mathcal{M}\}$ and $\{P : \check{P} \in \mathcal{M}\}$ are ideals of polynomials. Moreover, Floret-García [47] proved that for every quasi-normed ideal of polynomials $\wp$ there exists a quasi-normed ideal of multilinear mappings $\mathcal{M}$ such that $P \in \wp$ if and only if $\check{P} \in \mathcal{M}$.

There are different ways to construct an ideal of multilinear mappings and/or an ideal of polynomials from a given operator ideal $\mathcal{I}$: (i) The property enjoyed by the operators in $\mathcal{I}$ can be generalized to the multilinear and polynomial cases and the resulting classes of multilinear mappings and homogeneous polynomials happen to be ideals (that is the case of the ideal of weakly compact operators, as we shall see in section 5). The point is that, depending on the operator ideal, it may happen there is not a unique natural generalization of $\mathcal{I}$ to the multilinear and polynomials settings (that is what happens with the ideal of absolutely summing operators, as we shall see in section 5). (ii) The two methods outlined by Pietsch [85] which are described next.

The factorization method

Given an operator ideal $\mathcal{I}$, an n-linear mapping $A \in \mathcal{L}(E_1, \ldots, E_n; F)$ is said to be of *type* $\mathcal{L}(\mathcal{I})$, in symbols $A \in \mathcal{L}(\mathcal{I})(E_1, \ldots, E_n; F)$, if there are Banach spaces $G_1, \ldots, G_n$, linear operators $u_j \in \mathcal{I}(E_j; G_j)$, $j = 1, \ldots, n$, and a continuous n-linear mapping $B \in \mathcal{L}(G_1, \ldots, G_n; F)$ such that $A = B(u_1, \ldots, u_n)$.

If $\mathcal{I}$ is a normed operator ideal and $A \in \mathcal{L}(\mathcal{I})(E_1, \ldots, E_n; F)$ we define

$$\|A\|_{\mathcal{L}(\mathcal{I})} = \inf \|B\|\|u_1\|_{\mathcal{I}} \cdots \|u_n\|_{\mathcal{I}},$$

where the infimum is taken over all possible factorizations $A = B(u_1, \ldots, u_n)$ with u_j belonging to $\mathcal{I}$.

There are three natural ways to define an ideal of polynomials by the factorization method, namely, considering the polynomials P such that: (i) $\check{P}$ is of type $\mathcal{L}(\mathcal{I})$; (ii) $P = \hat{A}$ for some n-linear mapping A of type $\mathcal{L}(\mathcal{I})$; (iii) $P = Qu$ where u belongs to $\mathcal{I}$. Next we see that these conditions are equivalent.

9 Proposition. *Let $\mathcal{I}$ be an operator ideal and $P \in \mathcal{P}(^nE;F)$. The following are equivalent:*

(a) There exists an n-linear mapping $A \in \mathcal{L}(\mathcal{I})(^nE;F)$ such that $\hat{A} = P$.

(b) There is a Banach space G, a linear operator $u \in \mathcal{I}(E;G)$ and a polynomial $Q \in \mathcal{P}(^nG;F)$ such that $P = Qu$.

(c) $\check{P} \in \mathcal{L}(\mathcal{I})(^nE;F)$.

Proof. (a) $\Rightarrow$ (b) If $\hat{A} = P$ and $A = B(u_1, \ldots, u_n)$, where $u_j \in \mathcal{I}(E;G_j)$, $j = 1, \ldots, n$, and $B \in \mathcal{L}(G_1, \ldots, G_n;F)$, let $G = G_1 \times \cdots \times G_n$, $u(x) = (u_1(x), \ldots, u_n(x))$ for all $x \in E$, and $Q((y_1, \ldots, y_n)) = B(y_1, \ldots, y_n)$. In order to see that $u \in \mathcal{I}(E;G)$, let $i_j : G_j \to G$ be the canonical mapping and note that $u = \sum_{j=1}^n i_j u_j$ (see González-Gutiérrez [54, Proposition 13] and Braunss-Junek [30, Lemma 3.1]).
(b) $\Rightarrow$ (c) Since $P = Qu$, with the help of the polarization formula it is easy to see that $\check{P} = \check{Q}(u, \ldots, u)$.
(c) $\Rightarrow$ (a) Obvious. $\boxed{QED}$

A continuous n-homogeneous polynomial $P \in \mathcal{P}(^nE;F)$ is said to be of *type* $\mathcal{P}_{\mathcal{L}(\mathcal{I})}$, in symbols $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$, if P satisfies the equivalent conditions of Proposition 9. If $\mathcal{I}$ is a normed operator ideal and $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$ we define

$$\|P\|_{\mathcal{L}(\mathcal{I})} = \inf\{\|A\|_{\mathcal{L}(\mathcal{I})} : \hat{A} = P\}.$$

The linearization method

Given an operator ideal $\mathcal{I}$, an n-linear mapping $A \in \mathcal{L}(E_1, \ldots, E_n;F)$ is said to be *of type* $[\mathcal{I}]$, in symbols $A \in [\mathcal{I}](E_1, \ldots, E_n;F)$, if $I_i(A) \in \mathcal{I}(E_i;\mathcal{L}(E_1, \overset{[i]}{\ldots}, E_n;F))$, for every $i = 1, \ldots, n$. Of course, if $A \in \mathcal{L}(^nE;F)$ is symmetric, $A \in [\mathcal{I}](^nE;F)$ if $I(A) \in \mathcal{I}(E;\mathcal{L}(^{n-1}E;F))$.

If $\mathcal{I}$ is a normed operator ideal and $A \in [\mathcal{I}](E_1, \ldots, E_n;F)$ we define

$$\|A\|_{[\mathcal{I}]} = \max\{\|I_1(A)\|_{\mathcal{I}}, \cdots, \|I_n(A)\|_{\mathcal{I}}\}.$$

A continuous n-homogeneous polynomial $P \in \mathcal{P}(^nE;F)$ is said to be of *type* $\mathcal{P}_{[\mathcal{I}]}$, in symbols $P \in \mathcal{P}_{[\mathcal{I}]}(^nE;F)$, if $\check{P} \in [\mathcal{I}](^nE;F)$, that is, if $I(\check{P}) \in \mathcal{I}(E;\mathcal{L}(^{n-1}E;F))$.

If $\mathcal{I}$ is a normed operator ideal and $P \in \mathcal{P}_{[\mathcal{I}]}(^nE;F)$ we define

$$\|P\|_{[\mathcal{I}]} = \|\check{P}\|_{[\mathcal{I}]} = \|I(\check{P})\|_{\mathcal{I}}.$$

10 Proposition. *Let $\mathcal{I}$ be an operator ideal and $P \in \mathcal{P}(^nE;F)$. Then P is of type $\mathcal{P}_{[\mathcal{I}]}$ if and only if the linear operator*

$$\bar{P} : E \to \mathcal{P}(^{n-1}E;F) \ : \ \bar{P}(x)(y) = \check{P}(x,y,\dots,y)$$

belongs to $\mathcal{I}$.

Proof. If $J : \mathcal{L}(^{n-1}E;F) \to \mathcal{P}(^{n-1}E;F)$ and $c : \mathcal{P}(^{n-1}E;F) \to \mathcal{L}(^{n-1}E;F)$ are the linear operators given by $J(A) = \hat{A}$ and $c(Q) = \check{Q}$, it is easy to see that $\bar{P} = JI(\check{P})$ and $I(\check{P}) = c\bar{P}$. Now the result is a consequence of the ideal property. $\boxed{QED}$

11 Proposition. *Let $\mathcal{I}$ be an operator ideal.*

(a) $\mathcal{L}(\mathcal{I})$ and $[\mathcal{I}]$ are ideals of multilinear mappings; $\mathcal{P}_{\mathcal{L}(\mathcal{I})}$ and $\mathcal{P}_{[\mathcal{I}]}$ are ideals of polynomials.

(b) $\mathcal{P}_{\mathcal{L}(\mathcal{I})} \subseteq \mathcal{P}_{[\mathcal{I}]}$.

(c) If $\mathcal{I}$ is a closed operator ideal, then, for every $n \in \mathbb{N}$ and Banach spaces E and F, the components $\mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$ and $\mathcal{P}_{[\mathcal{I}]}(^nE;F)$ are closed subspaces of $\mathcal{P}(^nE;F)$.

(d) If $\mathcal{I}$ is a closed injective operator ideal, then $\mathcal{P}_{\mathcal{L}(\mathcal{I})} = \mathcal{P}_{[\mathcal{I}]}$.

(e) If $\mathcal{I}$ is a normed operator ideal, then $(\mathcal{L}(\mathcal{I}), \|\cdot\|_{\mathcal{L}(\mathcal{I})})$ (resp. $(\mathcal{P}_{\mathcal{L}(\mathcal{I})}, \|\cdot\|_{\mathcal{L}(\mathcal{I})})$ is a quasi-normed ideal of multilinear mappings (resp. quasi-normed ideal of polynomials) and $([\mathcal{I}], \|\cdot\|_{[\mathcal{I}]})$ (resp. $(\mathcal{P}_{[\mathcal{I}]}, \|\cdot\|_{[\mathcal{I}]})$ is a normed ideal of multilinear mappings (resp. normed ideal of polynomials).

Proof. (a) The fact that $\mathcal{L}(\mathcal{I})$ and $[\mathcal{I}]$ are ideals of multilinear mappings can be found in Braunss [27, Section 1.3] and Geiss [53, Satz 1.1 and Satz 1.2]. Now, the polynomial case follows easily from the correspondence $P \leftrightarrow \check{P}$. For the ideal property, one just have to note that $(tPu)^\vee = t\check{P}(u,\dots,u)$.
(b) The following argument is an adaptation of Geiss [53, Satz 1.3]: given $P \in \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$, from Proposition 4.3 we know that there is a Banach space G, a linear operator $u \in \mathcal{I}(E;G)$ and a polynomial $Q \in \mathcal{P}(^nG;F)$ such that $\check{P} = \check{Q}(u,\dots,u)$. Consider the linear operator

$$\varphi : G \to \mathcal{L}(^{n-1}E;F) \ : \ \varphi(z)(x_2,\dots,x_n) = \check{Q}(z,u(x_2),\dots,u(x_n)).$$

For every $x_1,\dots,x_n \in E$, it follows that

$$\begin{aligned}(\varphi u)(x_1)(x_2,\dots,x_n) &= \varphi(u(x_1))(x_2,\dots,x_n) = \check{Q}(u(x_1),u(x_2),\dots,u(x_n))\\ &= \check{Q}(u,\dots,u)(x_1,\dots,x_n) = \check{P}(x_1,\dots,x_n)\\ &= I(\check{P})(x_1)(x_2,\dots,x_n),\end{aligned}$$

what proves that $I(\check{P}) = \varphi u$. Since $u \in \mathcal{I}(E;G)$, it follows from the ideal property that $I(\check{P}) \in \mathcal{I}(E;\mathcal{L}(^{n-1}E;F))$, proving that P is of type $\mathcal{P}_{[\mathcal{I}]}$.

(c) From Braunss [27, Proposition 3.5] we have that $\mathcal{L}(\mathcal{I})(^nE;F)$ and $[\mathcal{I}](^nE;F)$ are closed in $\mathcal{L}(^nE;F)$. Again, the polynomial case follows from the isomorphism $P \leftrightarrow \check{P}$.

(d) See González-Gutiérrez [55].

(e) The case of $\mathcal{L}(\mathcal{I})$ and $\mathcal{P}_{\mathcal{L}(\mathcal{I})}$ can be found in Braunss-Junek [30], where it is proved that, for every n, $\|\cdot\|_{\mathcal{L}(\mathcal{I})}$ makes $\mathcal{L}(\mathcal{I})(E_1,\ldots,E_n;F)$ and $\mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$ complete $1/n$-normed spaces (in general, if $\mathcal{I}$ is a p-normed operator ideal, then $\mathcal{L}(\mathcal{I})(E_1,\ldots,E_n;F)$ and $\mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$ are complete p/n-normed spaces). The case of $[\mathcal{I}]$ and $\mathcal{P}_{[\mathcal{I}]}$ is trivial. $\boxed{QED}$

12 Remark. Particular cases of Proposition 11(d) were obtained before the final result due to González-Gutiérrez: the scalar-valued case for closed, injective and surjective ideals was proved by Geiss [52], another proof of the scalar-valued case for the ideal of weakly compact operators is due to Aron-Galindo [12, Theorem 1], and the case of compact operators was treated by Braunss [26] and González-Gutiérrez [54].

4 Ideals generated by weakly compact operators

Weakly compact polynomials

An n-linear mapping $A \in \mathcal{L}(E_1,\ldots,E_n;F)$ is said to be *weakly compact*, in symbols $A \in \mathcal{L}_{\mathcal{W}}(E_1,\ldots,E_n;F)$, if A maps $B_{E_1} \times \cdots \times B_{E_n}$ onto a relatively weakly compact subset of F. An n-homogeneous polynomial $P \in \mathcal{P}(^nE;F)$ is said to be *weakly compact*, in symbols $P \in \mathcal{P}_{\mathcal{W}}(^nE;F)$, if P maps B_E onto a relatively weakly compact subset of F. As we have already mentioned in the introduction, such classes have been extensively investigated.

Aron-Schottenloher defined in [14] the adjoint of a continuous n-homogeneous polynomial $P \in \mathcal{P}(^nE;F)$ to be the following linear operator

$$P^* : F' \to \mathcal{P}(^nE) \;:\; P^*(\varphi)(x) = \varphi(P(x)).$$

13 Proposition. *Let $P \in \mathcal{P}(^nE;F)$. The following are equivalent:*

(a) P is a weakly compact polynomial.

(b) For every bounded sequence (x_j) in E, $(P(x_j))$ has a weakly convergent subsequence.

(c) $\check{P}$ is a weakly compact n-linear mapping.

(d) P^* is a weakly compact operator.

(e) $P^{**}(\mathcal{P}(^nE)') \subseteq F$.

PROOF. (a) $\Leftrightarrow$ (b) This is the Eberlein-Smulian Theorem (see [37, Theorem V.13.1]).
(a) $\Rightarrow$ (c) See Pełczyński [75, Proposition 3].
(c) $\Rightarrow$ (a) This is trivial, because $P(B_E) \subseteq \check{P}(B_{E^n})$.
(a) $\Leftrightarrow$ (d) See Ryan [89, Proposition 2.1].
(a) $\Leftrightarrow$ (e) See Ryan [88, Proposition 4.3]. $\boxed{QED}$

Instead of three classes, namely, $\mathcal{P}_{\mathcal{W}}, \mathcal{P}_{\mathcal{L}(\mathcal{W})}$ and $\mathcal{P}_{[\mathcal{W}]}$, there are only two classes of polynomials generated by weakly compact operators to be studied. Next result is a straightforward combination of Proposition 5 and Proposition 11(d), but in our context it is interesting enough to be stated separately.

14 Theorem. $\mathcal{P}_{\mathcal{L}(\mathcal{W})} = \mathcal{P}_{[\mathcal{W}]}$.

The class $\mathcal{P}_{\mathcal{L}(\mathcal{W})} = \mathcal{P}_{[\mathcal{W}]}$ has been studied also in the context of Arens-regularity (see [6,8,11,12,45,51,62,63,96] and Remark 23(e)) and connections with the investigation of integral multilinear mappings can be found in [17]. Since a choice has to be made, we shall henceforth write $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ to denote both itself and $\mathcal{P}_{[\mathcal{W}]}$.

15 Proposition.

(a) $\mathcal{P}_{\mathcal{W}}$ *and* $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ *are ideals of polynomials.*

(b) The components $\mathcal{P}_{\mathcal{W}}(^nE;F)$ *and* $\mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F)$ *are closed subspaces of* $\mathcal{P}(^nE;F)$.

PROOF. (a) The case of $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ follows from Proposition 11(a). The case of $\mathcal{P}_{\mathcal{W}}$ is an easy exercise.
(b) The case of $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ follows from Proposition 5(a) and Proposition 11(c). The case of $\mathcal{P}_{\mathcal{W}}$ is due to Ryan [89, Corollary 2.2]. $\boxed{QED}$

16 Example. From Proposition 15(a) it follows that all polynomials of finite type belong to $\mathcal{P}_{\mathcal{W}}$ and to $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. From Proposition 15(b) it follows that all approximable polynomials belong to both of the classes, too. Soon we will see that there are polynomials belonging to both $\mathcal{P}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ which fail to be approximable.

Next proposition shows that weakly compact polynomials and polynomials of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ are easy to be produced. The proof is straightforward.

17 Proposition.

(a) A Banach space E *is reflexive if and only if, regardless of the Banach space* F *and* $n \in \mathbb{N}$, *every* n*-homogeneous polynomial from* E *to* F *is of type* $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

(b) A Banach space F is reflexive if and only if, regardless of the Banach space E and $n \in \mathbb{N}$, every n-homogeneous polynomial from E to F is weakly compact.

18 Remark. Let us see that, contrary to the case of linear operators, the implications

$$E \text{ is reflexive} \Rightarrow \mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{W}}(^nE;F) \text{ for every Banach space } F;$$
$$F \text{ is reflexive} \Rightarrow \mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F) \text{ for every Banach space } E;$$

are not always true. Given $n \geq 2$, consider the n-homogeneous polynomial

$$P_n : \ell_2 \to \ell_1 \ : \ P_n\Big((\alpha_i)_{i=1}^{\infty}\Big) = \Big((\alpha_i)^n\Big)_{i=1}^{\infty}, \ P_n \in \mathcal{P}(^n\ell_2;\ell_1).$$

ℓ_2 is reflexive, but P_n fails to be weakly compact because the sequence $(P_n(e_j))$ does not have a weakly convergent subsequence ((e_j) is the canonical basis of ℓ_p - see [20, Example 1]). On the other hand, later we will provide scalar-valued polynomials which fail to be of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

In Example 16 we saw that $\mathcal{P}_A \subseteq (\mathcal{P}_{\mathcal{W}} \cap \mathcal{P}_{\mathcal{L}(\mathcal{W})})$. The two examples described next show that the reverse inclusion does not hold.

19 Example (The scalar-valued case). Let $Q \in \mathcal{P}(^2\ell_2)$ be defined by $Q((\alpha_i)_{i=1}^{\infty}) = \sum_{i=1}^{\infty}(\alpha_i)^2$. On the one hand, Q is not weakly sequentially continuous, because (e_j) is a weakly null sequence and $Q(e_j) = 1$ for all j. Hence Q is not approximable (see also Tocha [94, Exemplos 2.5.13]). On the other hand, it is obvious that $Q \in (\mathcal{P}_{\mathcal{W}}(^2\ell_2) \cap \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2\ell_2))$.

20 Example (The vector-valued case). Let $n \geq 2$ and $p, q > 1$ be real numbers such that $nq = p$. ¿From Alencar [2, p.7] we have that the polynomial $P \in \mathcal{P}(^n\ell_p;\ell_q)$, which is defined in the same fashion of the polynomials introduced in Remark 18, fails to be approximable. Since ℓ_p and ℓ_q are reflexive, $P \in (\mathcal{P}_{\mathcal{W}}(^n\ell_p;\ell_q) \cap \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^n\ell_p;\ell_q))$.

¿From Theorem 7 we have that if $\mathcal{P}(^nE)$ is reflexive for some n, then E is reflexive, too. It is well known that the converse is not true, but it is interesting to note that the polynomials considered in Remark 18 and Example 20 provide two short proofs of the failure of the converse even for Hilbert spaces:

21 Corollary. *$\mathcal{P}(^n\ell_2)$ is not reflexive for every $n \geq 2$.*

Proof. First proof: Let $P_n : \ell_2 \to \ell_1$ be the polynomial defined in Remark 18. Since P_n is not weakly compact, from Proposition 13 we have that the linear operator $P_n^* : \ell_\infty \to \mathcal{P}(^n\ell_2)$ is not weakly compact. The result follows from Proposition 3.

Second proof: Theorem 7 shows that it is enough to prove the case $n = 2$. For having a Schauder basis, ℓ_2 has the approximation property [65, 1.e.1]. From Aron-Hervés-Valdivia [13, Corollary 2.11] we know that approximable homogeneous polynomials on ℓ_2 are weakly continuous on bounded sets. Since the 2-homogeneous polynomial on ℓ_2 defined in Example 20 is not approximable, it follows that there are 2-homogeneous polynomials on ℓ_2 which are not weakly continuous on bounded sets. The non-reflexivity of $\mathcal{P}(^2\ell_2)$ is now a consequence of Ryan [88, Proposition 5.3]. $\boxed{QED}$

Scalar-valued polynomials.

It is obvious that scalar-valued polynomials are weakly compact and that scalar-valued polynomials on reflexive spaces are of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. So, all that is left to be studied in the scalar-valued case is the class $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ on non-reflexive spaces. In this direction some coincidence results should be mentioned. For the basic theory of C^*-algebras see [37, Chapter VIII]. J denotes the classical James space, which is a non-reflexive Banach space isometric to its second dual (see [65, Example 1.d.2.]), and J' denotes its dual. T_J^* denotes the James space modelled on the original Tsirelson's space (see Aron-Dineen [10] or Dineen [45, Example 2.43]). The non-reflexivity of T_J^* can be found in [10, Proposition 13]. $\mathcal{A}$ denotes the disc algebra, that is, the Banach space of all functions analytic on the unit disc and continuous on the boundary with the sup norm (see [97, Chapter III.E]). $c_0(\{\ell_1^k\}_k) = \{(x_k)_k : x_k \in \ell_1^k \text{ and } \lim_{k\to\infty} \|x_k\| = 0\}$, with the sup norm. The definition of the Hardy space H^∞ can be found in [97, Section I.B.19]. For the definition of a Banach space of type 2 see [97, III.A.17].

22 Proposition.

(a) If E is either c_0 or T_J^*, then $\mathcal{P}(^nE) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE)$ for every n.

(b) If E is either a C^*-algebra or a space of type 2 or any of the following spaces: J, J', $\mathcal{A}$, $c_0(\{\ell_1^k\}_k)$, H^∞; then $\mathcal{P}(^2E) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2E)$.

Proof. (a) Let $\mathcal{P}_w(^nE)$ denote the space of all n-homogeneous polynomials on E which are weakly continuous on bounded sets. If E is either c_0 or T_J^* and $n \in \mathbb{N}$, we have that $\mathcal{P}(^nE) = \mathcal{P}_w(^nE)$ (for the case $E = c_0$ this is a result due to Pełczyński [75] - see also [45, Proposition 1.59] - for the case $E = T_J^*$ the result is due to Aron-Dineen [10, Proposition 14]). From Aron-Hervés-Valdivia [13, Theorem 2.9] it follows that, for every $P \in \mathcal{P}(^nE)$, $I(\check{P})$ is compact.
(b) Let E be a C^*-algebra. Given $P \in \mathcal{P}(^2E)$, since $I(\check{P})$ is a E'-valued linear operator defined on E, from a result due to Haagerup [59, p. 95] - see also [40, Corollary 19.6]) - it follows that $I(\check{P})$ factors through a Hilbert space. Therefore,

the weak compactness of $I(\check{P})$ implies that $P \in \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2E)$. If E has type 2, then E' has cotype 2 (see [97, Proposition III.A.22]). From a result due to Maurey (see [76, p. 251]) it follows - again - that $I(\check{P})$ factors through a Hilbert space. If $E = H^\infty$, from a result due to Bourgain [24, Corollary 3.10], it follows that $I(\check{P})$ is an absolutely 2-summing operator, hence weakly compact (see [42, Theorem 2.17]). The cases $E = J$ and $E = J'$ are due to Leung [63]. The case $E = c_0(\{\ell_1^k\}_k)$ is due to P. Harmand (see [11, Remark 1.4(d)] and [45, Ex. 6.55]). The case $E = \mathcal{A}$, which is due to Ülger [96, Theorem 2.9], is a consequence of another result due to Bourgain [23, p. 946] which asserts that every continuous bilinear form on $\mathcal{A}$ can be extended to a continuous bilinear form on the C^*-algebra $C(T)$, where T is the unit circle. $\boxed{QED}$

23 Remark. (a) From the proof of Proposition 22(a), it is obvious that the result can be rephrased as follows: if E is a Banach space in which all homogeneous polynomials are weakly continuous on bounded sets, then $\mathcal{P}(^nE) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE)$ for every n.

(b) Something more can be said about polynomials on c_0 and T_J^*: their duals ${c_0}' = \ell_1$ and $(T_J^*)'$ have the approximation property, because they have Schauder basis (see [45, p.1019]). Hence, an appeal to Aron-Hervés-Valdivia [13, Corollary 2.11] shows that $\mathcal{P}_A(^nE) = \mathcal{P}_w(^nE)$ for $E = c_0$ or $E = T_J^*$. From the proof of Proposition 22(a) it follows that $\mathcal{P}(^nc_0) = \mathcal{P}_A(^nc_0)$ and $\mathcal{P}(^nT_J^*) = \mathcal{P}_A(^nT_J^*)$ for every n.

(c) The coincidence $\mathcal{P}(^nc_0) = \mathcal{P}_A(^nc_0)$ is known as the Littlewood-Bogdanowicz-Pełczyński Theorem (see [7, p.215] or [50, 3.4.1]).

(d) The case $E = C(K)$ with $n = 2$ is also a consequence of the following fact: every operator from a $C(K)$ space into a Banach space which does not have a subspace isomorphic to c_0 is weakly compact (see [65, p.57]).

(e) The coincidence $\mathcal{P}(^2E) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2E)$ is explored in several situations (see [8,11,12,45,51,62,63,96]). According to the standard terminology, a Banach space E is said to be *regular* (resp. *symmetrically regular*) if $\mathcal{L}(^2E) = \mathcal{L}(\mathcal{W})(^2E)$ (resp. $\mathcal{P}(^2E) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2E)$). Aron-Cole-Gamelin [8, Theorem 8.3] provides a number of conditions that are equivalent to the equality $\mathcal{P}(^2E) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2E)$.

(f) A word must be said about multilinear mappings: it is obvious that $\mathcal{L}(^nE; F) = \mathcal{L}(\mathcal{W})(^nE; F) \Rightarrow \mathcal{P}(^nE; F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE; F)$, but the converse is not true. As we have already seen, Leung [63] proved that $\mathcal{P}(^2J') = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2J')$, but in the same paper it is proved that $\mathcal{L}(^2J') \neq \mathcal{L}(\mathcal{W})(^2J')$.

Later we will see that the result on C^*-algebras in Proposition 22(b) cannot be generalized to $n > 2$ (see Example 29). Actually, it is time to give examples of polynomials which are not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

The Dunford-Pettis property

A Banach space E has the *Dunford-Pettis property* if for all Banach spaces F, every weakly compact operator from E to F sends weakly convergent sequences onto norm convergent sequences.

24 Example. Spaces with the Dunford-Pettis property: $C(K)$-spaces, $L_1(\mu)$-spaces, spaces with the Schur property (weakly convergent sequences are norm convergent), closed subspaces of c_0, the space $M(K)$ of finite regular Borel measures on a compact space K, the disc algebra $\mathcal{A}$ and the Hardy space H^∞. Infinite dimensional reflexive spaces lack the Dunford-Pettis property. For more information see Diestel [41].

25 Proposition. *Let E be a Banach space. The following are equivalent:*

(a) E has the Dunford-Pettis property.

(b) For all Banach spaces F and $n \in \mathbb{N}$, $\mathcal{P}_{\mathcal{W}}(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F)$.

(c) For all Banach spaces F and $n \in \mathbb{N}$, $\mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F) \subseteq \mathcal{P}_{wsc}(^nE;F)$.

PROOF. (a) $\Rightarrow$ (b) This result, which is usually referred to as the equivalence between the Dunford-Pettis property and the so-called *Polynomial Dunford-Pettis property*, is due to Ryan ([87, Corollary 2.2], [88, Corollary 4.1]).
(b) $\Rightarrow$ (a) and (c) $\Rightarrow$ (a) are obvious with $n = 1$.
(a) $\Rightarrow$ (c) Given $P \in \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F)$, we know from Proposition 9 that $P = Qu$, where $u \in \mathcal{W}(E;G)$ and $Q \in \mathcal{P}(^nG;F)$ for some Banach space G. If $x_j \xrightarrow{w} x$ in E, since E has the Dunford-Pettis property, $u(x_j) \to u(x)$ in G. The continuity of Q yields $P(x_j) = Q(u(x_j)) \to Q(u(x)) = P(x)$. $\boxed{QED}$

Now we are in the position to give the first examples of polynomials which are not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

26 Example. Given $n \in \mathbb{N}$, let

$$Q_n : c_0 \to c_0 \ : \ Q_n\Big((\alpha_i)_{i=1}^\infty\Big) = \Big((\alpha_i)^n\Big)_{i=1}^\infty \ , \ Q_n \in \mathcal{P}(^nc_0;c_0).$$

Since $e_j \xrightarrow{w} 0$ and $(Q_n(e_j)) = (e_j)$ is not norm convergent, we have that $Q_n \notin \mathcal{P}_{wsc}(^nc_0;c_0)$. By Proposition 25 it follows that $Q_n \notin \mathcal{P}_{\mathcal{L}(W)}(^nc_0;c_0)$, because c_0 has the Dunford-Pettis property. Observe that, for the same reason, Q_n is not weakly compact.

If $\mathcal{K}$ denotes the ideal of compact linear operators and $\mathcal{P}_{\mathcal{K}}$ denotes the ideal of all compact homogeneous polynomials, it is easy to check that $\mathcal{P}_{\mathcal{L}(\mathcal{K})} \subseteq \mathcal{P}_{\mathcal{K}}$ but $\mathcal{P}_{\mathcal{K}}$ is not contained in $\mathcal{P}_{\mathcal{L}(\mathcal{K})}$. Our next purpose is to show that none of the inclusions $\mathcal{P}_{\mathcal{L}(\mathcal{W})} \subseteq \mathcal{P}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{W}} \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{W})}$ hold true.

27 Example ($\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ is not contained in $\mathcal{P}_{\mathcal{W}}$). This is the easy part. Given $n \geq 2$, let $P_n : \ell_1 \to \ell_1$ be the n-homogeneous polynomial defined by the same rule of the polynomials in Remark 18 and Example 26. The factorization

$$\ell_1 \xrightarrow{i} \ell_2 \xrightarrow{Q_n} \ell_1 \ : \ P_n = Q_n i,$$

where i is the formal inclusion and Q_n has the same definition of P_n, shows that $P_n \in \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^n\ell_1; \ell_1)$ because ℓ_2 is reflexive and $Q_n \in \mathcal{P}(^n\ell_2; \ell_1)$. The same argument used in Remark 18 shows that P_n is not weakly compact.

$\mathcal{P}_{\mathcal{W}}$ is not contained in $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$

First of all, we need examples of polynomials which are not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. The polynomials of Example 26 are of no help, because they are not weakly compact. Of course, a scalar-valued polynomial which is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ solves the problem, but such polynomials are not easy to be found. Three examples of scalar-valued polynomials which are not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ are described next.

28 Example (Aron-Cole-Gamelin [8]). Let B be the symmetric bilinear form on ℓ_1 defined by: for $x = (a_j)$ and $y = (b_j)$ in ℓ_1,

$$B(x, y) = \sum \{a_j b_k : j \text{ even}, 1 \leq k < j\} + \sum \{a_j b_k : k \text{ even}, 1 \leq j < k\}.$$

Aron-Cole-Gamelin proved in [8, p.83] that the linear operator associated to B, $I(B) : \ell_1 \to \ell_\infty$, is not weakly compact. Therefore $\hat{B}$ is a 2-homogeneous scalar-valued polynomial on ℓ_1, hence weakly compact, which is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. In Theorem 32 we will see the details of the argument used in [8].

29 Example (González-Gutiérrez [55]). Let u be a surjective operator from ℓ_∞ to ℓ_2 such that $B_{\ell_2} \subseteq u(B_{\ell_\infty})$. The existence of such an operator is a consequence of the following three facts: (i) every separable Banach space is isometric to a subspace of ℓ_∞ ([15, Theorem IV.II.2]); (ii) if the Banach space E contains an isomorphic copy of ℓ_1, then ℓ_2 is isomorphic to a quotient of E ([9]); (iii) the open mapping theorem [97, I.A.5] (see also [65, Remark 2.f.2]). If $u(x) = (u(x)_i)_{i=1}^\infty \in \ell_2$ for all $x \in \ell_\infty$, define

$$P(x) = \sum_{i=1}^{\infty} x_i (u(x)_i)^2, \text{ for } x = (x_i)_{i=1}^\infty \ , \ P \in \mathcal{P}(^3\ell_\infty).$$

González-Gutiérrez proved in [55, p.1729] that the operator $\bar{P} : \ell_\infty \to \mathcal{P}(^2\ell_\infty)$ is not weakly compact. From Proposition 10 and Theorem 14 it follows that P is a 3-homogeneous scalar-valued polynomial on ℓ_∞ which is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. This example shows that the result concerning C^*-algebras in Proposition 22(b) cannot be generalized to $n > 2$.

30 Example (Rennison [86]). Let $u : \ell_1 \to \ell_\infty$ be the linear operator defined by $u((\alpha_i)_{i=1}^\infty) = (\beta_i)_{i=1}^\infty$ for $(\alpha_i)_{i=1}^\infty \in \ell_1$, where

$$\beta_i = (-1)^{i+1}\sum_{j=1}^{i}\alpha_j + \sum_{j=i+1}^{\infty}(-1)^{j+1}\alpha_j, \text{ for all } i \in \mathbb{N}.$$

Let $A = I^{-1}(u)$. Now we prove that A is symmetric: if $u((\alpha_i)_{i=1}^\infty) = (\beta_i)_{i=1}^\infty$ and $u((\gamma_i)_{i=1}^\infty) = (\delta_i)_{i=1}^\infty$, then

$$\begin{aligned}
A((\alpha_i),(\gamma_i)) = u((\alpha_i))((\gamma_i)) &= \sum_{i=1}^{\infty}\beta_i\gamma_i \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{i}(-1)^{i+1}\alpha_j\gamma_i + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(-1)^{j+1}\alpha_j\gamma_i \\
&= \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}(-1)^{j+1}\alpha_i\gamma_j + \sum_{i=2}^{\infty}\sum_{j=i}^{i-1}(-1)^{i+1}\alpha_i\gamma_j \\
&= \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(-1)^{j+1}\gamma_j\alpha_i + \sum_{i=1}^{\infty}(-1)^{i+1}\gamma_i\alpha_i + \sum_{i=2}^{\infty}\sum_{j=i}^{i-1}(-1)^{i+1}\gamma_j\alpha_i \\
&= \sum_{i=1}^{\infty}\sum_{j=i}^{i}(-1)^{i+1}\gamma_j\alpha_i + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty}(-1)^{j+1}\gamma_j\alpha_i \\
&= \sum_{i=1}^{\infty}\delta_i\alpha_i = u((\gamma_i))((\alpha_i)) = A((\gamma_i),(\alpha_i)).
\end{aligned}$$

Then $A \in \mathcal{L}_s(^2\ell_1)$. Let $\varepsilon_2(A)$ denote the Aron-Berner extension of A to ℓ_1'', that is

$$\varepsilon_2(A) : \ell_1'' \times \ell_1'' \to \mathbb{C} \ : \ \varepsilon_2(A)(x'', y'') = u^{**}(x'')(y'') \ , \ \varepsilon_2(A) \in \mathcal{L}(^2\ell_1'').$$

Rennison [86, Theorem 3.5] proved that $\varepsilon_2(A)$ is not symmetric, thus it follows from [45, Proposition 6.13] that u is not weakly compact. Letting $P = \hat{A}$ we have that $I(\check{P}) = I(A) = u$. Therefore P is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

31 Remark. A fourth example can be found in Aron *et al.* [11, Remark 3.5(b)], where the authors provide a 2-homogeneous polynomial on the completed projective tensor product $\ell_2\hat{\otimes}_\pi\ell_2$ which is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. It is interesting to notice that the existence of such polynomial can be proved by the following (trivial) fact: if $\mathcal{I}$ is an operator ideal, G is a complemented subspace of E and $\mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nE;F)$, then $\mathcal{P}(^nG;F) = \mathcal{P}_{\mathcal{L}(\mathcal{I})}(^nG;F)$. By

Ryan [90, Example 2.10] we know that $\ell_2\hat{\otimes}_\pi\ell_2$ contains a complemented isometric copy of ℓ_1 and by Example 28 we have that $\mathcal{P}(^2\ell_1) \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2\ell_1)$, therefore $\mathcal{P}(^2\ell_2\hat{\otimes}_\pi\ell_2) \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2\ell_2\hat{\otimes}_\pi\ell_2)$.

All examples of polynomials which are not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ we have shown until now are compact (Examples 28, 29 and 30) or fail to be weakly compact (Example 26). So, weakly compact polynomials which are neither compact nor of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ are missing. That is our next aim.

A vector-valued version of the Aron-Cole-Gamelin bilinear form

It is obvious from the definition of the Aron-Cole-Gamelin bilinear form (Example 28) that the range of a vector-valued version must lie in a Banach algebra. In this section E will stand for a complex commutative Banach algebra (the basic theory of Banach algebras can be found in [37, Chapter VII]).

By $\ell_1(E)$ we denote the Banach space of all absolutely summable sequences in E endowed with the natural norm $\|(x_j)_{j=1}^\infty\| = \sum_{j=1}^\infty \|x_j\|$.

Now define $A : \ell_1(E) \times \ell_1(E) \to E$ by: for $x = (x_j)_{j=1}^\infty$ and $y = (y_j)_{j=1}^\infty$ set $A(x, y) =$

$$\sum\{x_j y_k : j \text{ even}, 1 \leq k < j\} + \sum\{x_j y_k : k \text{ even}, 1 \leq j < k\}. \tag{1}$$

Using the well-known fact that absolutely summable sequences in Banach spaces are unconditionally summable (see, e.g., [42, Proposition 1.1]), it is not difficult to show that A is well defined. It is easy to see that A is bilinear, continuous and symmetric.

The full application of next theorem requires a complex commutative reflexive infinite dimensional Banach algebra with identity. Here is a simple way to construct such an algebra: let E be a complex commutative reflexive infinite dimensional Banach algebra (for example, $\ell_p, 1 < p < +\infty$). There is a standard technique to give the set $E_1 = E \times \mathbb{C}$ the structure of a complex commutative Banach algebra with identity such that $\dim E_1/E = 1$ (see [37, Proposition VII.1.3]). It is obvious that E_1 is infinite dimensional, and, for having a reflexive subspace of finite codimension, E_1 is reflexive.

32 Theorem. *Let E be a complex commutative Banach algebra with identity. If $A : \ell_1(E) \times \ell_1(E) \to E$ is the symmetric bilinear mapping defined by (1), then*

(a) A is compact if and only if E is of finite dimension.

(b) A is weakly compact if and only if E is reflexive.

(c) The linear operator associated to A, $I(A) : \ell_1(E) \to \mathcal{L}(\ell_1(E); E)$, is not weakly compact.

PROOF. (a) The 'if' part is obvious. For the 'only if' part, suppose that E is infinite dimensional. Then there exists a bounded sequence $(x_m)_{m=1}^{\infty}$ in E which has no norm convergent subsequence. For each $m \in \mathbb{N}$ let $\beta_m = (x_m, e, 0, 0, 0, \ldots) \in \ell_1(E)$, where e is an identity. Therefore, $((\beta_m, \beta_m))$ is a bounded sequence in $\ell_1(E) \times \ell_1(E)$ and $(A(\beta_m, \beta_m)) = (2x_m)$ has no norm convergent subsequence in E, proving that A is not compact.
(b) Assume that A is weakly compact. Given a bounded sequence (x_m) in E, choose β_m in $\ell_1(E)$ exactly as in the proof of (a). Since A is weakly compact, $A(\beta_m, \beta_m)) = (2x_m)$ has a weakly convergent subsequence. Then every bounded sequence in E has a weakly convergent subsequence, what proves that E is reflexive. The converse is obvious.
(c) For every $m \in \mathbb{N}$, let $\alpha_m = (0, 0, \ldots, 0, e, 0, \ldots) \in \ell_1(E)$, where e is placed at the m-th position. In order to see that the sequence $(I(A)(\alpha_{2m}))$ has no weakly convergent subsequence in $\mathcal{L}(\ell_1(E); E)$, we will use the fact that the latter space has a copy of ℓ_∞ via the identification

$$(a_j) \in \ell_\infty \hookrightarrow (a_j)((x_j)) = \sum_{j=1}^{\infty} a_j x_j \in E, \text{ for all } (x_j) \in \ell_1(E).$$

Given $m \in \mathbb{N}$, seeing $I(A)(\alpha_{2m})$ as an operator in $\mathcal{L}(\ell_1(E); E)$, for every $(y_j) \in \ell_1(E)$ it follows that

$$\begin{aligned} I(A)(\alpha_{2m})((y_j)) &= A(\alpha_{2m}, (y_j)) = A((0, 0, \ldots, 0, e, 0, \ldots), (y_1, y_2, \ldots)) = \\ &= y_1 + y_2 + \cdots + y_{2m-1} + y_{2m+2} + y_{2m+4} + y_{2m+6} + \cdots . \end{aligned}$$

Thus, $I(A)(\alpha_{2m})$ can be regarded as an element of ℓ_∞ as follows:

$$I(A)(\alpha_{2m}) = (a_j) \in \ell_\infty, \text{ where } a_j = \begin{cases} 1, & 1 \leq j < 2m, \\ 0, & j = 2m, \\ 1, & j \text{ even}, \ j > 2m, \\ 0, & j \text{ odd}, \ j > 2m. \end{cases}$$

Assume that $(I(A)(\alpha_{2m}))$ has a weakly convergent subsequence in ℓ_∞. Using the sequences of ℓ_1 as elements of $(\ell_\infty)' = \ell_1''$, it is easy to see that such subsequence would converge weakly to the vector $\varphi = (1, 1, 1, \ldots) \in \ell_\infty$. In this case, there would be a sequence (λ_k), where each λ_k is a convex combination of some of the elements $(I(A)(\alpha_{2m}))$, such that (λ_k) converges in norm to φ. But this is absurd, because $\|\lambda_k - \varphi\| = 1$ for every k. Therefore, we conclude that $(I(A)(\alpha_{2m}))$ has no weakly convergent subsequence in ℓ_∞. Now the fact that $(I(A)(\alpha_{2m}))$ has no weakly convergent subsequence in $\mathcal{L}(\ell_1(E); E)$ is a consequence of the Hahn-Banach theorem. The proof is complete. $\boxed{QED}$

33 Remark. Let E be a complex commutative infinite dimensional reflexive Banach algebra with identity and A be the bilinear mapping defined by (1). Setting $P = \hat{A}$, from Theorem 32 we have that $P \in \mathcal{P}({}^2\ell_1(E); E)$ is a weakly compact polynomial which is neither compact nor of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

Polynomial reflexivity

This section deals with complete coincidence situations, that is, occurrences of the kind $\mathcal{P}({}^nE; F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}({}^nE; F) = \mathcal{P}_{\mathcal{W}}({}^nE; F)$. Such situations are rare, and according to what we are about to see, as rare as the reflexivity of the space $\mathcal{P}({}^nE)$.

34 Proposition. *Let E be a Banach space and $n \in \mathbb{N}$.*

(a) The space $\mathcal{P}({}^nE)$ is reflexive if and only if, regardless of the Banach space F, every n-homogeneous polynomial from E to F is weakly compact.

(b) If the space $\mathcal{P}({}^nE)$ is reflexive, then, regardless of the Banach space F, every n-homogeneous polynomial from E to F is of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$.

Proof. (a) Ryan [89, Theorem 3.8]: assume that $\mathcal{P}({}^nE)$ is reflexive. Given $P \in \mathcal{P}({}^nE; F)$, since P^* is a linear operator from F' to $\mathcal{P}({}^nE)$, it follows from the reflexivity of $\mathcal{P}({}^nE)$ that P^* is weakly compact. Now the weak compactness of P follows from Proposition 13.

Conversely, if δ_n is the map defined by

$$\delta_n : E \to \mathcal{P}({}^nE)' \ : \ \delta_n(x)(P) = P(x),$$

it follows that $\delta_n \in \mathcal{P}({}^nE; \mathcal{P}({}^nE)')$. Thus δ_n is weakly compact by assumption. From Proposition 13 we have that ${\delta_n}^* : \mathcal{P}({}^nE)'' \to \mathcal{P}({}^nE)$ is weakly compact. But the restriction of ${\delta_n}^*$ to $\mathcal{P}({}^nE)$ coincides with the identity operator on $\mathcal{P}({}^nE)$, then the reflexivity of $\mathcal{P}({}^nE)$ follows from the weak compactness of its identity operator.
(b) Call on Theorem 7 to obtain the reflexivity of E and apply Proposition 17(a) to complete the proof. $\boxed{QED}$

Let T^* denote the original Tsirelson's space, that is, the reflexive space discovered by B. S. Tsirelson in 1973 [95] which contains no ℓ_p, $1 < p < \infty$ (Casazza-Shura [33] is an excellent reference on Tsirelson's space).

35 Corollary.

(a) A Banach space E is polynomially reflexive, that is, $\mathcal{P}({}^nE)$ is reflexive for every $n \in \mathbb{N}$, if and only if, regardless of the Banach space F and $n \in \mathbb{N}$, $\mathcal{P}({}^nE; F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}({}^nE; F) = \mathcal{P}_{\mathcal{W}}({}^nE; F)$.

(b) $\mathcal{P}({}^nT^*;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}({}^nT^*;F) = \mathcal{P}_{\mathcal{W}}({}^nT^*;F)$ *for every Banach space* F *and* $n \in \mathbb{N}$.

(c) If $p > n$ *then* $\mathcal{P}({}^n\ell_p;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}({}^n\ell_p;F) = \mathcal{P}_{\mathcal{W}}({}^n\ell_p;F)$ *for every Banach space* F.

PROOF. (a) It is an immediate consequence of Proposition 34.
(b) Alencar-Aron-Dineen [3] proved that T^* is polynomially reflexive.
(c) From Alencar-Floret [4, Proposition 4.1] or Dimant-Zalduendo [44, Example 4.1] we have that $\mathcal{P}({}^n\ell_p)$ is reflexive for $p > n$. $\boxed{QED}$

36 Remark. (a) Note that the converse of Proposition 34(b) is not true: for every Banach space F and $n \geq 2$, $\mathcal{P}({}^n\ell_2;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}({}^n\ell_2;F)$ (Proposition 17(a)), but $\mathcal{P}({}^n\ell_2)$ is not reflexive (Corollary 21).
(b) With respect to the reflexivity of $\mathcal{P}({}^nE)$, remember that if E is not reflexive, from Theorem 7 it follows that $\mathcal{P}({}^nE)$ is not reflexive, either. For reflexive spaces, the situation is as follows: (i) if E is reflexive and $\mathcal{P}({}^nE) = \mathcal{P}_A({}^nE)$ then $\mathcal{P}({}^nE)$ is reflexive; (ii) if E is a reflexive space with the approximation property, then $\mathcal{P}({}^nE)$ is reflexive if and only if $\mathcal{P}({}^nE) = \mathcal{P}_A({}^nE)$ (see Mujica [73, Theorem 5.5]).

Factorization through reflexive spaces

Considering the equivalence (a) $\Leftrightarrow$ (e) in Proposition 4, it is natural to wonder about polynomials that factor through reflexive spaces. But there are two possible factorizations, namely, $P = Qu$ and $P = uQ$, where u is a linear operator and Q is a polynomial. Next result shows that these two possibilities correspond exactly to the two classes of polynomials we are working with.

37 Proposition. *Let* $P \in \mathcal{P}({}^nE;F)$.

(a) $P \in \mathcal{P}_{\mathcal{L}(W)}({}^nE;F)$ *if and only if there is a reflexive space* G*, a linear operator* $u \in \mathcal{L}(E;G)$ *and a polynomial* $Q \in \mathcal{P}({}^nG;F)$ *such that* $P = Qu$.

(b) $P \in \mathcal{P}_{\mathcal{W}}({}^nE;F)$ *if and only if there is a reflexive space* G*, a linear operator* $u \in \mathcal{L}(G;F)$ *and a polynomial* $Q \in \mathcal{P}({}^nE;G)$ *such that* $P = uQ$.

PROOF. (a) This is a simple combination of Proposition 9 [(a) $\Leftrightarrow$ (b)] and Proposition 4 [(a) $\Leftrightarrow$ (e)].
(b) First proof. Let $P \in \mathcal{P}_{\mathcal{W}}({}^nE;F)$. From Proposition 13 we have that the linear operator $P^* : F' \to \mathcal{P}({}^nE)$ is weakly compact and that $P^{**}(\mathcal{P}({}^nE)') \subseteq F$. Then $P^{**} : \mathcal{P}({}^nE)' \to F$ is a weakly compact operator, and by Proposition 4 we know that there exists a reflexive space G, and linear operators $v \in \mathcal{L}(\mathcal{P}({}^nE)';G)$, $u \in \mathcal{L}(G;F)$ such that $P^{**} = uv$. Let $\delta_n : E \to \mathcal{P}({}^nE)'$ be the n-homogeneous

polynomial introduced in the proof of Proposition 34(a). Given $\varphi \in F'$ and $x \in E$,

$$P^{**}(\delta_n(x))(\varphi) = \delta_n(x)(P^*(\varphi)(x) = \varphi(P(x)),$$

what shows that $P^{**}\delta_n(x) = P(x)$ for every $x \in E$. Hence $P = P^{**}\delta_n = uv\delta_n$. Letting $Q = v\delta_n$, $P = uQ$ gives the desired factorization.
Second proof. Let $Q(^nE)$ be the dual of $\mathcal{P}(^nE)$ with respect to the compact-open topology. By Mujica [72, Theorem 2.4] we know that there exists a polynomial $q_n \in \mathcal{P}(^nE; Q(^nE))$ such that for every F and every $P \in \mathcal{P}(^nE; F)$, there exists an operator $T_P : Q(^nE) \to F$ such that $P = T_P q_n$. Moreover, if $P \in \mathcal{P}_{\mathcal{W}}(^nE; F)$, by Mujica [72, Proposition 3.4] it follows that T_P is weakly compact. Proposition 4[(a) $\Leftrightarrow$ (e)] completes the proof.

The converse follows from the fact that continuous linear operators are weak-to-weak continuous. $\boxed{QED}$

38 Remark. (a) Proposition 37(b) was proved by R. Ryan ([88, Proposition 3.5] or [89, Theorem 3.7]) in the more general context of holomorphic mappings. Instead of an adaptation of Ryan's result, we preferred to provide a direct proof of the polynomial case (which is, of course, quite simpler than the holomorphic case proved by Ryan).
(b) The linearization technique used in the second proof of Proposition 37(b) was originally formulated by Ryan (see [88, Lemma 4.1]) in the language of symmetric tensor products.
(c) The result [72, Theorem 2.4] we used in the second proof of Proposition 37(b) is stated in that reference without proof. A detailed proof can be found in Çaliskan [38, Teorema 1.1.5].
(d) For the problem concerning the factorization of a holomorphic mapping f in the form $f = ug$, where g is another holomorphic mapping and u belongs to a closed surjective operator ideal, see González-Gutiérrez [56].
(e) Composition operators: given $P \in \mathcal{P}(^nE; F)$ and a Banach space G, define the linear operator $u_P : \mathcal{L}(F; G) \to \mathcal{P}(^nE; G)$ by $u_P(v) = vP$. Boyd [25] proved that $P \in \mathcal{P}_{\mathcal{W}}(^nE; F)$ if and only if u_P is a weakly compact operator.

Duality between the classes $\mathcal{P}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{L}(W)}$

As mentioned in the introduction, each class of polynomials defined as a generalization of a given operator ideal enjoys some of the properties of the ideal. That is exactly what happens with the ideal of weakly compact operators and the classes $\mathcal{P}_{\mathcal{W}}$ and $\mathcal{P}_{\mathcal{L}(W)}$. For example, Proposition 3 is equivalent to the following two implications:

(i) If E is reflexive, then $\mathcal{L}(E; F) = \mathcal{W}(E; F)$ for every Banach space F.

(ii) If F is reflexive, then $\mathcal{L}(E;F) = \mathcal{W}(E;F)$ for every Banach space E.

In Proposition 17 we saw that $\mathcal{P}_{\mathcal{L}(W)}$ generalizes (i) while $\mathcal{P}_{\mathcal{W}}$ generalizes (ii). Another example of this duality between the classes is given by the factorization through reflexive spaces, as explained above. The fact that some of the properties enjoyed by weakly compact operators are found in either $\mathcal{P}_{\mathcal{L}(W)}$ or $\mathcal{P}_{\mathcal{W}}$ is enough to show that each class deserves to be studied on its own. Nevertheless, besides the equivalence (b) $\Leftrightarrow$ (c) in Proposition 25, some relationships between these two classes are to be expected.

Let $P \in \mathcal{P}(^nE;F)$. The *derivative* dP is the $(n-1)$-homogeneous polynomial defined by

$$dP : E \to \mathcal{L}(E;F)\,,\; dP(x)(y) = n\check{P}(y,x,\ldots,x)\,;\; dP \in \mathcal{P}(^{n-1}E;\mathcal{L}(E;F)).$$

39 Proposition.

(a) Let E be a Banach space such that, for some $n \in \mathbb{N}$, $\mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{W}}(^nE;F)$ for every Banach space F. Then $\mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F)$ for every $n \in \mathbb{N}$ and every Banach space F.

(b) Suppose that $P \in \mathcal{P}(^nE)$ and dP is a weakly compact $(n-1)$-homogeneous polynomial from E to E'. Then P is of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$. Consequently, if $\mathcal{P}(^{n-1}E;E') = \mathcal{P}_{\mathcal{W}}(^{n-1}E;E')$, then $\mathcal{P}(^nE) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE)$.

(c) If every linear operator from E to $\mathcal{P}(^{n-1}E;F)$ is weakly compact, then $\mathcal{P}(^nE;F) = \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^nE;F)$.

PROOF. (a) Just combine Proposition 34, Theorem 7 and Proposition 17(a).
(b) Since $(dP)^*(\varphi)(x) = \varphi(dP(x))$ for every $\varphi \in E''$ and $x \in E$, if $x, y \in E \subseteq E''$ are given, then

$$(dP)^*(y)(x) = y(dP(x)) = dP(x)(y) = n\check{P}(y,x,\ldots,x) = n\bar{P}(y)(x),$$

what shows that the restriction of $(dP)^*$ to E coincides with $n\bar{P}$. From the weak compactness of dP and Proposition 13 we have that $(dP)^*$ is a weakly compact operator, hence $\bar{P}$ is weakly compact as well. Apply Proposition 10 to complete the proof. See also Aron-Galindo [12, Corollary 2].
(c) See Proposition 10. $\boxed{QED}$

The decreasing scale property

We prove that $\mathcal{P}_{\mathcal{L}(W)}$ and $\mathcal{P}_{\mathcal{W}}$ enjoy a property which is useful to extend non-coincidence results to higher degrees of homogeneity. It has been shown that this property is enjoyed by some other classes of polynomials (see [22, 4.3], [77, 1.1.8, 2.4.12] and [93, 1.3.5]). In Aron-Galindo [12, p.182] it is proved that if $\mathcal{L}(^nE) = \mathcal{L}(W)(^nE)$, then $\mathcal{L}(^mE) = \mathcal{L}(W)(^mE)$ for every $m \leq n$. We prove that this property can be extended to arbitrary operator ideals, to polynomials and to vector-valued mappings.

For a vector $a \in E$, let us fix the notation $a^m = (a, \ldots, a)$, where a appears m times. For an operator u, the notation u^m is defined analogously.

40 Lemma. *Let $\mathcal{I}$ be an operator ideal. If $Q \in \mathcal{P}_{\mathcal{L}(I)}(^mE; F)$, $\varphi \in E'$ and $P : E \to F$ is defined by $P(x) = \varphi(x)^n Q(x)$, then $P \in \mathcal{P}_{\mathcal{L}(I)}(^{m+n}E; F)$.*

Proof. Let $u \in \mathcal{I}(E; G)$ and $R \in \mathcal{P}(^mG; F)$ be such that $Q = Ru$. Defining $A : G^m \times \mathbb{K}^n \to F$ by

$$A(y_1, \ldots, y_m, \lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n \check{R}(y_1, \ldots, y_m),$$

and $B = A(u^m, \varphi^n)$ we have that $B \in \mathcal{L}(I)(^{m+n}E; F)$ and $\hat{B} = P$. Then $P \in \mathcal{P}_{\mathcal{L}(I)}(^{m+n}E; F)$. $\boxed{QED}$

41 Proposition. *Let $\mathcal{I}$ be an operator ideal.*

(a) If $\mathcal{L}(^nE; F) = \mathcal{L}(I)(^nE; F)$, then $\mathcal{L}(^mE; F) = \mathcal{L}(I)(^mE; F)$ for every $m \leq n$.

(b) If $\mathcal{P}(^nE; F) = \mathcal{P}_{\mathcal{L}(I)}(^nE; F)$, then $\mathcal{P}(^mE; F) = \mathcal{P}_{\mathcal{L}(I)}(^mE; F)$ for every $m \leq n$.

Proof. (a) It is enough to prove the case $m = n - 1$. Let $\varphi \in E'$, $\varphi \neq 0$, and $a \in E$ be such that $\varphi(a) = 1$. Given $A \in \mathcal{L}(^{n-1}E; F)$, define

$$B(x_1, \ldots, x_{n-1}, x_n) = \varphi(x_n) A(x_1, \ldots, x_{n-1}) \ , \ B \in \mathcal{L}(^nE; F).$$

By hypothesis, $B = C(u_1, \ldots, u_n)$, where $u_j \in \mathcal{I}(E; G_j)$, $j = 1, \ldots, n$, and $C \in \mathcal{L}(G_1, \ldots, G_n; F)$. Defining $D(y_1, \ldots, y_{n-1}) = C(y_1, \ldots, y_{n-1}, u_n(a))$ we have that $D \in \mathcal{L}(G_1, \ldots, G_{n-1}; F)$ and $A = D(u_1, \ldots, u_{n-1})$, proving that $A \in \mathcal{L}(I)(^{n-1}E; F)$.
(b) The following proof is an adaptation of an argument due to D. Pellegrino. As above, let $\varphi \in E'$, $\varphi \neq 0$, and $a \in E$ be such that $\varphi(a) = 1$.

Let $m = 1$. Given $u \in \mathcal{L}(E; F)$, define $P(x) = \varphi(x)^{n-1} u(x)$. By assumption, $P = Qv$, where $v \in \mathcal{I}(E; G)$ and $Q \in \mathcal{P}(^nG; F)$. Observe that

$$\check{P}(a^{n-1}, x) = \frac{1}{n}\Big(u(x) + (n-1)\varphi(x)u(a)\Big).$$

The operator $\varphi(\cdot)u(a)$ belongs to $\mathcal{I}$ because it is a finite rank operator. In order to see that the operator $\check{P}(a^{n-1},\cdot)$ belongs to $\mathcal{I}$, let $T : G \to F$ be defined by $T(y) = \check{Q}(v(a)^{n-1}, y)$, recall that v belongs to $\mathcal{I}$ and note that $\check{P}(a^{n-1}, x) = Tv(x)$. It follows that $u \in \mathcal{I}(E; F)$.

Let $m \leq n$ and suppose that $\mathcal{P}(^kE; F) = \mathcal{P}_{\mathcal{L}(I)}(^kE; F)$ for every $k = 1$, 2, ..., $m-1$. Let us prove that $\mathcal{P}(^mE; F) = \mathcal{P}_{\mathcal{L}(I)}(^mE; F)$. Given $Q \in \mathcal{P}(^mE; F)$, define $P(x) = \varphi(x)^{n-m}Q(x)$ for every $x \in E$. By assumption, $\check{P} \in \mathcal{L}(I)(^nE; F)$, say $P = Ru$, where $R \in \mathcal{P}(^nG; F)$ and $u \in \mathcal{I}(E; G)$. We prove that Q is of type $\mathcal{P}_{\mathcal{L}(I)}$ by showing that all other terms in the expression

$$\begin{aligned}\check{P}(a^{n-m}, x^m) = K_0Q(x) + K_1\varphi(x)\check{Q}(a, x^{m-1}) + K_2\varphi(x)^2\check{Q}(a, a, x^{m-2}) + \cdots \\ + \ K_{m-1}\varphi(x)^{m-1}\check{Q}(a^{m-1}, x) + K_m\varphi(x)^mQ(a),\end{aligned}$$

where each K_k is a constant with $K_0 \neq 0$, are of type $\mathcal{P}_{\mathcal{L}(I)}$. The case of $\varphi(\cdot)^mQ(a)$ is trivial because it is a polynomial of finite type. Defining $S : G \to F$ by $S(y) = \check{R}(u(a)^{n-m}, y^m)$, it follows that $Su(x) = \check{P}(a^{n-m}, x^m)$, showing that the polynomial in the left hand side of the expression above is of type $\mathcal{P}_{\mathcal{L}(I)}$. The remaining terms are all of the form $\varphi(\cdot)^jV(\cdot)$ where V is an $(m-j)$-homogeneous polynomial. By our assumption, V is of type $\mathcal{P}_{\mathcal{L}(I)}$, and from Lemma 40 it follows that $\varphi(\cdot)^jV(\cdot)$ is of type $\mathcal{P}_{\mathcal{L}(I)}$, too. The proof is complete. $\boxed{QED}$

42 Theorem.

(a) If $\mathcal{P}(^nE; F) = \mathcal{P}_{\mathcal{L}(W)}(^nE; F)$, *then* $\mathcal{P}(^mE; F) = \mathcal{P}_{\mathcal{L}(W)}(^mE; F)$ *for every* $m \leq n$.

(b) If $\mathcal{P}(^nE; F) = \mathcal{P}_{\mathcal{W}}(^nE; F)$, *then* $\mathcal{P}(^mE; F) = \mathcal{P}_{\mathcal{W}}(^mE; F)$ *for every* $m \leq n$.

Proof. (a) Put $\mathcal{W} = \mathcal{I}$ in Proposition 41.
(b) The proof follows the same steps of the proof of Proposition 41(b). For example, for $m = 1$, given $u \in \mathcal{L}(E; F)$, define P as in the proof of Proposition 41(b). Using the expression we obtained for $\check{P}(a^{n-1}, x)$, it is easy to see that u is weakly compact, because $\check{P}$ is weakly compact and $\varphi(\cdot)u(a)$ is a finite rank operator. The cases $m = 2, \ldots, n-1$ are proved with the help of the expression we obtained for $\check{P}(a^{n-m}, x^m)$ and repeated applications of the Bolzano-Weierstrass and the Eberlein-Smulian theorems. $\boxed{QED}$

Combining Theorem 42 with examples 28 and 29 we obtain:

43 Corollary.

(a) $\mathcal{P}(^n\ell_1) \neq \mathcal{P}_{\mathcal{L}(W)}(^n\ell_1)$ *for every* $n \geq 2$.

(b) $\mathcal{P}(^n\ell_\infty) \neq \mathcal{P}_{\mathcal{L}(W)}(^n\ell_\infty)$ *for every* $n \geq 3$.

44 Remark. A related 'increasing scale property' is not to be expected even for $n \geq 2$. By [44, Example 4.1] we have that $\mathcal{P}(^2\ell_3)$ is reflexive while $\mathcal{P}(^3\ell_3)$ fails to be reflexive (because it contains ℓ_∞), then it follows from Proposition 34 that $\mathcal{P}(^2\ell_3; F) = \mathcal{P}_W(^2\ell_3; F)$ but $\mathcal{P}(^3\ell_3; F) \neq \mathcal{P}_W(^3\ell_3; F)$, where $F = \mathcal{P}(^3\ell_3)'$. On the other hand, we know that $\mathcal{P}(^2\ell_\infty) = \mathcal{P}_{\mathcal{L}(W)}(^2\ell_\infty)$ (Proposition 22(b)) but $\mathcal{P}(^3\ell_\infty) \neq \mathcal{P}_{\mathcal{L}(W)}(^3\ell_\infty)$ (Example 29).

5 Absolutely summing polynomials

It is well known that every absolutely p-summing linear operator is weakly compact (see [42, Theorem 2.17]). Considering that this is an important property enjoyed by the ideal of weakly compact operators, we are concerned in this section with possible polynomial versions of this fact. Of course, this depends on the classes that are considered as polynomial generalizations of the ideals that are involved, but from [20, Example 1] we know that $\mathcal{P}_{\mathcal{W}}$ is not a good candidate. The aim of this section is to show that, at least for some polynomial generalizations of the ideal of absolutely p-summing operators, $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ is a better choice. For the general theory of absolutely summing operators the reader is referred to Diestel-Jarchow-Tonge [42].

Given $0 < p < \infty$, let Π_p be the quasi-Banach (Banach if $1 \leq p < \infty$) operator ideal of all absolutely p-summing linear operators between Banach spaces. From Proposition 11 we already have the ideals of polynomials $\mathcal{P}_{\mathcal{L}(\Pi_p)}$ and $\mathcal{P}_{[\Pi_p]}$ which are generated, respectively, by the factorization and the linearization methods. Next we introduce another polynomial generalization of absolutely summing operators which has been studied by several authors in recent years (see [5,18–20,22,31,34–36,49,66–68,70,77–81,91]).

Polynomials of dominated type

Let $p > 0$. An n-homogeneous polynomial $P \in \mathcal{P}(^nE; F)$ is said to be p-*dominated* if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{k} \|P(x_j)\|^{\frac{p}{n}}\right)^{\frac{n}{p}} \leq C \cdot \left[\sup_{\varphi \in B_{E'}} \left(\sum_{j=1}^{k} |\varphi(x_j)|^p\right)^{\frac{1}{p}}\right]^n ,$$

for every $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in E$. The space of all p-dominated n-homogeneous polynomials from E to F will be denoted by $\mathcal{P}_{d,p}(^nE; F)$. Such polynomials are sometimes called *absolutely p-summing polynomials* and *absolutely* $(\frac{p}{n};p)$-*summing polynomials*.

The class $\mathcal{L}_{d,p}(^nE;F)$ of p-dominated n-linear mappings from E^n to F is defined in the obvious way (see [18,70,77–80,91]). It is an easy exercise to check that $\mathcal{P}_{d,p}$ is an ideal of polynomials and that $\mathcal{L}_{d,p}$ is an ideal of multilinear mappings (see [80, Prop. 3.4 and Prop. 3.5]).

Dominated polynomials and dominated multilinear mappings became popular because, besides the fact that the case $n = 1$ coincides with the ideal of absolutely summing operators, such mappings enjoy a Pietsch-type domination theorem. Likewise the linear case, a complete characterization should involve the transformation of vector-valued sequences.

Vector-valued sequences

Let $p \in [1,\infty)$. $\ell_p(E)$ is the Banach space of all sequences $(x_j)_{j=1}^{\infty}$ in E which are *absolutely p-summable* with the norm

$$\|(x_j)_{j=1}^{\infty}\|_p = \left(\sum_{j=1}^{\infty}\|x_j\|^p\right)^{1/p}.$$

We denote by $\ell_p^w(E)$ the space of all sequences $(x_j)_{j=1}^{\infty}$ in E such that $(\varphi(x_j))_{j=1}^{\infty} \in \ell_p$ for every $\varphi \in E'$. Such sequences are said to be *weakly p-summable*. The norm

$$\|(x_j)_{j=1}^{\infty}\|_{w,p} = \sup_{\varphi \in B_{E'}} \|(\varphi(x_j))_{j=1}^{\infty}\|_p$$

makes $\ell_p^w(E)$ a Banach space. $p = \infty$ is just the case of bounded sequences and in $\ell_\infty(E)$ we consider the sup norm. A sequence $(x_j)_{j=1}^{\infty}$ in E is said to be *unconditionally p-summable* if

$$\lim_{k\to\infty} \|(x_j)_{j=k}^{\infty}\|_{w,p} = 0.$$

The space of all such sequences, denoted by $\ell_p^u(E)$, is a closed subspace of $\ell_p^w(E)$.

If $0 < p < 1$ we have p-norms instead of norms, and the resulting spaces are complete metrizable topological vector spaces.

45 Theorem. *Let $P \in \mathcal{P}(^nE;F)$. The following are equivalent:*

(a) P is a p-dominated polynomial.

(b) $(P(x_j))_{j=1}^{\infty} \in \ell_{\frac{p}{n}}(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^w(E)$.

(c) *P_w given by $P_w((x_j)_{j=1}^{\infty}) = (P(x_j))_{j=1}^{\infty}$ is a well defined continuous n-homogeneous polynomial from $\ell_p^w(E)$ into $\ell_{\frac{p}{n}}(F)$.*

(d) *$(P(x_j))_{j=1}^{\infty} \in \ell_{\frac{p}{n}}(F)$ whenever $(x_j)_{j=1}^{\infty} \in \ell_p^u(E)$.*

(e) *P_u given by $P_u((x_j)_{j=1}^{\infty}) = (P(x_j))_{j=1}^{\infty}$ is a well defined continuous n-homogeneous polynomial from $\ell_p^u(E)$ into $\ell_{\frac{p}{n}}(F)$.*

(f) *There are $C \geq 0$ and a regular probability measure μ on the Borel σ-algebra on $B_{E'}$ endowed with the weak star topology such that*

$$\|P(x)\| \leq C \cdot \left(\int_{B_{E'}} |\varphi(x)|^p d\mu(\varphi) \right)^{\frac{n}{p}}, \textit{ for all } x \in E.$$

(g) *$\check{P}$ is a p-dominated n-linear mapping.*

Proof. The equivalences between the conditions (a), (b), (c), (d) and (e) can be found in Matos [66, Proposition 2.4]. The equivalence (a) $\Leftrightarrow$ (f) can be found in Matos [66, Proposition 3.1]. (g) $\Rightarrow (a)$ is obvious and (b) $\Rightarrow$ (g) follows easily from the linear structure of the sequence spaces and the polarization formula (see [21, Corollary 3.4]). $\boxed{QED}$

Next result justifies our claim that, with respect to the relationship with absolutely summing polynomials, $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ is a better choice than $\mathcal{P}_{\mathcal{W}}$.

46 Proposition. *Let $p \geq 1$.*

(a) *$\mathcal{P}_{\mathcal{L}(\Pi_p)} = \mathcal{P}_{d,p} \subseteq \mathcal{P}_{[\Pi_p]} \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{W})}$.*

(b) *$\mathcal{P}_{d,1} \neq \mathcal{P}_{[\Pi_p]} \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}$.*

(c) *$\mathcal{P}_{\mathcal{W}}$ is not contained neither in $\mathcal{P}_{d,p}$ nor in $\mathcal{P}_{[\Pi_p]}$.*

(d) *$\mathcal{P}_{\mathcal{W}}$ does not contain neither $\mathcal{P}_{d,p}$ nor $\mathcal{P}_{[\Pi_p]}$.*

Proof. (a) Given $P \in \mathcal{P}_{\mathcal{L}(\Pi_p)}(^nE; F)$, from Proposition 9 we know that $P = Qu$, where G is a Banach space, $u \in \Pi_p(E; G)$ and $Q \in \mathcal{P}(^nG; F)$. From Pietsch's Domination Theorem we have a constant $C \geq 0$ and a regular probability measure μ on the Borel σ-algebra on $B_{E'}$ such that

$$\|P(x)\| \leq \|Q\| \cdot \|u(x)\|^n \leq \|Q\| \cdot C^n \cdot \left(\int_{B_{E'}} |\varphi(x)|^p d\mu(\varphi) \right)^{\frac{n}{p}},$$

for all $x \in E$. From Theorem 45 it follows that P is p-dominated. Now suppose that P is p-dominated. From Theorem 45 we have that $\check{P}$ is a p-dominated

n-linear mapping. From Peréz-García [80, Corolario 3.23] it follows that $\check{P} \in \mathcal{L}(\Pi_p)(^nE; F)$. Call on Proposition 9 to show that $P \in \mathcal{P}_{\mathcal{L}(\Pi_p)}(^nE; F)$. Hence $\mathcal{P}_{\mathcal{L}(\Pi_p)} = \mathcal{P}_{d,p}$. By Proposition 11(b) we have that $\mathcal{P}_{\mathcal{L}(\Pi_p)} \subseteq \mathcal{P}_{[\Pi_p]}$. To complete the proof, note that the relationship $\mathcal{P}_{[\Pi_p]} \subseteq \mathcal{P}_{\mathcal{L}(\mathcal{W})}$ is an immediate consequence of the fact that every p-summing operator is weakly compact (see [42, Theorem 2.17]).
(b) Given $0 < \varepsilon < \frac{1}{2}$, set $\alpha = \frac{1}{2} + \varepsilon$. Now define

$$P : \ell_2 \to \mathbb{C} \ : \ P((x_i)) = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} (x_i)^2 \ , \ P \in \mathcal{P}(^2\ell_2).$$

Pellegrino [77, Exemplo 5.2.5] shows that $P \in \mathcal{P}_{[\Pi_1]}(^2\ell_2)$ but P is not 1-dominated. Another example that shows that $\mathcal{P}_{d,2} \neq \mathcal{P}_{[\Pi_2]}$ can be found in Peréz-García [80, Ejemplo 4.11]. It is obvious that the identity operator on an infinite dimensional reflexive space shows that $\mathcal{P}_{[\Pi_p]} \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}$. Anyway, a non-linear counterexample is also easy to be produced: let Q be the 2-homogeneous scalar-valued polynomial on ℓ_2 defined in 5.7.1. Since $I(\check{Q})$ is the identity operator on ℓ_2, it follows that Q is of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$ but $P \notin \mathcal{P}_{[\Pi_p]}(^2\ell_2)$.
(c) Follows immediately from (a) and Examples 28, 29 and 30.
(d) Let $P_n = Q_n i : \ell_1 \to \ell_1$ be the homogeneous polynomial considered in Example 27, where i is the formal inclusion from ℓ_1 into ℓ_2. From Grothendieck's theorem [40, 17.14] we have that i is absolutely 1-summing, hence $P_n \in \mathcal{P}_{\mathcal{L}(\Pi_1)}(^n\ell_1; \ell_1)$. Since $p \geq 1$, $P \in \mathcal{P}_{\mathcal{L}(\Pi_p)}(^n\ell_1; \ell_1) = \mathcal{P}_{d,p}(^n\ell_1; \ell_1) \subseteq \mathcal{P}_{[\Pi_p]}(^n\ell_1; \ell_1)$. Since P_n is not weakly compact it follows that $\mathcal{P}_{\mathcal{W}}$ does not contain neither $\mathcal{P}_{d,p}$ nor $\mathcal{P}_{[\Pi_p]}$. $\boxed{QED}$

47 Remark. (a) In [20, Example 1] it is shown that P_n is n-dominated, an information which was improved in the proof of Proposition 46(d), where we showed that P_n is 1-dominated. Anyway, it is easy to see that the argument presented in [20, Example 1] can be improved, with the aid of Kahane's inequality [42, 11.1], in order to show that P_n is 1-dominated.
(b) The coincidence $\mathcal{P}_{\mathcal{L}(\Pi_p)} = \mathcal{P}_{d,p}$ is known since the beginning of the theory (see Geiss [52], Pietsch [85], Schneider [91]).

48 Corollary. *Let E be a complex commutative Banach algebra with identity and $p > 0$. Then $\mathcal{P}(^n\ell_1(E); E) \neq \mathcal{P}_{d,p}(^n\ell_1(E); E)$ for every $n \geq 2$. In particular, $\mathcal{P}(^n\ell_1) \neq \mathcal{P}_{d,p}(^n\ell_1)$ for every $p > 0$ and $n \geq 2$.*

Proof. Let $P = \hat{A}$, where A is the symmetric bilinear mapping of Theorem 32. Since P is not of type $\mathcal{P}_{\mathcal{L}(\mathcal{W})}$, $\mathcal{P}(^2\ell_1(E); E) \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^2\ell_1(E); E)$. From Proposition 41(b) we have that $\mathcal{P}(^n\ell_1(E); E) \neq \mathcal{P}_{\mathcal{L}(\mathcal{W})}(^n\ell_1(E); E)$ for every $n \geq 2$, and the result follows from Proposition 46(a). $\boxed{QED}$

49 Remark. (a) The non-coincidence $\mathcal{P}(^2\ell_1) \neq \mathcal{P}_{d,p}(^2\ell_1)$ was claimed in Meléndez-Tonge [70, p.208], but the reasoning seems to be incomplete.
(b) Some other polynomial/multilinear generalizations of the ideal of absolutely p-summing operators have been studied, e.g.: semi-integral mappings (Alencar-Matos [5], Botelho-Pellegrino [22], Pellegrino [77]), strongly summing mappings (Carando-Dimant [31], Dimant [43], Pellegrino [77]), fully (or multiple) summing mappings (Bombal-Peréz-García-Villanueva [16], Matos [69], Pellegrino [77], Peréz-García-Villanueva [81,82], Souza [92,93]).

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