Note di Matematica Vol. 19 - n. 1, 19-24 (1999)

# OVALS WITH 2-TRANSITIVE GROUPS FIXING A POINT

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## 1 Introduction

Let $\pi$ be a projective plane of order $n, \Omega$ an oval of $\pi$ and $G$ a collineation groups of $\pi$ leaving $\Omega$ invariant. When $G$ acts 2-transitively on the points of $\Sigma$, the structure of $G$ is known for odd $n$ [16], while for even $n$ the structure of $G$ has been extensively investigated but it is not yet determinated (e.g. see [7]). Recently Biliotti, Jha and Johnson investigated the possible structure of $G$ when $n$ is even, $G$ fixes a point of $\Omega$ and acts 2-transitively on the remaining points [2]. The aim of this paper is to carry on an analogous investigation in the case of $n$ odd.

So throughout the paper $\pi$ denotes a finite projective plane of odd order $n$. We refere to [8] for standard notions about projective planes. An oval $\Omega$ of $\pi$ is a set of $n+1$ points of $\pi$, no three of which are collinear. A line $r$ of $\pi$ is called an external line, a tangent or a secant of $\Omega$ according to whether $|r \cap \Omega| = 0, 1$ or 2. It is well known that a finite group $G$ possesses a unique maximal normal semisimple subgroup, which is called the layer of $G$ and is denoted by $L(G)$. If $F(G)$ is the Fitting subgroup of $G$ then $F^*(G) = L(G)F(G)$ is called the generalized Fitting subgroup of $G$.

Below we give some results which will be useful in the following.

Denote by $G$ a collineation group of $\pi$ that fixes a point $P$ on $\Omega$.

**Proposition 1** *An elementary abelian 2-subgroup of G has order at most 4.*

For a proof see [4] Result 1.4 (3).

**Proposition 2** *Let $O(G)$ be the maximal normal subgroup of G of odd order. Suppose that G is not irreducible over $\pi$ and put $\overline{G} = G/O(G)$. Then one of the following holds:*

*1. $\overline{G}$ is a 2-group with $m(\overline{G}) \leq 2$;*

*2. $\overline{G}$ is a $\{2,3\}$-group and $F^*(\overline{G}) = F(\overline{G}) \simeq Q_8, Q_8 \circ Z_{2^n}, Q_8 \circ D_{2^n}$, or $Q_8 \times Z_{2^n}$;*

*3. $\overline{G}$ is a $\{2,5\}$-group and $F^*(\overline{G}) = F(\overline{G}) \simeq Q_8 \circ D_8$ or $U_{64}$;*

*4. $\overline{G}$ is a $\{2,3,5\}$-group and $F^*(\overline{G}) = F(\overline{G}) \simeq Q_8 \circ D_8$;*

*5. $F^*(\overline{G}) \simeq SL(2,q) \times Z_{2^n}, SL(2,q) \circ Z_{2^n}, SL(2,q) \circ D_{2^n}$ or $F^*(\overline{G}) \simeq (A_7/Z_2) \times Z_{2^n}, (A_7/Z_2) \circ Z_{2^n}$ or $(A_7/Z_2) \circ D_{2^n}$.*

For a proof use [6] Theorem 5.6 (2) and Proposition 1.

Suppose $G$ acts as a finite doubly transitive permutation group on $\Omega - \{P\}$. Let us observe that, by Proposition 2, $F^*(\overline{G})$ does not contain any normal non abelian simple subgroup. So the unique minimal normal subgrouo of $G$ is elementary abelian. Therefore we are interested in 2-transitive permutation groups of the form $H = H_0A$ where $H_0$ denotes a one point stabilizer, $A$ is the minimal normal subgroup of $H, A$ is elementary abelian, $|A| = p^h$ with $p$ a prime. We may regard $A$ as a vector space over a suitable subfield of $GF(p^h)$ on which $H_0$ acts as a transitive linear group. Denote by $R$ the maximal solvable normal subgroup of $H$ and by $E(H)$ the group generated by all the minimal normal subgroups of $H/R$.

**Proposition 3** *If the minimal normal subgroup of $H$ is elementary abelian then either $H$ is solvable or $H$ has exactly one non solvable composition factor and $E(H)$ is non abelian simple.*

For a proof see [10] Satz 4 and [11] Theorem 6.1.

**Proposition 4** *Let $H$ be a doubly transitive solvable permutation group of odd degree $n$, then one of the following occurs:*
*1. $H \leq \Gamma L(1,n)$ and $n = p^h$*
*2. $SL(2,3) \lhd H_0$ and $n = 3^2, 5^2, 7^2, 11^2, 23^2$ or $3^4$.*

For a proof see [13].

**Proposition 5** *Let $H$ be a doubly transitive permutation group of odd degree $n$ so that the minimal normal subgroup is elementary abelian. If $E(H) \simeq PSL(2,q)$, with $q$ odd and $q \geq 5$, then one of the following occurs:*
*1. $SL(2,q) \lhd H_0$ and $n = q^2$*
*2. $SL(2,5) \lhd H_0$ and $n = 5^2, 11^2, 19^2, 29^2, 59^2$ or $3^4$*
*3. $SL(2,13) \lhd H_0$ and $n = 3^6$.*

For a proof use [11] Theorem 6.4 and the fact that the minimal normal subgroup of $H$ is elementary abelian.

**Proposition 6** *Let $G$ be a collineation group of $\pi$ which preserves an oval $\Omega$ and fixes a point $P$ of $\Omega$. If $G$ acts 2-transitively on $\Omega - \{P\}$ and $E(G)$ is non abelian simple, then $E(G) \simeq PSL(2,q)$ with $q$ odd and $q \geq 5$.*

**Proof.** By Proposition 3, $E(G)$ is the unique non-solvable composition factor of $G$ and it is non abelian simple. Looking at the composition factors of $\overline{G}$ in Proposition 2, we easily obtain the possible structure for $E(G)$. In the cases 1, 2 and 3 all composition factors of $G$ are solvable. In the case 4 the only possible occurrence is $E(G) \simeq A_5$, because $Aut(Q_8 \circ D_8) \simeq S_5$, while in the case 5 $E(G) \simeq PSL(2,q)$ or $A_7$. The case $E(G) \simeq A_7$ can be excluded by [11] Theorem 6.5 taking account that the mininal normal subgroup of $G$ must be elementary abelian of odd order. □

**Lemma 7** *Let $B$ be a collineation group of $\pi$ fixing a Baer subplane $\Phi$ pointwise and preserving an oval $\Omega$ of $\pi$. Then $|B| \leq 2$.*

**Proof.** Let $Q$ be a point of $\Omega - \Phi$, $Q$ lies in a line of $\Phi$ which is a tangent or a secant of $\Omega$, that is $|Q^B| = 1$ or 2 and $Q$ is fixed only by the identity of $B$ since $Q \notin \Phi$. Hence $|B| = |Q^B||B_Q| \leq 2$. □

**Lemma 8** *Let $G$ be a collineation group of $\pi$ of degree $n$ which preserves an oval $\Omega$ and fixes a point $P$ of $\Omega$. Let $A$ be the minimal normal elementary abelian subgroup of $G$. If $G$ acts 2-transitively on $\Omega - \{P\}$ and $n = p^2$ with $p$ a prime, then $|G_{OX}| \leq 2$, where $G_{OX}$ is the stabilizer of the point $X$ of $\Omega - \{P, O\}$.*

**Proof.** The minimal normal subgroup of $G$ is a vector space of dimension two over $GF(p)$ and $G_O$ acts as a transitive linear group on it. Every collineation $\alpha \in G_{OX}$ fixes a subspace of dimension one of $A$ pointwise, that means that $\alpha$ leaves invariant $p+1$ points of $\Omega$. Therefore $\alpha$ fixes a Baer subolane of $\pi$ pointwise. By Lemma 7 it following $|G_{OX}| \leq 2$.

Now we can prove the main theorem of the paper. □

**Theorem 9** *Let $\pi$ be a projective plane of odd oder $n$ admitting a collineation group $G$ which preserves an oval $\Omega$ of $\pi$ and fixes a point $P$ of $\Omega$. Suppose $G$ acts 2-transitively on $\Omega - \{P\}$ and $G_O$ denotes the stabilizer of the point $O$ of $\Omega - \{P\}$, then $n$ is a prime power $p^h$ and one of the following holds:*

| | | | | | |
|---|---|---|---|---|---|
| (1) | $G \leq \Gamma L(1,n)$ | $n = p^h$ | | | |
| (2) | $SL(2,5) \lhd G_O$ | $n = p^2$ | and | $p =$ | $29, 59$ |
| (3) | $SL(2,3) \lhd G_O$ | $n = p^2$ | and | $p =$ | $5, 7, 11, 23$ |
| | | $n = p^4$ | and | $p =$ | $3$ |
| (4) | $G_O \simeq SL(2,13)$ | $n = p^6$ | and | $p =$ | $3$ |

**Proof.** When $G$ is a solvable group we may apply Proposition 4 to obtain (1) and (3). Note that the case $n = 3^2$ of Proposition 4.1 cannot occur. Indeed by [13] there are two non isomorphic groups of degree $3^2$, namely in Huppert's notation

$$G^O_{3^2,1} = \{A, B, C, D, \} \qquad G^O_{3^2,2} = \{A, B, C\}$$

where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, D = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

over $GF(3)$.

Since $G_{OX}$ has order 3 inside $G^O_{3^2,2}$, this case cannot occur by Lemma 8.

Now we consider $E(G)$ non abelian simple. By Proposition 1, $E(G) \simeq PSL(2,q)$, so we have (2) and (4) by Proposition 5. Indeed the following cases of Proposition 5 cannot occur.

1. $SL(2,q) \lhd G_0$ and $n = q^2$ where $q$ is the power of a prime $p$.

In this case $SL(2,q)$ acts in the natural manner over a 2-dimensional vector space $A$ over $GF(q)$. A $p$-element $\alpha \in SL(2,q)$ fixes a 1-dimensional subspace of $A$ pointwise and hence $\alpha$ is a Baer collineation. Since $p$ is an odd prime this contradicts Lemma 7.

2. $SL(2,5) \lhd G_0$ and $n = 11^2, 19^2, 3^4$.

(2.i) $n = 11^2$.

$G_{OX}$ has order 5, contrary to Lemma 8. (see [8] 5.2.5)

(2.ii) $n = 19^2$

By [18] $G_{OX}$ has an abelian subgroup of order 9 and so it cannot be regular in $\Omega - \{P, O\}$ by [17], Theorem 18.1. Therefore there is an element $\alpha$ in $G_O$ of period at least 3 that fixes a point of $\Omega - \{P, O\}$. By Lemma 8 we have a contradiction.

(2.iii) $n = 3^4$.

By [1], §6 we have that $G_O$ has an element $\alpha$ of period 3 which fixes a 2-dimensional vector space over $GF(3)$ pointwise. Hence $\alpha$ is a Baer collineation and we can exclude this case by Lemma 7.

This complete the proof. □

Below we list the admissible degrees for which some, but not all, of the corresponding groups can be excluded. For sake of completeness we list, for each of these degrees, all the possible groups.

1) $n = 5^2$:

$$G^O_{5^2,1} = \{A,B,C,D,2E\} \quad G^O_{5^2,2} = \{A,B,C,2E\} \quad G^O_{5^2,3} = \{A,B,C\}$$

where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}, D = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$$

over $GF(5)$.

$G_{OX}$ has order 4 inside $G^O_{5^2,1}$ contrary to Lemma 8.

2) $n = 7^2$:

$$G^O_{7^2,1} = \{A,B,C,D,2E\} \quad G^O_{7^2,2} = \{A,B,C,D\}$$

where

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}, B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & -2 \\ 4 & -1 \end{pmatrix}, D = \begin{pmatrix} 3 & -1 \\ 3 & -3 \end{pmatrix}$$

over $GF(7)$.

$G_{OX}$ has order 3 inside $G^O_{7^2,1}$ and this contradicts Lemma 8.

3) $n = 3^4$:

$$G^O_{3^4,1} = \{A,B,C,D,F,G\} \quad G^O_{3^4,2} = \{A,B,C,D,F,G^2\} \quad G^O_{3^4,3} = \{A,B,C,D,F\}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, F = \begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

over $GF(3)$.

Let us note that in $G^O_{3^4,2}$ there is a ciclic group of order 4, genereted by $G^2$, that fixes a subspace of dimension two pointwise and hence it induces a baer collineation in $\pi$. Hence by Lemma 7 the cases $G^O_{3^4,2}$ and $G^O_{3^4,1}$ cannot occur.

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Received January 19, 1999
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