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# Zero sets and linear dependence of multilinear forms

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**Abstract.** We study the relation between common zero sets of multilinear forms and linear dependence of the forms. We also describe a related problem concerning sets of linear operators on $\mathbb{K}^n$, whose values at each $0 \neq y \in \mathbb{K}^n$, span $\mathbb{K}^n$.

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*Dedicated to the memory of Professor Klaus Floret*

It is well known that if we have linear forms $T_1, T_2, \dots T_n, T$ with $\bigcap_{i=1}^n T_i^{-1}(0) \subseteq T^{-1}(0)$, then $T = \sum_{i=1}^n \alpha_i T_i$ for some scalars $\alpha_i$. A natural question is whether or not multilinear forms have a similar property. We offer first a short counterexample in the general case, and then prove a positive result for a special case.

**1 Example.** Let $A_1,\ A_2$, and $B$ be bilinear forms mapping $\mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}$ defined by

$$\begin{aligned} A_1(x,y) &= x_1y_1 + x_2y_2 \\ A_2(x,y) &= x_1y_2 \\ B(x,y) &= x_1y_1 + x_1y_2. \end{aligned}$$

It is an elementary exercise to show that $A_1^{-1}(0) \cap A_2^{-1}(0) \subset B^{-1}(0)$, even though $B$ is clearly not a linear combination of $A_1$ and $A_2$.

Although Example 1 provides a counterexample to the general problem for multilinear forms, we do have a positive result in the case where $n = 1$. We note here that the result was previously known, [1, p. 97], in the bilinear case, but under the stronger assumptions that the two forms shared a common zero set and were defined on finite dimensional spaces. We should also point out that attempts to linearize the problem, by replacing the multilinear forms acting on $E \times \cdots \times E$ by the associated elements of $(E \bigotimes \cdots \bigotimes E)'$, seem doomed to failure, as Example 1 shows.

**2 Theorem.** *Let $A$ and $B$ be two $n$-linear forms on the product $E_1 \times \cdots \times E_n$ of $n$ vector spaces, with $A^{-1}(0) \subseteq B^{-1}(0)$. Then $B = \alpha A$ for some $\alpha \in \mathbb{K}$.*

Proof. We use induction on $n$, noting that the theorem is the linear case when $n = 1$ and that the result is trivial if $B = 0$. So assume $B \neq 0$ and say that $B(e_1, \ldots e_n) = 1$. Then, by our assumption, $A(e_1, \ldots e_n) = \beta \neq 0$, and by scaling $A$, we may without loss of generality assume that $\beta = 1$. We will show that $A = B$.

We first show that $A(x_1, \ldots, x_n) = B(x_1, \ldots, x_n)$ for any $(x_1, \ldots, x_n)$ where $x_j = e_j$ for some $j$. Consider $A_j : E_1 \times \cdots E_{j-1} \times E_{j+1} \times \cdots \times E_n$ defined by

$$A_j(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots y_n) \equiv A(y_1, \ldots, y_{j-1}, e_j, y_{j+1}, \ldots, y_n).$$

We define $B_j$ in the same manner. Then $A_j$ and $B_j$ are $(n-1)$-linear forms with the property that $A_j^{-1}(0) \subseteq B_j^{-1}(0)$. By the induction hypothesis, $B_j = \alpha A_j$ for some scalar $\alpha$. But $B_j(e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n) = B(e_1, \ldots, e_n) = A(e_1, \ldots, e_n) = A_j(e_1, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n)$, so it must be that $\alpha = 1$. Hence $B_j = A_j$ or $A(x_1, \ldots, e_j, \ldots, x_n) = B(x_1, \ldots, e_j, \ldots, x_n)$ and the first step is proved.

We now show that $A(x_1, \ldots, x_n) = B(x_1, \ldots, x_n)$ for any $(x_1, \ldots, x_n) \in E_1 \times \cdots \times E_n$. For a subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$, let $A[i_1, \ldots, i_k] \equiv A(v_1, \ldots, v_n)$ where

$$v_i = \begin{cases} x_i & if \quad i \notin \{i_1, \ldots i_k\} \\ e_i & if \quad i \in \{i_1, \ldots i_k\} \end{cases}$$

and let $J_k$ denote the set of subsets of $\{1, \ldots, n\}$ of length $k$.

Using the multilinearity of $A$ we have that for any vector $(c_1, \ldots, c_n) \in \mathbb{K}^n$,

$$\begin{array}{rcl} A(x_1 + c_1 e_1, \ldots, x_n + c_n e_n) & = & A(x_1, \ldots, x_n) + \\ & + & \sum_{j \in J_1} c_j A[j] \\ & + & \sum_{\{i_1, i_2\} \in J_2} c_{i_1} c_{i_2} A[i_1, i_2] \\ & & \vdots \\ & + & \sum_{J_{n-1}} c_{i_1} \ldots c_{i_{n-1}} A[i_1, \ldots i_{n-1}] \\ & + & c_1 \ldots c_n A[1, \ldots n]. \end{array}$$

Letting $k$ be the smallest positive integer so that $A[j_1,\dots j_k] \neq 0$ for some $\{j_1,\dots,j_k\} \in J_k$, we choose the $c_j$'s as follows:

$$\begin{aligned} c_{j_1} &= -\frac{A(x_1,\dots,x_n)}{A[j_1,\dots,j_k]}, \\ c_{j_2} &= c_{j_3} = \dots = c_{j_k} = 1, \\ c_j &= 0 \quad if \quad j \notin \{j_1,\dots,j_k\}. \end{aligned}$$

Then $A(x_1 + c_1e_1,\dots,x_n + c_ne_n) = 0$, and so by the induction hypothesis and the first step we have:

$$\begin{aligned} 0 &= B(x_1 + c_1e_1,\dots,x_n + c_ne_n) \\ &= B(x_1,\dots,x_n) + \sum_{j\in J_1} c_jB[j] + \sum_{\{i_1,i_2\}\in J_2} c_{i_1}c_{i_2}B[i_1,i_2] \\ &\quad + \dots + \\ &\quad + \sum_{J_{n-1}} c_{i_1}\dots c_{i_{n-1}}B[i_1,\dots i_{n-1}] + c_1\dots c_nB[1,\dots n] \\ &= B(x_1,\dots,x_n) - A(x_1,\dots,x_n). \end{aligned}$$

Hence, $B(x_1,\dots,x_n) = A(x_1,\dots,x_n)$, and the theorem is proved. $\boxed{QED}$

Despite Example 1 and Theorem 2, there remains the following related question: What if the forms are all required to be symmetric? We have a counterexample for this question also, but this time only in the real case. Perhaps more interesting than the example itself is the method of constructing this and more general examples. We first present a few definitions and an explanation of the ideas involved, saving the example for later. To simplify notation, in what follows we restrict our attention to bilinear forms.

**3 Definition.** For a positive integer $n$ and scalar field $\mathbb{K}$, we define $M(n,\mathbb{K})$ to be $(\mathbb{K}^n \times \{0\}) \bigcup (\{0\} \times \mathbb{K}^n)$.

Notice that $M(n,\mathbb{K})$ is minimal in that it is contained in the zero set of every bilinear form on $\mathbb{K}^n \times \mathbb{K}^n$. The general idea for what follows is that we start with a collection $\{A_1,\dots,A_m\}$ of $m < n^2$ bilinear forms on $\mathbb{K}^n \times \mathbb{K}^n$ with $\bigcap_{i=1}^m A^{-1}(0) = M(n,\mathbb{K})$. Since the dimension of the vector space of bilinear forms on $\mathbb{K}^n \times \mathbb{K}^n$ is $n^2$, it follows that there is a bilinear form $B \notin \operatorname{span}\{A_1,\dots,A_m\}$, and which clearly satisfies $\bigcap_{i=1}^m A_i^{-1}(0) \subseteq B^{-1}(0)$.

Of course, the problem is how to find such examples. To do so, we transport the idea above from the setting of bilinear forms to that of linear operators, via $B : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K} \rightsquigarrow f_B : \mathbb{K}^n \to \mathbb{K}^n$, where $\langle f_B(y), x\rangle = B(x,y)$.

**4 Definition.** Let $f_1, \ldots, f_m \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)$. We say that $\{f_1, \ldots, f_m\}$ has property $(*)$ if for all nonzero $y \in \mathbb{K}^n$, $\{f_1(y), \ldots, f_m(y)\}$ spans $\mathbb{K}^n$.

The next proposition gives the connection between a family of bilinear forms having the minimal set as their common zero set and the corresponding linear operators having property $(*)$.

**5 Proposition.** *Let $A_1, \ldots, A_m$ be bilinear forms on $\mathbb{K}^n \times \mathbb{K}^n$. Then*

$$\bigcap_{i=1}^{m} A_i^{-1}(0) = M(n, \mathbb{K})$$

*if and only if $\{f_{A_1}, \ldots, f_{A_m}\}$ has property $(*)$.*

Proof. To simplify notation, we will replace $f_{A_i}$ by $f_i$. Assume that $\bigcap_{i=1}^m A_i^{-1}(0) = M(n, \mathbb{K})$, but that for some $y \neq 0$, $\{f_1(y), \ldots, f_m(y)\}$ does not span $\mathbb{K}^n$. Let $x \neq 0$ with $x \perp f_i(y)$ for each $i = 1, \ldots, m$. Since $A_i(x, y) = \langle f_i(y), x \rangle = 0$ for all $i = 1, \ldots, m$, we have a contradiction. Thus, $\{f_1(y), \ldots, f_m(y)\}$ must span $\mathbb{K}^n$ for every nonzero $y$. The proof of the converse implication is similar. $\boxed{QED}$

We are now ready to give the counterexample to the symmetric case which was mentioned earlier.

**6 Example.** Let $f_1$, $f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, be linear maps defined by

$$\begin{aligned} f_1(y) &= (y_1, -y_2) \\ f_2(y) &= (y_2, y_1). \end{aligned}$$

Since $\det \begin{bmatrix} y_1 & -y_2 \\ y_2 & y_1 \end{bmatrix} = y_1^2 + y_2^2 > 0$ for all $(y_1, y_2) \neq (0, 0)$, we see that $\{f_1, f_2\}$ has property $(*)$. (Note that it is here that we need that the field is $\mathbb{R}$.)

Now, the corresponding bilinear forms are

$$\begin{aligned} A_1(x, y) &= x_1 y_1 - x_2 y_2 \\ A_2(x, y) &= x_1 y_2 + x_2 y_1, \end{aligned}$$

and by Proposition 5, $\bigcap_{i=1}^2 A_i^{-1}(0) = M(2, \mathbb{R})$. Furthermore, $B(x, y) = x_1 y_1 \notin \operatorname{span}\{A_1, A_2\}$, so that $A_1$, $A_2$ and $B$ comprise a counterexample to the real bilinear symmetric case.

Our interest now turns from the relation between common zero sets and linear dependence among multilinear forms to simply that of common zeros. We will continue to make use of the corresponding linear setting. We need one more definition to help with the terminology.

**7 Definition.** For a positive integer $n$ and scalar field $\mathbb{K}$, we define $m(n, \mathbb{K})$ to be the smallest positive integer $k$ for which there exists a collection $\{f_1, \ldots, f_k\}$ of $k$ linear maps in $\mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)$ having property $(*)$.

With the above definition, we have the following corollary to Proposition 5:

**8 Corollary.** *Let $n$ and $k$ be positive integers and $A_1, \dots, A_k$ be bilinear forms on $\mathbb{K}^n \times \mathbb{K}^n$. If $k < m(n, \mathbb{K})$, then $A_1, \dots, A_k$ have a nontrivial common zero, i.e. a zero not in $M(n, \mathbb{K})$.*

Corollary 8 shows the merit of knowing and/or estimating the values $m(n, \mathbb{K})$ for a given $n$ and $\mathbb{K}$, and this is where our focus will now turn. We know the exact values for a few cases, but we first give some general estimates.

**9 Lemma.** *Let $n > 1$. If $n$ is odd or if $\mathbb{K} = \mathbb{C}$, then $m(n, \mathbb{K}) \geq n + 1$.*

Proof. Let $f_1, \dots, f_n$ be any $n$ linear operators in $\mathcal{L}(\mathbb{K}^n, \mathbb{K}^n)$. For $y \in \mathbb{K}^n$, consider the vectors $f_1(y), \dots, f_n(y)$ as rows in an $n \times n$ matrix $M$ as in Example 6. Then $\{f_1, \dots, f_n\}$ has property $(*)$ if and only if $\det M \neq 0$ for all $y \in \mathbb{K}^n$, $y \neq 0$. But $\det M$ is an $n$-homogeneous polynomial in the $n$ variables $y_1, \dots, y_n$ and therefore, under the given hypothesis, has a non-trivial zero. Hence, $\{f_1, \dots, f_n\}$ cannot have property $(*)$, and we see then that $m(n, \mathbb{K}) \geq n + 1$. $\boxed{QED}$

**10 Lemma.** *For every $n$ and $\mathbb{K}$, $m(n, \mathbb{K}) \leq 2n - 1$.*

Proof. Consider the following $(2n - 1) \times n$ matrix:

$$\begin{pmatrix} y_1 & 0 & 0 & \dots & 0 \\ y_2 & y_1 & 0 & \dots & 0 \\ y_3 & y_2 & y_1 & \dots & 0 \\ & & \vdots & & \\ y_n & y_{n-1} & y_{n-2} & \cdots & y_1 \\ 0 & y_n & y_{n-1} & \cdots & y_2 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & y_n \end{pmatrix}$$

As before, each of the $2n-1$ rows represents a linear operator. If $y = (y_1, \dots, y_n)$, $y \neq 0$ with $y_j$, say, being the first non-zero coordinate of $y$, then the determinant of the sub-matrix formed by taking rows $j$ through $n + j - 1$ is $y_j^n$, and we are done. $\boxed{QED}$

We now give the values of $m(n, \mathbb{K})$ which are known. Note that by Example 6, $m(2, \mathbb{R}) = 2$.

**11 Example.** $m(3, \mathbb{R}) = 4$. To see this, consider the matrix

$$\begin{pmatrix} y_1 & y_2 & y_3 \\ -y_2 & y_1 & 0 \\ -y_3 & 0 & y_1 \\ 0 & -y_3 & y_2 \end{pmatrix}$$

whose rows correspond to the operators $I$, $P_{1,2} \circ R_3$, $P_{1,3} \circ R_2$, and $P_{2,3} \circ R_1$ (where $P_{i,j}$ denotes the projection onto the $(x_i, x_j)$ -plane and $R_k$ denotes rotation about the $x_k$ -axis). The determinant of the first three rows is $y_1(y_1^2 + y_2^2 + y_3^2)$. Hence, if $y_1 \neq 0$, we are done. Assuming then that $y_1 = 0$, the determinant of the first, second and fourth rows is $y_2(y_2^2 + y_3^2)$. Again, if $y_2 \neq 0$, we are done. Assuming then that $y_1 = y_2 = 0$, we must have that $y_3 \neq 0$, and the determinant of rows one, three and four is $y_3^3$ which is nonzero. We see then that the four operators represented in the given matrix have property $(*)$. Given Lemma 9, we see that $m(3, \mathbb{R}) = 4$.

**12 Example.** $m(4, \mathbb{R}) = 4$. Consider the following matrix:

$$\begin{pmatrix} y_1 & -y_2 & -y_3 & -y_4 \\ y_2 & y_1 & -y_4 & y_3 \\ y_3 & y_4 & y_1 & -y_2 \\ y_4 & -y_3 & y_2 & y_1 \end{pmatrix}$$

Once again, the rows represent linear operators in $\mathcal{L}(\mathbb{R}^4, \mathbb{R}^4)$. The determinant of the above matrix is $(y_1^2 + y_2^2 + y_3^2 + y_4^2)^2$, which is nonzero for any nonzero $y$ in $\mathbb{R}^4$. Thus, these operators have property $(*)$.

Note that the matrices in Examples 6 and 12 reflect, in some sense, the complex numbers and the quaternions, respectively. To be explicit, if we associate $y_1$ with 1, $y_2$ with $i$, $y_3$ with $j$, and $y_4$ with $k$, then the successive columns in the previous example are obtained by right multiplication of Column 1 by $-i, -j$, and $-k$, respectively. We do not know whether there is an analogous matrix for the octonians and, in fact, an exact description of the numbers $m(n, \mathbb{K})$ remains unknown. The question of whether a counterexample to the original problem concerning multilinear forms exists in the complex symmetric case is also open.

## References

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