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GROUPS WHOSE INFINITE PROPER SUBGROUPS ARE T-GROUPS

MARIA ROSARIA CELENTANI, ULDERICO DARDANO

1. INTRODUCTION AND MAIN RESULT

A group is said to be a T-group if its subnormal subgroups are normal. Classes of «generalized» T-groups have been the object of much attention and have been studied intensively. In [10] D.J.S. Robinson studied the class of non-T groups all of whose *proper* subgroups are T-groups. In the usual terminology for group classes these are minimal-non-T groups. It is a well established pattern to study group classes by studying minimal (in the above sense) classes. This kind of investigations began with a paper by G.A. Miller and H.C. Moreno [6].

Under (weak) hypotheses of generalized solubility or finiteness minimal-non-T groups turn out to be finite, whence soluble. In the case of p-groups they coincide with minimal-non-abelian groups if we substitute for the quaternion group $\mathcal{Q}_8$ of order 8 that of order 16 (type I). In the non-primary cases they also have a restricted structure as their order is divisible only by two primes p, q. In fact they have the form $F = \langle x \rangle \ltimes P$ where x has order q^m and P is a Sylow p-subgroup of F which is either isomorphic to $\mathcal{Q}_8$ (type II) or non-cyclic elementary abelian, the only G-subgroups of P are the characteristic ones (type II and IV), unless $p \equiv 1 \pmod q$ and P has order p^2 (type III) (see [10]).

This paper, inspired by the above-quoted of Robinson, tries to show to which extent similar ideas work in infinite groups. We study the class of non-T groups whose *proper infinite* subgroups are T-groups. If we call $\tilde{T}-groups$ the groups whose *all infinite* subgroups are T-groups, we can say that we are interested in minimal-non-$\tilde{T}$ groups (see [1]). We are able to give a description of these groups under solubility assumption. Observe that any direct product of a Tarski p-group by a dihedral group of order 8 is a minimal-non-$\tilde{T}$ group, as in any Hall extension of a Prüfer group by a minimal-non-T group. We will see that, in fact, with the exception of type II, all minimal-non-T groups are subgroups of minimal-non-$\tilde{T}$ groups also in less trivial ways. This enables us to recall some more detailed information on minimal-non-T groups while stating Theorem A , our main result. In the last section of the paper we will describe non-abelian (infinite) soluble groups whose proper infinite subgroups are abelian (see Theorem B), a result of independent interest.

Notation and *terminology* are mostly standard. We refer to [2] and [5]. In particular:

- letters p, q, r denote only prime numbers,
- $n|m$ means that n divide m,
- $\mathbb{Z}(p)$ is the ring $\mathbb{Z}/p\mathbb{Z}$,
- $\mathbb{Z}(p^\infty)$ is a Prüfer p-group containing $\mathbb{Z}(p)$,
- $\mathbf{Z}_p$ is the ring of p-adic integers, i.e. the endomorphism ring of $\mathbb{Z}(p^\infty)$,
- $\mathbf{Q}_p$ is the field of fractions of $\mathbf{Z}_p$,

- $|x|$ is the order of the element x,
- $Soc\ G$ is the subgroup generated by all elements with prime order of the abelian group G,
- $\pi(G)$ is the set of prime numbers p for which the group G has an element of order p,
- if N is a normal subgroup a group G such that $\pi(N) \cap \pi(G/N) = \emptyset$ then we say that N is a *Hall subgroup* of G or that G is a *Hall extension* of N,
- a *power automorphism* of a group is an automorphism mapping every subgroup onto itself.

Let us state now our main result:

Theorem A. *An infinite soluble group is a non-T group whose infinite proper subgroups are T-groups if and only if it is of one of the following four types. Such groups are all Chernikov groups.*

1. *Non-(Prüfer-by-finite) groups.* Groups with the form $G = \langle x \rangle R$, where R is a radicable abelian p-group with finite rank $n > 1$ and is normal in G, x has order q^m and acts on R by means of the matrix:

$$\Theta = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 & \dots & -\alpha_n \end{pmatrix}$$

with entries in $\mathbf{Z}_p$, which is the companion matrix of the minimal polynomial $\mu(t) = t^n + \alpha_n t^{n-1} + \dots + \alpha_1$ of a q^f-th primitive root of 1 where $f \leq m$ is the greatest positive integer such that q^{f-1} divides $p-1$ (see section 2 for details).

We recall that the degree n of $\mu(t)$ is equal to the multiplicative order of p mod q^f if $q \neq p$ and to $p-1$ otherwise. Moreover, soluble minimal-non-T groups of type IV, have all the form $F = \langle x \rangle P$ where $P = Soc\ R$ and $G = \langle x \rangle R$ is above. We have Robinson's subtypes IVa and IVb according to $f = 1$ or $f > 1$. Observe that G cannot be a 2-group, since $p = q = 2$ implies $n = 1$.

2. *Prüfer-by-finite non-primary groups.* Groups of one of the following subtypes where R is a Prüfer p-group:

(a) Hall extensions of R by a minimal-non-T group;

(b) $G = F \times R$ where F is a non-primary minimal-non-T group such that $p \in \pi(F)$ and which is of type IVa if the Sylow p-subgroup P of F is normal;

(c) $G = \langle x \rangle \ltimes (\langle a \rangle \times R), a^p = 1, a^x = a^\zeta, c^x = c^\eta$, for any $c \in R, \zeta$ and η are p-adic integers such that ζ is a primitive q^f-th root of 1 with $0 < f \leq m, \eta = \zeta^{1+kq^{f-1}}$ and $0 < k < q$;

(d) $G = (\langle x \rangle R) \ltimes P$, where $\langle x \rangle R$ is a non-abelian 2-group with property T and $F = \langle x \rangle \ltimes P$ is a minimal-non-T group (of type III or IVb).

Moreover soluble minimal-non-T groups of type III are all of type $F = \langle x \rangle \ltimes (\langle a \rangle \times Soc\ R)$ where $G = \langle x \rangle \ltimes (\langle a \rangle \times R)$ is of type 2c. About case (d) we recall that $\langle x \rangle R$ is a non-abelian T-group if and only if x induces the inversion map on R and $|x| \leq 4$. About case (a) see Lemma 7.

3. *Prüfer-by-finite primary groups which are central-by-finite.* Groups of the form $G = FR$ where F is a finite minimal-non-abelian p-group, R is a Prüfer p-group and $[F, R] = 1$.

4. *Prüfer-by-finite* 2*-groups which are not central-by-finite.* See groups of types $(i) - (x)$ in the statement of Proposition 13 below.

2. PROOF OF THEOREM A

In this section we will give the proof of our main result. As shorthand notation we shall say that G is an **X**-*group* whenever G is an infinite non-T group all of whose infinite proper subgroup are T-groups. We start by noting that it is a trivial fact that a minimal non-T group is a $\mathcal{B}_2$-group, i.e. a group in which subnormal subgroup have defect at most 2, and that the same can be said for **X**-groups.

Proposition 1. *An* **X**-*group is a* $\mathcal{B}_2$-*group.*

Proof. Let G be an **X**-group and S be a non-normal subnormal subgroup of G. Thus S^G is a proper subgroup of G; if it is infinite then it is a T-group and $S \triangleleft S^G$. If S^G is finite, then the normalizer $N_G(S)$ of S has finite index in G and is contained in a maximal subgroup M. Then applying Lemma 7.3.16 of [5] to the finite group G/M_G one gets $S^G \leq M$ and so $N_G(S)S^G \leq M$. Thus $N_G(S)S^G$ is a T-group and $S \triangleleft S^G$, what we wanted. □

In hypothesis of solubility, **X**-groups have a strong finiteness condition; in fact they are Chernikov groups, as we are going to show in Proposition 3. We note that since by [10] a soluble minimal-non-T group is finite we may state: *if G is a soluble* **X**-*group, then there is a finite subgroup F of G which is a minimal-non-T group.* This fact will play a major rôle in our arguments and will be stated later in greater details (see Lemma 6).

We need a result from [1] (see Theorem 3.2). Recall that a group all of whose subgroups are T-groups is said to be a $\overline{T}-group$.

Proposition 2. *Let G be an infinite soluble group. Then G has property $\tilde{T}$ and not $\overline{T}$ if and only if $G = (S \times E \times B) \ltimes A$, where:*

(i) A and B are finite abelian groups with coprime odd orders;
(ii) E is an elementary abelian 2-group;
(iii) every subgroup of A is normal in G;
(iv) either $S = \langle z, R\rangle$, with $R \triangleleft S$, or $S = Q \ltimes R$, where R is a Prüfer 2-group, z has order 2 or 4, Q is isomorphic to the quaternion group of order 8 and $[S, R] \neq 1$.

Proposition 3. *Let G be a soluble* **X**-*group. Then G is a Chernikov group whose finite residual has no proper infinite G-subgroups.*

Proof. We first show that G is periodic. For suppose that a is an element of G with infinite order. If G were not finitely generated then for any pair of non-commuting elements x and y of G the subgroup $\langle a, x, y\rangle$ would be an infinite non-abelian T-group, a contradiction (see [8], Theorem 3.3.1). Thus G must be a finitely generated soluble $\mathcal{B}_2$-group and hence, by Theorem A of [9], it is finite-by-nilpotent. Then by a well-known property of nilpotent groups there is a normal subgroup N of G such that G/N is infinite cyclic, say $G = \langle x, N\rangle$. Futhermore for any positive integer s the group $\langle x^s, N\rangle$ is finitely generated and so abelian; this implies that G itself is abelian, again a contradiction. Therefore G is periodic.

Assume now that G is not a Chernikov group. Since G is soluble it does not satisfy the minimal condition on abelian subnormal subgroups (see [11]) and so it has a subnormal subgroup A which is the direct product of an infinite family of subgroups with prime order. Since A^G is different from G, it is a T-group generated by subnormal subgroups with prime order and therefore it has the same form as A. Hence we may assume that A is normal in G. Let now F be a finite non-T-subgroup of G. There exists a subgroup B with finite index in A such that $|F| < |A : B|$ and $F \cap B = 1$. If $FA \neq G$ then FA is a T-group and $B \triangleleft FA$. It follows that $F \simeq FB/B$ is a T-group, which is absurd. Then $FA = G$ and B has finite index in G; moreover $|FB_G : B_G| = |F| < |A : B| \leq |G : B_G|$ shows that $F \simeq FB_G/B_G$ is a T-group, which is impossible. Thus G is a Chernikov group.

Let finally R be the finite residual of G and F as before. To obtain a contradiction we suppose that there is an infinite proper p-subgroup H of R which is normal in G. Of course we may suppose that H is radicable. Then FH is a $\tilde{T}$-group with a non-T subgroup and so its finite residual H is a 2-group by Proposition 2. It follows that R is a 2-group. Let x be an element of $R\backslash H$ with order 8. Then by the same result we get the contradiction that $\langle x, FH\rangle$ is not a $\tilde{T}$-group and conclude that R has no infinite proper G-subgroups. □

We observe that since both soluble T-groups and soluble minimal-non-T groups have bounded derived length by the previous proposition the same holds for soluble **X**-groups.

Because of Proposition 3 from now on we will be concerned with Chernikov groups; the proof of Theorem A will be split into cases according to the introduction. Before continuing it we recall a useful result due to Robinson which gives a necessary condition for a group to have property T (see [8], Lemma 5.2.2).

Lemma 4. *Let the periodic group G have a Hall normal subgroup N such that every subnormal subgroup of N is normal in G and G/N is a T-group, then G itself is a T-group.*

Case 1 - Non-(Prüfer-by-finite) groups.

We shall show that a soluble (Chernikov) **X**-group which is not Prüfer-by-finite is a group of type **1**. We state first a proposition giving a group-theoretical characterization of the groups we consider in this case.

Proposition 5. *Let the group G be not Prüfer-by-finite and R be its finite residual. Then G is an* **X**-*group if and only if $G = \langle x, R\rangle$ and the following hold:*
(i) x has order q^m, where $m > 0$ and q is a prime;
(ii) no infinite proper subgroup of R is normalized by x;
(iii) every subgroup of R is normalized by x^q;
(iv) G is not a 2-group.

Proof. To see the sufficiency of the condition note that if H is an infinite subgroup of G not contained in $M = \langle x^q, R\rangle$ then $G = HR$ and $H \cap R \triangleleft G$. It follows that $R \leq H$ and $H = G$. Thus it suffices to observe that M is a $\overline{T}$-group. If $p = q$ then M is clearly abelian, for there are no power automorphisms of R with order p as p is odd. If $p \neq q$ then M is a $\overline{T}$-group by Lemma 4.

Conversely, if G is an **X**-group then every subgroup of R is normal in each maximal subgroup of G, so that, by Proposition 3, G has only one maximal subgroup and G/R is cyclic with prime-power order. So $G = \langle x, R\rangle$ and (i)-(iii) clearly hold. Moreover if G is a 2-group then x^2 induces by conjugation on R a power automorphism, which is clearly either the identity or the inversion map. In the latter case $\langle x^2, R\rangle$ would not be a $\tilde{T}$-group, by Proposition 2. In the former case, if R_1 is a Prüfer subgroup of R then $R_1^{1+x} = \{aa^x | a \in R_1\}$ is a normal subgroup of G, a contradiction. □

We want now to give an explicit description of the action by conjugation of x on R and get information on the rank n of R. Let ϑ be the automorphism which x induces by conjugation on R and $R^\# = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}(p^\infty), R)$; regard $V = \mathbf{Q}_p \otimes_{\mathbf{Z}_p} R^\#$ as a right $\mathbf{Q}_p\langle\vartheta\rangle$-module in the natural way. It is well-known that *V has no proper non-trivial $\mathbf{Q}_p\langle\vartheta\rangle$-submodules if and only if ϑ may be represented by a matrix Θ as in the introduction where $\mu(t) = t^n + \alpha_n t^{n-1} + \ldots + \alpha_1$ is an irreducible polynomial with coefficients in $\mathbf{Z}_p$* (on this matter see [3] and also [7]).

Since ϑ^q is a power automorphism of R, while ϑ is not and has order q^f where $f \leq m$, we have that $\mu(t)$ is an irreducible non-linear factor of $t^q - \lambda$, where λ is the p-adic unit with order q^{f-1} such that $a^{\vartheta^q} = a^\lambda$ for any $a \in R$. Furthermore $q^{f-1} | p-1$ if p is odd and $q^{f-1} | 2$ if $p = 2$.

If $\lambda = 1$ (and this is the case if $q = p \neq 2$ or, more generally, we force G to have all proper infinite subgroups abelian) then $f = 1$ and $\mu(t)$ is the minimal polynomial of a primitive q-th root of 1, whose degree n is equal to $p-1$ if $q = p$ and to the multiplicative order of p *mod* q otherwise. In the former case clearly $\mu(t) = t^{p-1} + t^{p-2} + \ldots + 1$ and Θ has entries in $\mathbb{Z}$. If $\lambda \neq 1$ and λ is a q-th root in $\mathbf{Z}_p$, or equivalently q^f divises $p-1$, then $t^q - \lambda$ splits over $\mathbf{Z}_p$ into linear factors, contradicting to $n > 1$. Thus f is the greatest positive integer such that $q^{f-1}|p-1$ and $\mu(t) = t^q - \lambda$ is irreducible.

Let us observe that the argument in the last lines of the proof of Proposition 5 is actually an application of the facts we have just stated. In fact, if $q = p = 2$ and x^2 centralizes R, then in the above notation ϑ has order 2 and $\lambda = 1$. Therefore $\mu(t)$ divides $t^2 - 1$ and its degree n is 1, a contradiction. By the way we also note that the consideration of the automorphism acting on $\mathbb{Z}(2^\infty) \bigoplus \mathbb{Z}(2^\infty)$ by means of the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

shows that condition (iv) in the statement of the above proposition may not be relaxed.

We have proved that non-(Prüfer-by-finite) **X**-groups are of type **1** in the introduction.

Case 2 - Prüfer-by-finite non-primary groups.

In this case we are concerned with Chernikov groups whose finite residual R is a Prüfer r-group (where r is a prime). The next lemma completes an observation we have made before.

Lemma 6. *Let G a soluble* **X***-group whose finite residual R is a Prüfer group. Then G has a (finite) minimal-non-T subgroup F. Furthermore $G = FR$, unless G is a 2-group and R is non-central. Moreover if R is a Hall subgroup of G we can choose F such that $G = F \ltimes R$.*

Proof. Suppose first that R is a Hall subgroup and let F be a complement of R. Hence $F \simeq G/R$ is not a T group by Lemma 4 and so is a minimal-non-T group. Thus we may assume that R is not a Hall subgroup.

The first part of the statement about the existence of a (finite) non-T subgroup F of G has already been settled. Assume $G \neq FR$. Then FR has property $\tilde{T}$ but not $\overline{T}$ and so, by Proposition 2, R is a Prüfer 2-group not contained in the centre of G. Let D be the set of all elements of G with odd order. If D is *not* a subgroup then $G = \langle D, R\rangle$, because in a torsion T-group the elements of odd order fill a subgroup. Furthermore, since R has no non-trivial automorphism of odd order, R in central in $G = \langle D, R\rangle$, a contradiction. Thus D is a subgroup of G and $G = S \ltimes D$, where S is a Sylow 2-subgroup of G. Since R is not a Hall subgroup of G, i.e. $R < S$, we have $D \simeq_G DR/R < G/R$; it follows that D is a T-group. Moreover G/D also is a T-group as it is isomorphic to S. If by contradiction G/R were a T-group then by the above G-isomorphism every (sub)normal subgroup of D

would be normal in G and, in view of Lemma 4, G would be a T-group, a contradiction. So G/R is a non-primary minimal-non-T group, $\pi(G/R) = \{p, 2\}$, $D = P$ is the Sylow p-subgroup of G and $S = \langle x, R\rangle$ for some element x. Thus $G = S \ltimes P$, $F_1 = \langle x\rangle \ltimes P$ is a minimal-non-T group and of course every infinite subgroup of S is a T-group. □

As a consequence of this lemma we see that if G is an $\mathbf{X}$-group then $\pi(G) = \{p, q, r\}$ has at most 3 elements. Let us exploit the action of a non-primary minimal-non-T group on a Prüfer group.

Lemma 7. *Let G be a group such that $G = FR$, where R ia a Prüfer r-group and $F = \langle x\rangle \ltimes P$ is a non-primary minimal-non-T group (notation as in the introduction). If G is an $\mathbf{X}$-group then:*
(i) $[x, R] \neq 1$ implies that either $q \nmid r-1$ or $q = r = 2$;
(ii) $[P, R] \neq 1$ implies that F is of type III (therefore $q|p-1$ and $P = \langle a\rangle \times \langle b\rangle$) and: $a^x = a^\zeta$, where ζ is a $q-th$ primitive root of 1 in $\mathbf{Z}_p$, $b^x = b, [a, R] = 1$.

Proof. Clearly (i) is trivial. If $[P, R] \neq 1$ then $P/C_P(R)$ is a non-trivial cyclic group, hence it has order p. On the one hand, since $C_P(R) \lhd F$ we get that P is neither minimal normal in F nor isomorphic with the quaternion group of order 8; in other words F is of type III and $P = \langle a\rangle \times \langle b\rangle$, where $\langle a\rangle$ and $\langle b\rangle$ are normal in F. On the other hand under these circumstances we have $\langle a\rangle = [x, \langle a\rangle] \leq F' \leq C_F(R)$, as the automorphism group of R is abelian. Thus $C_P(R) = \langle a\rangle$ and $b^{-1+x} = [b, x] \in C_P(R) \cap \langle b\rangle = 1$. The statement now follows. □

We can now settle this case by the following Proposition.

Proposition 8. *Let the non-primary soluble group G have a Prüfer r-subgroup R with finite index. Then G is an $\mathbf{X}$-group if and only if G is of type **2** in the introduction (where $p = r$).*

Proof. We first show the necessity of the condition, which is trivially verified if R is a Hall subgroup of G (type 2a). So let $r \in \pi(G/R)$. From Lemma 6 we get the existence of a minimal-non-T group F such that $G = FR$. Then F is not a primary group and has the form $F = \langle x\rangle \ltimes P$, where x is a q-element and P is a p-group either isomorphic to the quaternion group of order 8 (type II) or elementary abelian. Recall that in the latter case P is minimal normal in F if F is of type IV, otherwise F is of type III. Obviously either $p = r$ or $q = r$.

If $q = r \neq 2$ then $[x, R] = 1$ and $\langle x\rangle \cap R = F \cap R$, as $\langle x\rangle$ is a Sylow q-subgroup of F. Assume by contradiction that $[P, R] \neq 1$. Then $p|r-1 = q-1$ and by Lemma 7 we get the contradiction F is of type III and $q|p-1$. Let now $\langle x, R\rangle = \langle y\rangle \times R$ and $F_1 = \langle y\rangle \ltimes P$. Since y acts on P the same way as x does, then F_1 is a minimal-non-T-group and $G = F_1 \times R$ is of type 2b. On the other hand if $q = r = 2$ and still $[x, R] = 1$ arguing as before we get that G again is of type 2b; finally if $[x, R] \neq 1$, i.e. x induces the inversion map on R, then

G must be of type 2d. Let us move now to the case $p = r$.

If $p = r$ it follows $[P, R] = 1$. In fact if $[P, R] \neq 1$ then by Lemma 7 we get that F is of type III and so $p \neq 2$, then it is a trivial fact that P centralizes R. Let us examine all possibilities for F. We see that F is not of type II, because PR is abelian. If F is of type III then $P = \langle a \rangle \times \langle b \rangle$ where $\langle a \rangle$ and $\langle b \rangle$ are normal in G. If, for a contradiction, both $\langle x, a, R \rangle$ and $\langle x, b, R \rangle$ are different from G then they are T-groups and the (universal) power automorphisms induced by x on $\langle a, R \rangle$ and $\langle b, R \rangle$ respectively have the same exponent; it would follow that x induces on $P = \langle a, b \rangle$ a power automorphism and F is a T-group. Thus G is as in 2c, where relations beetween ζ and η are due to the fact that they hold mod p (see [10]) and there is only one periodic p-adic integer which lifts a non-zero element of $\mathbb{Z}(p)$, since $p \neq 2$. If F is of type IVa then $q \nmid p-1$ and $[x, R] = 1$; on the other hand P is minimal normal in F and $F \cap R = P \cap R = 1$, hence G is again of type 2b. Finally, if by contradiction F is of type IVb and f is the greatest positive integer such that q^{f-1} divides $p-1$ then x^q induces on R an automorphism of order q^{f-1}, since $G \neq \langle x^q, R \rangle$. It would follow that x induces on R an automorphism of order q^f and $q^f | p-1$, contradicting the choice of f. The necessity of the condition is now shown.

To show the sufficiency we observe that a group of type 2a is trivially an **X**-group by Lemma 4. Then observe that groups of type 2c are not T-groups because x induces on the abelian normal subgroup $\langle a, R \rangle$ a non-power automorphisml and that all other groups described in the statement have a finite factor group (namely G/R) which is not a T-group. Thus, since subgroup with finite index of soluble T-groups are still T-groups (see [4]), we just need to check whether the maximal subgroups of G have property T; let G_1 be one of them.

Let G be of type 2b and $\pi(F) = \{p, q\}$; then $G_1 = F_1 R$ where F_1 is a maximal subgroup of F. If $q = r$ let $P_1 = G_1 \cap P = F_1 \cap P$, then $G_1/P_1 \simeq G_1 P/P$ is abelian and F_1 (and with it G_1) induces on P_1 a cyclic group of power automorphisms; by Lemma 4 G_1 is a T-group. Similarly if $p = r$ then either $G_1 = \langle x^q, PR \rangle$ or G_1 is conjugate to $\langle x \rangle R$. In both cases it is trivial that G_1 is a T-group.

If G is of type 2c then the maximal subgroups of G are conjugate to either $\langle x \rangle R$ or $\langle x^q, a, R \rangle$. The former group is a T-group by Lemma 4 and the latter has property T as x^q induces on $\langle a, R \rangle$ a power automorphism of exponent $\zeta^q = \eta^q$. Finally if G is of type 2d then $G_1 = S_1 \ltimes P_1$, where S_1 is conjugate to a subgroup of $S = \langle x \rangle R$ (and therefore a T-group) and $P_1 = G_1 \cap P$. Thus G_1 has property T, again by Lemma 4. □

Case 3 - Prüfer-by-finite primary-groups which are central-by-finite.

In this case we deal with a p-group G whose finite residual R is central. Clearly if $p \neq 2$ then every soluble **X**-group which is a primary group falls in this case.

Recall that if p is an odd prime for soluble p-groups property T is equivalent to commutativity. Thus a soluble p-group G (p odd) has property **X** if and only if it is non-abelian but all its infinite proper subgroups are. Futhermore the same hold for soluble 2-groups in which the largest normal abelian divisible subgroup is central. Case 3 is easily settled by the following proposition.

Proposition 9. *Let G be a Prüfer-by-finite p-group whose finite residual R is central. Then G is an* **X***-group if and only if $G = FR$, where F is a minimal-non-abelian group.*

Proof. The sufficiency of the condition is self-evident. If G is an **X**-group then by Lemma 6 we have $G = FR$ where F is a minimal-non-T group. If F is not minimal-non-abelian, as we claim, then F is isomorphic to the quaternion group of order 16 and has a subgroup F_1 isomorphic to the quaternion group of order 8, which is minimal-non-abelian. The proof is complete once we observe that $G \neq F_1 R$ yields the contradiction that $F_1 R$ is a non-abelian T-group. □

Note that if we consider the direct product G of a finite minimal-non-abelian p-group F and a Prüfer p-group R amalgamating the commutator subgroup of F and the socle of R we see that G is an **X**-group and G/R is abelian, so G does not split over R.

Case 4 - Prüfer-by-finite 2-groups which are not central-by-finite.

This is the final step of the proof of the main result. We are going to deal only with 2-groups whose finite residual R is non-central. Although in [8] a complete description of 2-groups with property T is to be found we recall from that paper a statement which fits for our arguments (see Theorem 3.1.1).

Lemma 10. *Let the group G have an abelian subgroup C with index 2 and an element $x \notin C$ such that $c^x = c^{-1}$, for all $c \in C$. Then G is a T-group if and only if $C^2 \leq \langle x^2, C^4 \rangle$.*

Observe that if G is non-abelian the subgroup C is identified as the Fitting subgroup $Fit\, G$ of G. As a direct consequence of this lemma we get:

Lemma 11. *Let $G = \langle x, R \rangle$ be a Prüfer-by-cyclic 2-group with finite residual R. Then G is an* **X***-group if and only if*
(i) $c^x = c^{-1}$, for all $c \in R$;
(ii) $|x| \geq 8$.

The next lemma shows that the remaining groups form a restricted class.

Lemma 12. *Let the 2-group G have property* **X** *and a non-central Prüfer subgroup R such that G/R is a finite non-cyclic group. Then $G = \langle x, \langle y \rangle \times R \rangle$ and the following hold:*
(i) $y^8 = x^4 = 1$, $[x^2, y] = 1$
(ii) $y^x \in \{y^k, y^k a, y^k x^2, y^k x^2 a\}$, where $\langle a \rangle = Soc\, R$, k is an odd integer and, if $|y| = 8$, $k \equiv -1 \pmod 4$.

Proof. Let $C = C_G(R)$ and $x \in G \backslash C$ (clearly $|G : C| = 2$). Since $R \leq Z(C)$ and C is a T-group, $C = C_1 \times R$ is abelian. Assume by contradiction that for any $c \in C_1$ the subgroup $\langle x, c, R\rangle$ is proper and so a T-group. Thus G/R has exponent at most 4, $c^x = c^{-1}$ and $c^2 = x^2$ if $|c| = 4$, by results in [8]. By Lemma 10 we get the contradiction that G is a T-group. Therefore $G = \langle x, \langle y\rangle \times R\rangle$, for some $y \in C_1$ chosen of minimal order, and $C = \langle x^2, y, R\rangle$.

Since $\langle x, R\rangle$ is a proper subgroup of G it is a non-abelian T-group and $x^4 = 1$. By the same reason also $G_1 = \langle x, y^2, R\rangle$ is a non-abelian T-group. Thus $y^{2x} = y^{-2}$ and G_1/R has exponent at most 4, hence $y^8 = 1$. From $C \triangleleft G$ it follows $y^x = x^i y^k b$, where k is an integer, $i = 0, 2$ and $b \in R$. From $y^{2x} = y^{-2}$ it follows easily $b^2 = 1$ and the stated condition on k. Furthermore $y^x = x^i b$ means $y = x^i b$ which in turn implies $G = \langle x, R\rangle$, a contradiction. Therefore (ii) holds. □

Because of the previous lemma we introduce some terminology. Let us say that a group G is of type $X(J, i, k)$, where $J \in \{I, II, III\}$, $i \in \{1, 2, 3\}$, $k \in \{1, -1, 3\}$ if and only if:

$G = \langle x, \langle y\rangle \times R\rangle$ where $x^4 = 1, |y| = 2^i, R \simeq \mathbb{Z}(2^\infty), \langle a\rangle = Soc\, R, c^x = c^{-1} \forall c \in R, J = I, II, III$ according to: $y^x = y^k$, $y^x = y^k a$, $y^x = y^k x^2 a^e$ (with $e = 0, 1$)$[x^2, y] = 1$ and

$$(\bullet) \qquad y^4 \in \langle x^2, R\rangle.$$

If we omit condition $(\bullet)$ from the above definition, Lemma 12 may be restated by saying that a 2-group G with a non-central Prüfer subgroup R such that G/R is a finite non-cyclic group and property **X** is a group of type $X(J, i, k)$ for some admissible 3-tuple (J, i, k). Observe that the type $X(J, i, k)$ does not identify the isomorphism type of the group.

Proposition 13. *Let G be a Prüfer-by-finite 2-group and R its finite residual. Then G is a* **X**-*group and R is non-central if and only if G is of one of the following types:*
(i) $\langle x, R\rangle$ where $c^x = c^{-1}$ for all $c \in C$ and $|x| \geq 8$.
(ii) $X(I, 2, 1)$
(iii) $X(I, 2, -1)$ and $y^2 \notin \langle x^2, R\rangle$
(iv) $X(I, 3, -1)$
(v) $X(II, 1, *)$
(vi) $X(II, 2, *)$
(vii) $X(II, 3, -1)$
(viii) $X(III, 1, *)$ and $[x, y] \neq 1$
(ix) $X(III, 2, *)$ and $y^x \neq y^{-1}$
(x) $X(III, 3, 3)$.

Proof. By the above all we have to do is to show that in the hypotheses and notation of Lemma 12 condition $(\bullet)$ holds and to determine which groups of type $X(J, i, k)$ are **X**-groups. Let $G = \langle x, \langle y\rangle \times R)\rangle$ be one of them. Then the Frattini subgroup $\Phi(G)$ of G

is $\langle x^2, y^2, R\rangle$, the (three) maximal subgroups of G are $G_0 = C = \langle y, \Phi(G)\rangle = \langle x^2, y, R\rangle$ (which is abelian by construction), $G_1 = \langle x, \Phi(G)\rangle = \langle x, y^2, R\rangle$, $G_2 = \langle xy, \Phi(G)\rangle = \langle xy, y^2, R\rangle$ and $Fit\, G_1 = Fit\, G_2 = \Phi(G)$. Furthermore $C^2 = \Phi(G)$, $C^4 = \Phi(G)^2 = \langle y^4, R\rangle$, $\Phi(G)^4 = R$. By Lemma 10, G is a not a T-group if and only if:

$$(\dagger) \qquad y^x \neq y^{-1} \qquad \text{or} \qquad y^2 \notin \langle x^2, y^4, R\rangle$$

furthermore G_1 and G_2 are T-groups if and only if:

$$(\ddagger) \qquad y^{2x} = y^{-2} \qquad \text{and} \qquad y^4 \in \langle x^2, R\rangle \cap \langle x^2 y^x y, R\rangle$$

(to get $(\ddagger)$ we have used the equality $(xy)^2 = x^2 y^x y$). Since we do not have used $(\bullet)$ to get $(\ddagger)$, we can add $(\bullet)$ to the necessary condition of Lemma 12. Moreover, since in a soluble T-group subgroups with finite index are still T-groups (see [4]), a group G as above is an **X**-group if and only if $(\dagger)$ and $(\ddagger)$ hold. We proceed now by cases (and subcases).

Case $X(I, i, k)$: This is the case $y^x = y^k$. If $i = 1$ then $(\dagger)$ does not hold.

If $i = 2$ then $k \equiv 1$ or $k \equiv -1$ (mod 4), in the latter eventuality to satisfy $(\dagger)$ we must have $y^2 \notin \langle x^2, R\rangle$. Then we see that both types (ii) and (iii) have **X**.

If $i = 3$ then $k \equiv -1$ or $k \equiv 3$ (mod 8). In the former case we have $(\ddagger)$ if and only if $y^4 \in \langle x^2, R\rangle$; once we observe that under these circumstances $y^2 \notin \langle x^2, y^4, R\rangle$ actually follows from $y^4 \in \langle x^2, R\rangle$ we include (iv) in the list. Finally the case $k \equiv 3$ may not occur as $y^4 \in \langle x^2, R\rangle \cap \langle x^2 y^4, R\rangle$ is incompatible with the choice of y.

Case $X(II, i, k)$: This is the case $y^x = y^k a$. Of course $(\dagger)$ holds and if $i = 1, 2$ there is nothing to say. If $i = 3$ proceed as above.

Case $X(III, i, k)$: If $y^x = y^k x^2$ we again have to consider just the case $i = 3$ and $(\ddagger)$. Since $x^2 y^k y = y^{k+1}$ then $(\ddagger)$ holds if and only if $k = 3$ and $y^4 \in \langle x^2, R\rangle$. Thus we get type (x). The case $y^x = y^k x^2 a$ is handled similarly. □

The proof of Theorem A is now complete.

3. NON-ABELIAN GROUPS WITH ALL INFINITE PROPER SUBGROUPS ABELIAN

This last section is a by-product of the previous ones. We note here that using our main result, Proposition 2 and the discussion following Proposition 5, we are able describe non-abelian infinite soluble groups whose all infinite proper subgroups are abelian.

Theorem B. *Let G an infinite soluble non-abelian group. Then all infinite proper subgroups of G are abelian if and only if G has an abelian divisible normal $\mathfrak{p}$-subgroup R and one of the following hold:*

(a) $G = \langle x, R \rangle$, *where* x *has prime power order* q^m, $[x^q, R] = 1$ *and, if* R *has rank greater than* 1, *then* G *is of type* **1** *in the introduction with* $f = 1$;
(b) $G = FR$, *where* R *has rank* 1, F *is a minimal-non-abelian group and* $[F, R] = 1$. *Moreover* F *can be chosen such that* $G = F \times R$, *provided* G *is not a* p*-group.*

Proof. The sufficiency of the condition is clear because every infinite proper subgroup of G is contained in $\langle x^q \rangle R$ in case (a) or contains R in case (b). To prove the necessity first observe that if G is not a T-group then it is an **X**-group and we just have to read through the list in the introduction looking for groups having the property we are interested in. Let then G be a T-group. By the same arguments of Proposition 3 we see that G is a Chernikov group whose finite residual has no proper infinite G-subgroups, hence G is Prüfer-by-finite. If G is not a $\overline{T}$-group then by Proposition 2 we get the result (case (a)). If G is a $\overline{T}$-group then it has a finite subgroup F which is a minimal-non-abelian group and has property T. Moreover $G = FR$. If G/R has prime-power order we have type (a). Otherwise R is central and, if G is not a p-group, $F = \langle x \rangle \ltimes P$ where P is an elementary abelian non-central primary Sylow subgroup which is minimal normal in F and x has prime-power order. Therefore $P \cap R = 1$. Finally, if $\langle x, R \rangle = \langle y \rangle \times R$, then the subgroup $\langle y, P \rangle$ may be chosen for F. □

REFERENCES

[1] M.R. CELENTANI, U. DARDANO, *Some sensitivity conditions for infinite groups*, Rend. Mat. (7), **12** (1992).

[2] P.M. COHN, *Algebra*, 2nd ed., John Wiley and Sons, New York 1989.

[3] B. HARTLEY, *A dual approach to Chernikov modules*, Math. Proc. Cambridge Phil. Soc., **82** (1977), 215-239.

[4] H. HEINEKEN, J.C. LENNOX, *Subgroups of finite index in T-groups*, Boll. Un. Mat. It. (6), **4-B** (1985), 829-841.

[5] J.C. LENNOX, S.E. STONEHEWER, *Subnormal Subgroups of Groups*, Oxford Mathematical Monographs, Clarendon Press, Oxford 1987.

[6] G.A. MILLER, H.C. MORENO, *Non-abelian groups in which every subgroup is abelian*, Trans. Amer. Math. Soc., **4** (1909), 398-404.

[7] M.L. NEWELL, *Supplements in abelian-by-nilpotent groups*, J. London Math. Soc. (6), **11** (1975), 74-80.

[8] D.J.S. ROBINSON, *Groups in which normality is a transitive relation*, Proc. Cambridge Philos. Soc., **60** (1964), 21-38.

[9] D.J.S. ROBINSON, *On finitely generated soluble groups*, Proc. London Math. Soc. (3), **15** (1965), 508-516.

[10] D.J.S. ROBINSON, *Groups which are minimal with respect to normality being intransitive*, Pac. J. Math., **31 (3)** (1969), 777-785 .

[11] D.J.S. ROBINSON, *Finiteness conditions and generalized soluble groups*, Springer Verlag, New York-Heidelberg-Berlin, 1972.

[12] D.I. ZAICEV, *The complementation of subgroups in extremal groups. Investigation of groups with prescribed property of subgroups*, Mathematics Institute of the Academy of Sciences of the Ukrainian (SSSR), Kiev, 1974 (in Russian).

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M.R. Celentani, U. Dardano
Dipartimento di Matematica e Appl. «R. Caccioppoli»
Università di Napoli
Via Cintia, Compl. Monte S. Angelo
I-80126 Napoli
Italy