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Tensor topologies on spaces of symmetric tensor products

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Abstract. Here we show how, in the general context of locally convex spaces, it is possible to get an n-tensor topology (on spaces of n-tensor products) from an n-tensor topology on spaces of symmetric n-tensors products. Indeed, given an n-tensor topology on the spaces of symmetric n-tensor products we construct an n-tensor topology on the spaces of all n-tensor products whose restriction to the symmetric ones gives the original topology. Moreover, we prove that when one starts with an n-tensor topology, restricts it to symmetric tensors and then extends it, the original topology is obtained when it is symmetric, and we also obtain some results on complementation with applications to spaces of polynomials. Part of these results generalize to the context of locally convex spaces some Floret's results in [17] and [18].

Keywords: Tensor product, symmetric tensor topology, locally convex space.

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Dedicated to the memory of Klaus Floret (1941-2002)

Introduction

From the observation by Ryan in [22] that the space of continuous homogeneous polynomials on a locally convex space is the dual of the space of symmetric tensor products endowed with the projective topology, the study of spaces of symmetric tensors has became of great interest and several results related with them have recently appeared in papers and books, e.g. [2–4,6,8,11,14,17,18].

Natural topologies on spaces of symmetric tensor products give rise to natural spaces of polynomials [14]. The standard topologies on spaces of tensor products induce natural topologies on spaces of symmetric tensor products and the main goal of this paper is to prove that the natural topologies on spaces of symmetric tensor products come from natural topologies on spaces of tensor products. Moreover we obtain some applications of this. This has been done by K. Floret [17,18] in the context of normed spaces and we use his ideas to generalize his results to the context of locally convex spaces.

1 Notations and Definitions

Let us fix some notations: $E; E_1, \ldots, E_n$ will be locally convex spaces over the same field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\otimes_{j=1}^n E_j$ will denote the tensor product $E_1 \otimes \cdots \otimes E_n$ and $\otimes^n$ the canonical mapping $\Pi_{j=1}^n E_j \to \otimes_{j=1}^n E_j$. When $E_1 = \cdots = E_n = E$ we use the notation $\otimes^n E$ and for $x \in E$, $\otimes^n x := x \otimes \cdots \otimes x$.

$\otimes_s^n E$ will represent the space of symmetric n-tensor products on E. Its elements are finite sums $\sum_{l=1}^m \delta_l \otimes^n x_l,\ x_l \in E$ and $\delta_l = \pm 1,\ l = 1, \ldots, m$. If $\mathbb{K} = \mathbb{C}$, the δ_l can be assumed equal to 1. Given $x_1, \ldots, x_n \in E$, we denote by $x_1 \vee \cdots \vee x_n$ the symmetrization of the tensor product $x_1 \otimes \cdots \otimes x_n$; that is,

$$x_1 \vee \cdots \vee x_n = \frac{1}{n!} \sum_{\eta \in S_n} x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)}.$$

(S_n denotes the group of permutations of $\{1, \ldots, n\}$). As proved in [17,22],

$$x_1 \vee \cdots \vee x_n = \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \otimes^n (\varepsilon_1 x_1 + \cdots + \varepsilon_n x_n).$$

Given a locally convex space E we denote by i_E^n the inclusion $\otimes_s^n E \hookrightarrow \otimes^n E$ and by σ_E^n the linearization of the n-linear mapping

$$\otimes_s^n : (x_1, \ldots, x_n) \in E^n \mapsto x_1 \vee \cdots \vee x_n \in \otimes_s^n E.$$

Given n locally convex spaces $E_1, \ldots, E_n$ we denote by $J_{E_1,\ldots,E_n}$ the composition of the mapping $I_1 \otimes \cdots \otimes I_n$, where I_k denotes the natural inclusion of E_k into $\Pi_{j=1}^n E_j$, with $\sqrt{n!}\sigma^n_{\Pi_{j=1}^n E_j}$. It is defined from $\otimes_{j=1}^n E_j$ into $\otimes_s^n \left(\Pi_{j=1}^n E_j\right)$. We note that, for every $x_j \in E_j,\ j = 1, \ldots, n$,

$$J_{E_1,\ldots,E_n}(x_1 \otimes \cdots \otimes x_n) = \sqrt{n!}(x_1, 0, \ldots, 0) \vee \cdots \vee (0, \ldots, 0, x_n).$$

Finally $Q_{E_1,\ldots,E_n}$ will denote the mapping

$$\otimes_s^n \left(\Pi_{j=1}^n E_j\right) \xrightarrow{i^n_{\Pi_{j=1}^n E_j}} \otimes^n \left(\Pi_{j=1}^n E_j\right) \xrightarrow{\sqrt{n!} P_1 \otimes \cdots \otimes P_n} \otimes_{j=1}^n E_j$$

being $P_k, k = 1, \ldots, n$, the projections $\Pi_{j=1}^n E_j \to E_k$. For $z = \otimes^n (x_1, \ldots, x_n) \in \otimes_s^n \left(\Pi_{j=1}^n E_j\right)$,

$$Q_{E_1,\ldots,E_n}(z) = \sqrt{n!} x_1 \otimes \cdots \otimes x_n.$$

Note that $Q_{E_1,\dots,E_n} \circ J_{E_1,\dots,E_n}$ is the identity on $\otimes_{j=1}^n E_j$.

These notations have been considered in [17,18], the introduction of $J_{E_1,\dots,E_n}$ is motivated by [7, Lemma 8].

In [3] the following definition is given, it is a generalization of the concept of tensor norm, see [10,15,16,23].

1 Definition. Let n be a natural number. A *tensor topology of order* n (or n*-tensor topology*) is a map which assigns to each n locally convex spaces $E_1, \dots, E_n$ a topology τ on $\otimes_{j=1}^n E_j$ such that

(1) The canonical mapping $\otimes^n : \Pi_{j=1}^n E_j \to (\otimes_{j=1}^n E_j, \tau)$ is separately continuous.

(2) If D_j, $j = 1, \dots, n$, are equicontinuous subsets of E_j', then

$$\{\varphi_1 \otimes \cdots \otimes \varphi_n : \varphi_j \in D_j\} \subset \left(\otimes_{j=1}^n E_j\right)^*$$

is $\tau-$equicontinuous.

(3) The mapping property: If E_j and F_j, $j = 1, \dots, n$, are locally convex spaces and $T_j \in \mathcal{L}(E_j, F_j)$, then

$$\otimes_{j=1}^n T_j : \otimes_{j=1}^n E_j \to \otimes_{j=1}^n F_j$$

is continuous with respect to the corresponding τ topologies.

Later on we will need to introduce a subscript to enhance the dependence on n.

Examples. The injective topology ε of uniform convergence on the sets of the form $\{\varphi_1 \otimes \cdots \otimes \varphi_n : \varphi_j \in D_j\}$, with $D_j \subset E_j'$ equicontinuous, $j = 1, \dots, n$, and the inductive topology i, which is the finest locally convex topology on $\otimes_{j=1}^n E_j$ that makes the canonical mapping $\otimes^n : \Pi_{j=1}^n E_j \to \otimes_{j=1}^n E_j$ separately continuous, are tensor topologies of order n. The projective topology π, which is the finest locally convex topology on $\otimes_{j=1}^n E_j$ that makes the canonical mapping $\otimes^n$ continuous, is also an n-tensor topology. See [16] or [21] for $n = 2$. In [3] other examples of n-tensor topologies are given.

Conditions (1) and (2) are equivalent to say that $\varepsilon \le \tau \le i$.

We can adapt for locally convex spaces the above definition to the symmetric case, generalizing Floret's definition for normed spaces:

2 Definition. Let n be a natural number. An *s-tensor topology of order* n is a map which assigns to each locally convex space E a topology τ_s on $\otimes_s^n E$ such that

(1) The canonical mapping $\otimes_s^n : E^n \to (\otimes_s^n E, \tau_s)$ is separately continuous.

(2) If D is an equicontinuous subset of E', then

$$\otimes_s^n D := \{\otimes^n \varphi : \varphi \in D\} \subset (\otimes_s^n E)^*$$

is τ_s-equicontinuous.

(3) The symmetric mapping property: If E and F are locally convex spaces and $T \in \mathcal{L}(E, F)$, then,

$$\otimes^n T : \otimes_s^n E \to \otimes_s^n F$$

is continuous with respect to the corresponding τ_s topologies.

Examples. The topology ε_s of uniform convergence on the sets of the form $\otimes_s^n D$, D an equicontinuous subset of E' and i_s, here defined as the finest locally convex topology on $\otimes_s^n E$ that makes the canonical mapping $\otimes_s^n : E^n \to \otimes_s^n E$ separately continuous, are s-tensor topologies of order n. The same happens for the topology π_s (see [14,18]), defined as the finest locally convex topology on $\otimes_s^n E$ that makes $\otimes_s^n$ continuous. The restriction of any n-tensor topology to symmetric tensors gives an s-tensor topology of order n.

Conditions (1) and (2) are equivalent to say that $\varepsilon_s \leq \tau_s \leq i_s$.

3 Definition. An n-tensor topology τ is said to be *symmetric* if for every locally convex spaces $E_1, \dots, E_n$ and every $\eta \in S_n$ the mapping

$$\sum_{l=1}^{m} x_{1,l} \otimes \cdots \otimes x_{n,l} \in \otimes_{j=1}^n E_j \mapsto \sum_{l=1}^{m} x_{\eta(1),l} \otimes \cdots \otimes x_{\eta(n),l} \in \otimes_{j=1}^n E_{\eta(j)}$$

is continuous with respect to the corresponding τ topologies. The n-tensor topologies ε, π and i are symmetric.

2 The results

To prove the main result (Theorem 5 below) we will use the following lemma.

4 Lemma. *For the n-tensor topologies ε and i we have the following equalities:*

$$\varepsilon|_{\otimes_s^n E} = \varepsilon_s, \quad i|_{\otimes_s^n E} = i_s$$

for every locally convex space E.

PROOF. By the definitions of ε_s and ε it follows straightforward that $\varepsilon_s \leq \varepsilon|_{\otimes_s^n E}$ on $\otimes_s^n E$ for every locally convex space E. On the other hand, given a locally convex space E let $\varepsilon_{\otimes^n D}$ be a continuous seminorm on $(\otimes^n E, \varepsilon)$, where we assume D is a balanced, convex and equicontinuous subset of E',

$$\varepsilon_{\otimes^n D}(z) = \sup\{|(\varphi_1 \otimes \cdots \otimes \varphi_n)(z)| : \varphi_1, \ldots, \varphi_n \in D\}, \ z \in \otimes^n E.$$

When $\varepsilon_{\otimes^n D}$ is applied to a symmetric tensor $z = \sum_{l=1}^m \delta_l \otimes^n x_l$, we get

$$\varepsilon_{\otimes^n D}(z) = \sup\left\{\left|\sum_{l=1}^m \delta_l \varphi_1(x_l) \cdots \varphi_n(x_l)\right| : \varphi_1, \ldots, \varphi_n \in D\right\} \overset{(*)}{=}$$

$$\sup\left\{\left|\sum_{l=1}^m \delta_l \frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \left[(\varepsilon_1 \varphi_1 + \cdots + \varepsilon_n \varphi_n)(x_l)\right]^n\right| : \varphi_1, \ldots, \varphi_n \in D\right\} =$$

$$\sup\left\{\left|\frac{1}{2^n n!} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n n^n \sum_{l=1}^m \delta_l \left[\psi(x_l)\right]^n\right| : \psi \in D\right\} \leq$$

$$\sup\left\{\left|\frac{n^n}{n!} \sum_{l=1}^m \delta_l \left[\psi(x_l)\right]^n\right| : \psi \in D\right\} = \frac{n^n}{n!} \varepsilon_{\otimes_s^n D}(z).$$

(*) is a consequence of the polarization formula that can be seen, for instance, in [14, Section 1.1].

We now prove that for every locally convex space E, $i|_{\otimes_s^n E}$ is the finest locally convex topology on $\otimes_s^n E$ that makes separately continuous the mapping $\otimes_s^n$. Let us fix $n-1$ elements, $x_1, \ldots, x_{n-1}$ in E. The mapping

$$x \mapsto x_1 \vee \cdots \vee x_{n-1} \vee x$$

is the sum of applications of the following type:

$$x \mapsto \frac{1}{n!} y_1 \otimes \cdots \otimes y_n,$$

where, for some $j = 1, \ldots, n$, $y_j = x$, and the others are $x_1, \ldots, x_n$. These mappings are continuous from E into $(\otimes^n E, i)$ because $\otimes^n : E^n \to (\otimes^n E, i)$ is separately continuous. Let $\mathcal{T}$ be a locally convex topology on $\otimes_s^n E$ such that the above mapping is continuous. Then, for every $x^* := (x_1, \ldots, x_{n-1}) \in E^{n-1}$ and for every convex balanced open neighborhood W of 0 in $(\otimes_s^n E, \mathcal{T})$ there is a convex and balanced neighborhood V_{x^*} of 0 in E such that

$$\{x_1\} \vee \cdots \vee \{x_{n-1}\} \vee V_{x^*} \subset W.$$

Then, since

$$\Gamma\left[\bigcup_{x^*}(\{x_1\}\vee\cdots\vee\{x_{n-1}\}\vee V_{x^*})\right]=$$
$$\Gamma\left[\bigcup_{x^*}\left[\frac{1}{n!}\left(V_{x^*}\otimes\{x_1\}\otimes\cdots\otimes\{x_{n-1}\}+\cdots+\{x_1\}\otimes\cdots\otimes\{x_{n-1}\}\otimes V_{x^*}\right)\right]\right]$$
$$\cap\otimes_s^n E,$$

(Γ denotes convex and balanced hull) is contained in W, we have that W contains a neighborhood of 0 for $i|_{\otimes_s^n E}$ and then $i|_{\otimes_s^n E}$ is finer than $\mathcal{T}$. $\boxed{QED}$

5 Theorem. *Let τ_s be an s-tensor topology of order n, then there exists a tensor topology $\widetilde{\tau_s}$ of order n, which is symmetric, such that $\widetilde{\tau_s}|_{\otimes_s^n E}=\tau_s$ on $\otimes_s^n E$ for every locally convex space E.*

Proof. For every n locally convex spaces $E_1,\ldots,E_n$ and every τ_s continuous seminorm α on $\otimes_s^n\left(\Pi_{j=1}^n E_j\right)$ let

$$\widetilde{\alpha}(z)=\alpha\left(J_{E_1,\ldots,E_n}(z)\right);\ z\in\otimes_{j=1}^n E_j.$$

Then $\widetilde{\alpha}$ is a seminorm on $\otimes_{j=1}^n E_j$ and $\{\widetilde{\alpha}:\alpha$ continuous seminorm on $(\otimes_s^n\left(\Pi_{j=1}^n E_j\right),\tau_s)\}$ defines an n-tensor topology $\widetilde{\tau_s}$ on $\otimes_{j=1}^n E_j$ with the properties mentioned in the theorem. Let us prove conditions (1), (2) and (3) in Definition 1:

(1): We decompose the identity $\left(\otimes_{j=1}^n E_j,i\right)\to(\otimes_{j=1}^n E_j,\widetilde{\tau_s})$ in the following mappings:

$$\left(\otimes_{j=1}^n E_j,i\right)\xrightarrow{J_{E_1,\ldots,E_n}}\left(\otimes_s^n\left(\Pi_{j=1}^n E_j\right),i_s\right)\xrightarrow{\mathrm{Id}}$$
$$\left(\otimes_s^n\left(\Pi_{j=1}^n E_j\right),\tau_s\right)\xrightarrow{Q_{E_1,\ldots,E_n}}\left(\otimes_{j=1}^n E_j,\widetilde{\tau_s}\right).$$

The first is continuous because of the mapping property, the symmetry of i and the Lemma; the second, which is the identity, is continuous because τ_s is an s-tensor topology. The continuity of the third follows in this way: For every continuous seminorm α on $\left(\otimes_s^n\left(\Pi_{j=1}^n E_j\right),\tau_s\right)$, and $z=\sum_{l=1}^m\delta_l\otimes^n(x_{1,l},\ldots,x_{n,l})\in\otimes_s^n\left(\Pi_{j=1}^n E_j\right)$,

$$\widetilde{\alpha}(Q_{E_1,\ldots,E_n}(z))=\alpha\left(\sum_{l=1}^m\frac{1}{2^n}\delta_l\sum_{\varepsilon_j=\pm1}\varepsilon_1\cdots\varepsilon_n\otimes^n(\varepsilon_1x_{1,l},\ldots,\varepsilon_nx_{n,l})\right).$$

Since the mappings

$$(y_1, \ldots, y_n) \in \Pi_{j=1}^n E_j \mapsto (\varepsilon_1 y_1, \ldots, \varepsilon_n y_n) \in \Pi_{j=1}^n E_j$$

are continuous, the desired continuity follows from the symmetric mapping property of τ_s.

(2): The identity $\left(\otimes_{j=1}^n E_j, \widetilde{\tau_s}\right) \to (\otimes_{j=1}^n E_j, \varepsilon)$ can be decomposed as:

$$\left(\otimes_{j=1}^n E_j, \widetilde{\tau_s}\right) \xrightarrow{J_{E_1,\ldots,E_n}} \left(\otimes_s^n (\Pi_{j=1}^n E_j), \tau_s\right) \xrightarrow{\mathrm{Id}}$$
$$\left(\otimes_s^n (\Pi_{j=1}^n E_j), \varepsilon_s\right) \xrightarrow{Q_{E_1,\ldots,E_n}} (\otimes_{j=1}^n E_j, \varepsilon).$$

The first mapping is continuous by the definition of $\widetilde{\tau_s}$, the second because τ_s is an s-tensor topology and the third because the Lemma and the mapping property for ε.

(3): The mapping property for $\widetilde{\tau_s}$. Let $T_j : E_j \to F_j$ be linear continuous mappings between locally convex spaces E_j and F_j, $j = 1, \ldots, n$. Then,

$$\otimes_{j=1}^n T_j : \otimes_{j=1}^n E_j \to \otimes_{j=1}^n F_j$$

is continuous for the corresponding $\widetilde{\tau_s}$ topologies because it is the composition of the continuous mappings:

$$\left(\otimes_{j=1}^n E_j, \widetilde{\tau_s}\right) \xrightarrow{J_{E_1,\ldots,E_n}} \left(\otimes_s^n \left(\Pi_{j=1}^n E_j\right), \tau_s\right) \xrightarrow{\otimes^n (T_1 \times \cdots \times T_n)}$$
$$\left(\otimes_s^n \left(\Pi_{j=1}^n F_j\right), \tau_s\right) \xrightarrow{Q_{F_1,\ldots,F_n}} \left(\otimes_{j=1}^n F_j, \widetilde{\tau_s}\right).$$

To see the symmetry of $\widetilde{\tau_s}$ we have to prove that for every $\eta \in S_n$, the mapping

$$\sum_{l=1}^m x_{1,l} \otimes \cdots \otimes x_{n,l} \in \otimes_{j=1}^n E_j \mapsto \sum_{l=1}^m x_{\eta(1),l} \otimes \cdots \otimes x_{\eta(n),l} \in \otimes_{j=1}^n E_{\eta(j)}$$

is continuous for the corresponding $\widetilde{\tau_s}$ topologies. But this mapping is the composition of the mappings:

$$(\otimes_{j=1}^n E_j, \widetilde{\tau_s}) \xrightarrow{J_{E_1,\ldots,E_n}} \left(\otimes_s^n (\Pi_{j=1}^n E_j), \tau_s\right) \xrightarrow{\otimes^n T_\eta}$$
$$(\otimes_s^n (\Pi_{j=1}^n E_{\eta(j)}), \tau_s) \xrightarrow{Q_{E_{\eta(1)},\ldots,E_{\eta(n)}}} (\otimes_{j=1}^n E_{\eta(j)}, \widetilde{\tau_s})$$

and we have already seen that the first and the third of them are continuous. The second one is also continuous because the mapping

$$T_\eta : (x_1, \ldots, x_n) \in \Pi_{j=1}^n E_j \mapsto (x_{\eta(1)}, \ldots, x_{\eta(n)}) \in \Pi_{j=1}^n E_{\eta(j)}$$

is linear and continuous and τ_s has the symmetric mapping property.

Let us see how $\widetilde{\tau}_s|_{\otimes_s^n E} = \tau_s$ on $\otimes_s^n E$ for every locally convex space E.
The identity

$$(\otimes_s^n E, \widetilde{\tau}_s|_{\otimes_s^n E}) \to (\otimes_s^n E, \tau_s)$$

is the composition of the mappings:

$$(\otimes_s^n E, \widetilde{\tau}_s|_{\otimes_s^n E}) \xrightarrow{J_{E,\dots,E}\circ i_E^n} (\otimes_s^n E^n, \tau_s) \xrightarrow{\sigma_E^n \circ Q_{E,\dots,E}} (\otimes_s^n E, \tau_s).$$

The first one is continuous because of the definition of $\widetilde{\tau}_s$ and the second is continuous because of the symmetric mapping property of τ_s: It is the linear mapping on $\otimes_s^n E^n$ generated by

$$\otimes^n (x_1, \dots, x_n) \in (\otimes_s^n E^n, \tau_s) \mapsto$$
$$\frac{1}{2^n \sqrt{n!}} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_n \otimes^n (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n) \in (\otimes_s^n E, \tau_s).$$

On the other hand the identity

$$(\otimes_s^n E, \tau_s) \to (\otimes_s^n E, \widetilde{\tau}_s|_{\otimes_s^n E})$$

is the composition of the mappings:

$$(\otimes_s^n E, \tau_s) \xrightarrow{J_{E,\dots,E}\circ i_E^n} (\otimes_s^n E^n, \tau_s) \xrightarrow{\sigma_E^n \circ Q_{E,\dots,E}} (\otimes_s^n E, \widetilde{\tau}_s|_{\otimes_s^n E}).$$

The first is continuous because the symmetric mapping property of τ_s, since each of the mappings

$$x \in E \mapsto (\varepsilon_1 x, \dots, \varepsilon_n x) \in E^n$$

is continuous, and the continuity of the second can be obtained as follows:

For every continuous seminorm α on $(\otimes_s^n E^n, \tau_s)$ and every $(x_1, \dots, x_n) \in E^n$,

$$\widetilde{\alpha}(\sigma_E^n \circ Q_{E,\dots,E}(\otimes^n (x_1, \dots, x_n))) \le \frac{1}{\sqrt{n!} 2^n} \sum_{\varepsilon_j = \pm 1} \widetilde{\alpha}(\otimes^n (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)) =$$
$$\frac{1}{n! 2^{2n}} \sum_{\varepsilon_j, \gamma_j = \pm 1} \alpha \left(\otimes^n (\gamma_1 (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n), \dots, \gamma_n (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)\right)).$$

Since the mappings

$$(x_1, \dots, x_n) \in E^n \mapsto (\gamma_1 (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n), \dots, \gamma_n (\varepsilon_1 x_1 + \dots + \varepsilon_n x_n)) \in E^n$$

are continuous, the symmetric mapping property of τ_s gives that there is a τ_s continuous seminorm β on $\otimes_s^n E^n$ such that

$$\widetilde{\alpha}(\sigma_E^n \circ Q_{E,\dots,E}(\otimes^n(x_1,\dots,x_n))) \le \beta(\otimes^n(x_1,\dots,x_n))$$

and the same for linear combinations, that is, for tensors in $\otimes_s^n E^n$. $\boxed{QED}$

Follow several consequences of the above theorem:

6 Corollary. *For every locally convex space E, every $n \in \mathbb{N}$, and every s-tensor topology τ_s of order n, $(\otimes_s^n E, \tau_s)$ is a complemented subspace of $(\otimes^n E, \widetilde{\tau_s})$.*

PROOF. The continuity of the inclusion $i_E^n : (\otimes_s^n E, \tau_s) \to (\otimes^n E, \widetilde{\tau_s})$ follows from Theorem 5. On the other hand, for every $\eta \in S_n$ the symmetry of $\widetilde{\tau_s}$ gives that the mapping

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_s}) \mapsto x_{\eta(1)} \otimes \cdots \otimes x_{\eta(n)} \in (\otimes^n E, \widetilde{\tau_s})$$

is continuous, so is the mapping generated by

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_s}) \mapsto x_1 \vee \cdots \vee x_n \in (\otimes_s^n E, \tau_s),$$

(using again Theorem 5). Moreover the composition of i_E^n with it is the identity. $\boxed{QED}$

7 Corollary. *For every locally convex spaces $E_1, \dots, E_n$, $(\otimes_{j=1}^n E_j, \widetilde{\tau_s})$ is a complemented subspace in $(\otimes_s^n(\Pi_{j=1}^n E_j), \tau_s)$.*

Indeed, the maps $J_{E_1,\dots,E_n}$ and $Q_{E_1,\dots,E_n}$ are continuous as we have seen in the proof of Theorem 5, and $Q_{E_1,\dots,E_n} \circ J_{E_1,\dots,E_n} = Id$.

In the next theorem we prove that when one restricts an n-tensor topology τ to the space of symmetric tensors and then extends it with the procedure given in the above Theorem 5 obtains an n-tensor topology finer than τ. Both topologies are the same when τ is symmetric[1].

8 Theorem. *For every n-tensor topology τ, the extension of its restriction to symmetric n-tensors is an n-tensor topology finer than τ. Moreover if τ is symmetric, both topologies coincide.*

PROOF. Let us denote by $\tau|_s$ the restriction of τ to spaces of symmetric tensors. The identity $(\otimes_{j=1}^n E_j, \widetilde{\tau|_s}) \to (\otimes_{j=1}^n E_j, \tau)$, can be factorized as:

$$(\otimes_{j=1}^n E_j, \widetilde{\tau|_s}) \xrightarrow{J_{E_1,\dots,E_n}} \left(\otimes_s^n \left(\Pi_{j=1}^n E_j\right), \tau|_s\right) \xrightarrow{Q_{E_1,\dots,E_n}} (\otimes_{j=1}^n E_j, \tau).$$

[1]If τ is no symmetric the equality has no sense because $\widetilde{\tau|_s} = \tau$ y $\widetilde{\tau|_s}$ is symmetric.

The first mapping is continuous by the definition of $\widetilde{\tau|_s}$, and the second is $Q_{E_1,\ldots,E_n}$ which is continuous as has been proved in a previous similar situation.

Assume now that τ is symmetric. The identity

$$(\otimes_{j=1}^n E_j, \tau) \to (\otimes_{j=1}^n E_j, \widetilde{\tau|_s})$$

can be factorized as:

$$(\otimes_{j=1}^n E_j, \tau) \xrightarrow{J_{E_1,\ldots,E_n}} (\otimes_s^n(\Pi_{j=1}^n E_j), \tau|_s) \xrightarrow{Q_{E_1,\ldots,E_n}} (\otimes_{j=1}^n E_j, \widetilde{\tau|_s}).$$

The first is continuous by the mapping property and the symmetry of τ and the second is also continuous for those spaces as we know. $\boxed{QED}$

From the above Theorem 8 and Corollary 7 we get

9 Corollary. *For every locally convex space E and $n \in \mathbb{N}$, $\mathcal{L}_I(^nE) := (\otimes^n E, \varepsilon)'_\beta$ is a complemented subspace of the strong dual $\mathcal{P}_I(^nE^n) := (\otimes_s^n E^n, \varepsilon_s)'_\beta$.*

The elements in $\mathcal{L}_I(^nE)$ (resp. $\mathcal{P}_I(^nE^n)$) are called integral n-linear mappings (resp. integral n -homogeneous polynomials) and appear in several papers and books, among others: [1,9,13,14,17,19,20].

Remark. Having in mind the above Theorem 8 we notice that Corollary 7 for $\tau_s = \pi_s$ has been proved in [7, Lemma 8] for $n = 2$ and in [6] for arbitrary n.

Other results on complementation for spaces of polynomials have been obtained in [7] and [12].

3 Some applications

To give an application of Theorems 5 and 8 above we recall the definition of tensor topology given in [3] and introduce the concept of s-tensor topology.

10 Definition. A *tensor topology* is a sequence $\tau = (\tau_n)_{n\in\mathbb{N}}$, where each τ_n is an n-tensor topology, which is associative. That is, for all m and $n \in \mathbb{N}$, with $m < n$, and for every n locally convex spaces $E_j, j = 1, \ldots, n$, the equality

$$((\otimes_{j=1}^m E_j, \tau_m) \otimes (\otimes_{j=m+1}^n E_j, \tau_{n-m}), \tau_2) = (\otimes_{j=1}^n E_j, \tau_n).$$

holds topologically. A tensor topology $\tau = (\tau_n)_{n\in\mathbb{N}}$ is called *symmetric* if all τ_n are symmetric according Definition 3 above.

The natural topologies ε, π and i are symmetric tensor topologies [3].

11 Definition. An *s-tensor topology* is a sequence $\tau_s = (\tau_{s,n})_{n\in\mathbb{N}}$ where each $\tau_{s,n}$ is an s-tensor topology of order n such that $\tau := (\widetilde{\tau_{s,n}})_{n\in\mathbb{N}}$ is a tensor topology. The topologies ε_s , π_s and i_s are s-tensor topologies.

12 Corollary. *For every locally convex space E and every s-tensor topology $\tau_s = (\tau_{s,n})_{n\in\mathbb{N}}$, $(\otimes_s^n E, \tau_{s,n})$ is a complemented subspace of $(\otimes_s^{n+1} E, \tau_{s,n+1})$ for each $n \in \mathbb{N}$.*

Proof. We will use several times the above Theorem 5. Fix $n \in \mathbb{N}$, $e \in E$ and $\varphi \in E'$ such that $\varphi(e) = 1$. The extension by linearity of the mapping J considered in [6, Th. 3] defined by

$$J(\otimes^n x) = \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} \varphi(x)^{k-1} e \vee \overset{k}{\cdots} \vee e \vee x \vee \overset{n-k+1}{\cdots} \vee x$$

is continuous between $(\otimes_s^n E, \tau_{s,n})$ and $(\otimes_s^{n+1} E, \tau_{s,n+1})$. Indeed, it is a sum of linear combinations of mappings of the following type (note that $\widetilde{\tau} = (\widetilde{\tau_{s,n}})_{n\in\mathbb{N}}$ is associative and each $\widetilde{\tau_{s,n}}$ is symmetric):

For $k = 1$:

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_{s,n}}) \mapsto e \otimes x_1 \otimes \cdots \otimes x_n \in (E \otimes (\otimes^n E, \widetilde{\tau_{s,n}}), \widetilde{\tau_{s,2}}),$$

which is continuous by property (1) of 2-tensor topologies.

For $1 < k < n+1$:

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_{s,n}}) \mapsto$$
$$\varphi(x_1) \cdots \varphi(x_{k-1}) e \otimes \overset{k}{\cdots} \otimes e \otimes x_k \otimes \cdots \otimes x_n \in \left(\otimes^{n+1} E, \widetilde{\tau_{s,n+1}}\right)$$

which is continuous by properties (1) and (3) of tensor topologies of a fixed degree applied to the mappings:

$$x \in E \mapsto \varphi(x) e \in E \quad (k-1 \text{ times})$$

$$x_k \otimes \cdots \otimes x_n \in (\otimes^{n-k+1} E, \widetilde{\tau_{s,n-k+1}}) \mapsto$$
$$e \otimes x_k \otimes \cdots \otimes x_n \in (E \otimes (\otimes^{n-k+1} E, \widetilde{\tau_{s,n-k+1}}), \widetilde{\tau_{s,2}}).$$

For $k = n+1$:

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_{s,n}}) \mapsto \varphi(x_1) \cdots \varphi(x_n) e \otimes \overset{n+1}{\cdots} \otimes e \in (\otimes_s^{n+1} E, \tau_{s,n+1}),$$

which is the composition of the mappings

$$x_1 \otimes \cdots \otimes x_n \in (\otimes^n E, \widetilde{\tau_{s,n}}) \mapsto \varphi(x_1) \cdots \varphi(x_n) e \otimes \overset{n}{\cdots} \otimes e \in (\otimes_s^n E, \tau_{s,n})$$

$$\otimes^n x \in (\otimes_s^n E, \tau_{s,n}) \mapsto (\otimes^n x) \otimes e \in ((\otimes_s^n E, \tau_{s,n}) \otimes E, \widetilde{\tau_{s,2}})$$

and both mappings are continuous by properties (1) and (3) of tensor topologies of a fixed degree.

A projection Π can be defined as the linear map generated by

$$\otimes^{n+1} x \in (\otimes_s^{n+1} E, \tau_{s,n+1}) \mapsto \varphi(x) \otimes^n x \in (\otimes_s^n E, \tau_{s,n}).$$

Π is continuous as a consequence of the mapping property applied to the identity $(\otimes^n E, \widetilde{\tau_{s,n}}) \to (\otimes^n E, \widetilde{\tau_{s,n}})$ and $x \in E \to \varphi(x) \in \mathbb{K}$. Note that $(F \otimes \mathbb{K}, \widetilde{\tau_{s,2}}) = F$ for every s-tensor topology of order 2 and every locally convex space F. In [6, Th. 3] the equality $\Pi \circ J = Id$ on $\otimes_s^n E$ is checked. $\boxed{QED}$

Remark. The above corollary generalizes to every s-tensor topology Blasco's result for the projective topology [6, Th. 3] and gives a new proof of it without using any seminorm description.

13 Corollary. *$\mathcal{P}_I(^n E)$ is a complemented subspace of $\mathcal{P}_I(^{n+1} E)$ endowed with the strong topologies as dual spaces.*

Analogous results on the spaces of all continuous polynomials have been obtained in [5] for the normed case and in [6] for locally convex spaces.

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