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A DETERMINISTIC INVENTORY SYSTEM WITH WEIBULL DISTRIBUTION DETERIORATION AND RAMP TYPE DEMAND RATE

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Abstract: *A continuous order-level inventory model is developed for deteriorating items with a ramp type demand function of time. A two parameter Weibull distribution is taken to represent the time to deterioration. The model is solved analytically by enumerating two possible shortage models to obtain the optimal solution of the problem. The method is illustrated by two numerical examples and sensitivity analysis of the optimal solutions with respect to the parameters of the system is carried out.*

***Keywords**: Inventory, Deterioration, Weibull distribution, Ramp Type Demand*

1. Introduction

In formulating inventory models, two facts of the problem have been of growing interest, one being the deterioration of items, the other being the variation in the demand rate. Time-varying demand patterns are usually used to reflect sales in different phases of the product life cycle in the market. For example, the demand for inventory items increases over time in the growth phase and decreases in the decline phase. An inventory model with a linear trend in demand was initially developed by [11]. After that, many researchers (see for example, [10], [21],[19], [28], [15], [3], [2], [14], [33], [8], [18], [20], [16], [17], [4], [7], [31], [27], [6] and [29]) have incorporated a time varying demand rate into their models for deteriorating items with or without shortages under a variety of circumstances.
The effect of deterioration of physical goods cannot be disregarded in many inventory systems. Deterioration is defined as decay, damage and spoilage. Food items, photographic films, drugs, pharmaceuticals, chemicals, electronic components and radioactive substances are some

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examples in which sufficient deterioration may occur during the normal storage period of the items and consequently this loss must be taken into account while analyzing the inventory system. One of the earliest research works on a continuously decaying inventory for a constant demand was analyzed in [12]. An order-level inventory model for deteriorating items with a constant rate of deterioration was considered in [30]. An order-level inventory model was developed in [1] by correcting and modifying the errors in the analysis of [30]. An inventory model with the assumptions of variable deterioration rate of two-parameter Weibull distribution, constant demand rate and no shortages was formulated in [9]. A more general model was developed in [14] by taking a time-proportional deterioration rate, finite production rate proportional to the demand rate, time-dependent demand rate and shortages. An extensive survey of literature concerning inventory models for deteriorating items was discussed in [24] and [26]. It was observed in [5] while studying the difficulties of fitting empirical data to mathematical distributions, that both leakage failure of dry batteries and life expectancy of ethical drugs could be expressed in terms of Weibull distribution.

In these cases the rate of deterioration increased with age or longer the items remain unused, higher the rate at which they failed. The work [5] prompted [9] to develop an inventory model for deteriorating items with variable rate of deterioration. They used the two parameter Weibull distribution to represent the distribution of the time to deterioration. The instantaneous rate function Z(t) for a two parameter Weibull distribution is given by:

$$Z(t) = \alpha \beta t^{(\beta-1)}$$

where α is the scale parameter, $\alpha > 0$; β is the shape parameter, $\beta > 0$; t is the time of deterioration, $t > 0$. This model was further generalized in [25] by taking a three-parameter Weibull distribution deterioration rate. An inventory model with a finite rate of replenishment and a two parameter Weibull distribution deterioration rate was developed in [23]. The models developed by [9], [23] and [25] did not allow shortages in inventory and used a constant demand rate. Recently an inventory model with a time-quadratic demand rate and shortages was developed in [13]. They also used a two parameter Weibull distribution to represent the distribution of the time to deterioration.

An order-level inventory model for deteriorating items, where the demand rate is a ramp type function of time was discussed in [22]. This type of demand rate is generally seen in the case of any new brand of consumer goods coming to the market. The demand rate for such items increases with time up to a certain time and then ultimately stabilizes and becomes constant. It is believed that this type of demand rate is quiet realistic. An order level inventory system for deteriorating items with a ramp type demand function of time and two possible types of shortages was developed in [32].

In the present paper, we have developed three continuous order-level inventory models for deteriorating items with shortages. In all these models, the demand rate is taken as a ramp type function of time and deterioration rate is assumed to follow a two-parameter Weibull distribution. Analytical solutions of the models are discussed and are illustrated with the help of numerical examples. Sensitivity of the optimal solutions with respect to changes in different parameter values is also examined.

2. Notations and Modeling Assumptions

The mathematical models of the deterministic inventory replenishment problems are developed with the following notations and assumptions:

i. The demand rate R(t) is assumed to be a ramp type function of time :
$R(t) = D_0[\, t - (t - \mu)\, H(\, t - \mu)\,], \qquad D_0 > 0$
Where $H(t - \mu)$ is the well known Heaviside's function defined as follows:
$H(t - \mu) = 1, \quad t \geq \mu$
$\qquad\qquad = 0, \quad t < \mu$

ii. C_h is the inventory holding cost per unit per unit of time.

iii. *A* is the replenishment cost per cycle.

iv. C_s is the shortage cost per unit per unit of time.

v. C_d is the unit deterioration cost.

vi. Replenishment is instantaneous and lead time is zero.

vii. *T* is the fixed length of each ordering cycle.

viii. *S* is the maximum inventory level of each ordering cycle.

ix. $I(t)$ is the on-hand inventory at time t over [0 , T].

x. Shortages are allowed and are fully backlogged.

xi. The distribution of the time to deterioration follows a two parameter Weibull distribution:
$Z(t) = \alpha\, \beta\, t^{(\beta-1)}$
where α is the scale parameter, $\alpha > 0$; β is the shape parameter, $\beta > 0$; *t* is the time of deterioration, $t > 0$.

3. Mathematical Models and its analysis

The objective of the inventory problem here is to determine the optimal order quantity so as to keep the total relevant cost minimum. Based on whether the inventory starts with shortages or not, there are three possible models under the assumptions described above.

3.1 Model I: The Inventory model starts without shortages

In this subsection, we will analyze the deterministic inventory model for deteriorating items where the inventory starts without shortages. Replenishment is made at time $t = 0$ when the inventory level is at its maximum, *S*. From $t = 0$ to $t = t_1$ time units, the inventory level decreases due to demand and deterioration. At time t_1, the inventory level reaches zero, thereafter, shortages are allowed to occur during the time interval (t_1, T) and all of the demand during this period is backlogged. The total number of backlogged items is replaced by the next replenishment.

The inventory level of the system at any time *t* over [0, *T*] can be described by the following equations:

$$\frac{dI(t)}{dt} + \alpha \beta t^{(\beta-1)} I(t) = -R(t)\,, \quad I(0) = S\,,\ I(t_1) = 0, \quad 0 \leq t \leq t_1$$

$$\frac{dI(t)}{dt} = -R(t)\,,\; I(t_1) = 0,\; t_1 \le t < T$$

Let us assume that $0 < \mu < t_1$;
Therefore, the above governing equations become:

$$\frac{dI(t)}{dt} + \alpha\beta t^{(\beta-1)}\, I(t) = -D_o t\,,\quad I(0) = S,\quad 0 \le t \le \mu \tag{1}$$

$$\frac{dI(t)}{dt} + \alpha\beta t^{(\beta-1)}\, I(t) = -D_o \mu,\quad I(t_1) = 0,\quad \mu \le t \le t_1 \tag{2}$$

$$\frac{dI(t)}{dt} = -D_o \mu,\quad t_1 \le t < T \tag{3}$$

From differential equation (1), we have:

$$I(t)\, e^{\alpha t^{\beta}} = -D_0 \int_0^t t\, e^{\alpha t^{\beta}} dt + S$$
$$= -D_0 \int_0^t t\left(1 + \alpha t^{\beta} + \frac{\alpha^2\, t^{2\beta}}{2}\right) dt + S$$

neglecting α^3 and higher powers of α (since $0 < \alpha << 1$).

$$I(t) = S - \frac{D_0 t^2}{2} - \alpha S t^{\beta} + $$
$$+ \frac{\alpha\, D_0\, \beta\, t^{(\beta+2)}}{2(\beta+2)} + \frac{\alpha^2\, S\, t^{2\beta}}{2} - \frac{(\alpha\,\beta)^2\, D_0 t^{(2\beta+2)}}{4(\beta+1)(\beta+2)}\,,\; 0 \le t \le \mu \tag{4}$$

From differential equation (2), we have:

$$I(t)\, e^{\alpha t^{\beta}} = -D_0\, \mu \int_{\mu}^t t\, e^{\alpha t^{\beta}} dt + C_1$$
$$= -D_0\, \mu \int_{\mu}^t \left(1 + \alpha t^{\beta} + \frac{\alpha^2\, t^{2\beta}}{2}\right) dt + C_1$$

neglecting α^3 and higher powers of α (since $0 < \alpha << 1$).

Now $I(t_1) = 0$ gives:

$$C_1 = D_0 \mu \int_{\mu}^{t_1} (1+\alpha t^{\beta} + \frac{\alpha^2 \; t^{2\beta}}{2}) \, dt$$

$$I(t) = D_0 \mu \; [A_1 - t - \left(\alpha \; t_1 \; + \frac{\alpha^2 \; t_1^{(\beta+1)}}{(\beta+1)} \right) t^{\beta} + + \frac{\alpha \; \beta \; t^{\;(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 \; t_1 \; t^{2\beta}}{2} - \frac{(\alpha \; \beta \;)^2 \; t^{\;(2\beta+1)}}{(\beta+1)(2\beta+1)}] \tag{5}$$

where $A_1 = t_1 + \frac{\alpha \, t_1^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 \, t_1^{(2\beta+1)}}{2(2\beta+1)}$, $\mu \le t \le t_1$

From differential equation (3), we have:

$$I(t) = - D_0 \mu \; (t - t_1), \; t_1 \le t < T \tag{6}$$

Now the differential equations (1) and (2) should give the same values of I(t) at $t = \mu$. So after simplification we have:

$$S = D_0 \mu \; [A_1 - \frac{\mu}{2} - \frac{\alpha \; \mu^{\;(\beta+1)}}{(\beta+1)(\beta+2)} - \frac{\alpha^{\;2} \; \mu^{\;(2\beta+1)}}{2(2\beta+1)(2\beta+2)}]$$

The total number of items deteriorated during [0, t_1] is:

D_T = Initial inventory – Total demand during [0, t_1]

$$= S - [\int_0^{\mu} D_0 \; t \; dt \; + \int_{\mu}^{t_1} D_0 \; \mu \; dt \;]$$

$$= S - D_0 \mu \; (t_1 - \frac{\mu}{2}) \tag{7}$$

The inventory accumulated over the period [0, t_1] is:

$$H_T = \int_0^{t_1} I(t) \; dt \; = \int_0^{\mu} I(t) \; dt \; + \int_{\mu}^{t_1} I(t) \; dt \; = I_1 + I_2 \tag{8}$$

Using equations (4) and (5) and evaluating the integrals, we get:

$$I_1 = S\mu - \frac{D_0 \mu^3}{6} - \alpha S \; \frac{\mu^{\beta+1}}{(\beta+1)} + \frac{\alpha \; D_0 \; \beta \; \mu^{\;(\beta+3)}}{2(\beta+2)(\beta+3)} + \frac{\alpha^2 \; S \; \mu^{2\beta+1}}{2(2\beta+1)} - \frac{(\alpha \; \beta \;)^2 \; D_0 \mu^{\;(2\beta+3)}}{4(\beta+1)(\beta+2)(2\beta+3)},$$

$$I_2 = D_0\,\mu\,[\,A_1\,(t\,-\mu)\,-\frac{(t_1{}^2-\mu^2)}{2}-\left(\alpha\,t_1\,+\frac{\alpha^2\,t_1{}^{(\beta+1)}}{(\beta+1)}\right)\frac{(t_1{}^{\beta+1}-\mu^{\beta+1})}{(\beta+1)}$$

$$+\,\frac{\alpha\,\beta\,(t_1{}^{\beta+2}-\mu^{\beta+2})}{(\beta+1)(\beta+2)}+\frac{\alpha^2\,t_1\,(t_1{}^{2\beta+1}-\mu^{2\beta+1})}{2(2\beta+1)}-\frac{(\alpha\,\beta\,)^2\,(t_1{}^{2\beta+2}-\mu^{2\beta+2})}{(\beta+1)(2\beta+1)(2\beta+2)}\,]$$

Using equation (6), the shortage accumulated during the period $[t_1, T)$ is:

$$B_T = -\int_{t_1}^{T} I(t)dt \;=\; \frac{D_0\;\mu\;(T-t_1)^2}{2}, \tag{9}$$

Using equations (4) to (9), we can get the total relevant cost of the system during the time interval $[0, T)$ which is:

$$X = A + C_d\,D_T + C_h\,H_T + C_s\,B_T$$

Therefore average total cost of the system per unit time is:

$$C_1(t_1, T) = \frac{X}{T} \tag{10}$$

To minimize C_1 the optimal value of t_1 and T (denoted by t_1* and T*) can be obtained by solving the equations:

$$\frac{\partial C_1}{\partial t_1} = 0,\ \frac{\partial C_1}{\partial T} = 0,$$

provided t_1* and T* satisfy the following convexity condition:

$$\begin{pmatrix} \dfrac{\partial^2 C_1}{\partial t_1{}^2} & \dfrac{\partial^2 C_1}{\partial t_1 \partial T} \\ \dfrac{\partial^2 C_1}{\partial T \partial t_1} & \dfrac{\partial^2 C_1}{\partial T^2} \end{pmatrix} \text{ is positive definite.}$$

The total backorder amount at the end of the cycle from equation (8) is $D_0\mu(T^* - t_1^*)$. Therefore the optimal order quantity, Q* is:

$$Q^* = S^* + D_0\,\mu\,(T^* - t_1^*)$$

3.1 Model II: The Inventory model starts with shortages

Here we have considered the continuous deterministic inventory model for deteriorating items, where the inventory is allowed to start with shortages. Depending on the procurement time t_1, two different circumstances may arise: (*i*) $\mu < t_1$ and (*ii*) $\mu > t_1$. The inventory system starts with zero inventory level at $t = 0$ and shortages are permitted to accumulate up to t_1. Replenishment is done at time t_1. The quantity received at t_1 is used partly to make up for the shortages accumulated in the previous cycle from time 0 to t_1. The rest of the procurement accounts for the demand and deterioration in $[t_1, T]$. The inventory level gradually falls down to zero at time T.
The inventory level of the system at time t over the period $[0, T]$ can be modeled by the following equations:

$$\frac{dI(t)}{dt} = - R(t)\ , \ 0 \le t < t_1$$

$$\frac{dI(t)}{dt} + \alpha \beta t^{(\beta-1)}\ I(t) = - R(t)\ , \ t_1 \le t \le T$$

Situation I: ($\mu < t_1$)

In this situation, the above equations become:

$$\frac{dI(t)}{dt} = - D_o t, \quad I(0) = 0, \ 0 \le t \le \mu \tag{11}$$

$$\frac{dI(t)}{dt} = - D_o \mu, \quad \mu \le t < t_1 \tag{12}$$

$$\frac{dI(t)}{dt} + \alpha \beta t^{(\beta-1)}\ I(t) = - D_o \mu, \quad I(T) = 0, \ t_1 \le t \le T \tag{13}$$

The solutions of the differential equations (11) – (13) are:

$$\text{I(t)} = - \frac{D_0 t^2}{2}, \ 0 \le t \le \mu \tag{14}$$

$$= D_0 \mu \left(\frac{\mu}{2} - t \right), \mu \le t < t_1 \tag{15}$$

$$= D_0 \mu \left[A_2 - t - \left(\alpha T + \frac{\alpha^2 T^{(\beta+1)}}{(\beta+1)} \right) t^{\beta} + \frac{\alpha \beta t^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 T t^{2\beta}}{2} - \frac{(\alpha \beta)^2 t^{(2\beta+1)}}{(\beta+1)(2\beta+1)} \right]$$

$$A_2 = T + \frac{\alpha T^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 T^{(2\beta+1)}}{2(2\beta+1)}, \ t_1 \le t \le T \tag{16}$$

Since $I(t_1) = S$,

$$S = D_0 \mu \left[A_2 - t_1 - \left(\alpha T + \frac{\alpha^2 T^{(\beta+1)}}{(\beta+1)} \right) t_1^{\beta} + \frac{\alpha \beta t_1^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 T t_1^{2\beta}}{2} - \frac{(\alpha \beta)^2 t_1^{(2\beta+1)}}{(\beta+1)(2\beta+1)} \right]$$

As discussed in the previous Situation I, using equations (14) – (16) we have

$$D_T = S - D_0 \mu (T - t_1) \tag{17}$$

$$H_T = D_0 \mu \left[A_2 (T - t_1) - \frac{(T^2 - t_1^2)}{2} - \left(\alpha T + \frac{\alpha^2 T^{(\beta+1)}}{(\beta+1)} \right) \frac{(T^{\beta+1} - t_1^{\beta+1})}{(\beta+1)} \right.$$
$$\left. + \frac{\alpha \beta (T^{\beta+2} - t_1^{\beta+2})}{(\beta+1)(\beta+2)} + \frac{\alpha^2 T (T^{2\beta+1} - t_1^{2\beta+1})}{2(2\beta+1)} - \frac{(\alpha \beta)^2 (T^{2\beta+2} - t_1^{2\beta+2})}{(\beta+1)(2\beta+1)(2\beta+2)} \right] \tag{18}$$

$$B_T = \frac{D_0 \mu}{6} \{ \mu^2 + 3 t_1 (t_1 - \mu) \} \tag{19}$$

Using equations (17) – (19) the average total cost of the system per unit time is:

$$C_2(t_1, T) = \frac{\left(A + C_d D_T + C_h H_T + C_s B_T \right)}{T} \tag{20}$$

To minimize C_2 the optimal value of t_1 and T can be obtained by solving the equations:

$$\frac{\partial C_2}{\partial t_1} = 0, \quad \frac{\partial C_2}{\partial T} = 0$$

provided t_1^* and T^* satisfy the following convexity condition:

$$\begin{pmatrix} \frac{\partial^2 C_2}{\partial t_1^2} & \frac{\partial^2 C_2}{\partial t_1 \partial T} \\ \frac{\partial^2 C_2}{\partial T \partial t_1} & \frac{\partial^2 C_2}{\partial T^2} \end{pmatrix} \text{ is positive definite.}$$

The total backorder amount for the entire cycle from equations (14) and (15) is:

$\frac{D_0\mu^2}{2} + D_0 \mu\ (t_1^* -\ \mu\ ).$

Therefore the optimal order quantity, Q^* is:

$$Q^* = S^* + \frac{D_0\mu^2}{2} + D_0 \mu\ (t_1^* -\ \mu\ )$$
$$= S^* + D_0 \mu\ (t_1^* - \frac{\mu}{2})$$

Situation II: ($\mu > t_1$)

In this situation, the above equations become:

$$\frac{dI(t)}{dt} = - D_o\, t, \quad 0 \le t < t_1 \tag{21}$$

$$\frac{dI(t)}{dt} + \alpha\, \beta\, t^{(\beta-1)}\ I(t) = - D_o\, t,\ t_1 \le t \le \mu \tag{22}$$

$$\frac{dI(t)}{dt} + \alpha\, \beta\, t^{(\beta-1)}\ I(t) = - D_o\, \mu,\ \mu \le t \le T \tag{23}$$

The solutions of the differential equations (21) – (23) with the boundary conditions I(0) = I(T) = 0 are:

$$I(t) = - \frac{D_0 t^2}{2},\ 0 \le t < t_1 \tag{24}$$

$$= D_0 \left[\mu A_2 - A_3 - \frac{t^2}{2} + \alpha\mu \left\{ \frac{\mu}{2} - T - \frac{\alpha}{(\beta+1)}\left(T^{\beta+1} - \frac{\mu^{(\beta+1)}}{(\beta+2)} \right) \right\} t^{\beta} + \frac{\alpha\ \beta\ t^{(\beta+2)}}{2(\beta+2)} + \frac{\alpha^2\ \mu(2T-\mu)t^{2\beta}}{4} - \frac{(\alpha\ \beta)^2\ t^{(2\beta+2)}}{4(\beta+1)(\beta+2)} \right],\ t_1 \le t \le \mu \tag{25}$$

$$= D_0 \mu\ \left[A_2 - t - \left(\alpha\ T + \frac{\alpha^2\ T^{(\beta+1)}}{(\beta+1)} \right) t^{\beta} + \frac{\alpha\ \beta\ t^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2\ T\, t^{2\beta}}{2} - \frac{(\alpha\ \beta)^2\ t^{(2\beta+1)}}{(\beta+1)(2\beta+1)} \right]$$

$$\mu \le t \le T \tag{26}$$

where $A_2 = T + \frac{\alpha T^{(\beta+1)}}{(\beta+1)} + \frac{\alpha^2 T^{(2\beta+1)}}{2(2\beta+1)}$ and $A_3 = \frac{\mu^2}{2} + \frac{\alpha\ \mu^{(\beta+2)}}{(\beta+1)(\beta+2)} + \frac{\alpha^2\ \mu^{(2\beta+2)}}{2(2\beta+1)(2\beta+2)}$

Since I(t_1) =S, from equation (25):

$$S = D_0 \left[\mu A_2 - A_3 - \frac{t_1^{\,2}}{2} + \alpha \mu \left\{ \frac{\mu}{2} - T - \frac{\alpha}{(\beta+1)} \left(T^{\beta+1} - \frac{\mu^{(\beta+1)}}{(\beta+2)} \right) \right\} t_1^{\beta} \right.$$

$$\left. + \frac{\alpha \beta t_1^{(\beta+2)}}{2(\beta+2)} + \frac{\alpha^2 \mu (2T - \mu) t_1^{2\beta}}{4} - \frac{(\alpha \beta)^2 t_1^{(2\beta+2)}}{4(\beta+1)(\beta+2)} \right]$$

Proceeding as in the earlier case, using equations (24) – (26) we get:

$$D_T = S + \frac{D_0}{2} (\mu^2 + t_1^{\,2} - 2 \mu T) \tag{27}$$

$$H_T = \int_{t_1}^{\mu} I(t)\, dt + \int_{\mu}^{T} I(t)\, dt \tag{28}$$

$$B_T = \frac{D_0 t_1^{\,3}}{6} \tag{29}$$

Therefore average total cost of the system per unit time is:

$$C_3(t_1, T) = \frac{\left(A + C_d D_T + C_h H_T + C_s B_T\right)}{T} \tag{30}$$

To minimize C_3 the optimal value of t_1 and T can be obtained by solving the equations:

$$\frac{\partial C_3}{\partial t_1} = 0, \; \frac{\partial C_3}{\partial T} = 0$$

provided t_1* and T* satisfy the convexity condition. The total backorder amount for the cycle from equation (24) is:

$$\frac{D_0 t_1 *^2}{2}.$$

Therefore the optimal order quantity, Q* is:

$$Q^* = S^* + \frac{D_0 t_1 *^2}{2}$$

4. Numerical Examples

Example 1: To illustrate the theory developed above, the following numerical example has been considered. Let the input parameters are as follows:

A= 1500, C_h= 3, C_s= 15, C_d= 5, D_0= 100, α= .001, β= 2, μ= 0.8

For α = .001, β = 2 > 1 the deterioration rate of the items on stock gradually increases with time. Here Model I describes an inventory model which starts without shortages and Model II describes an inventory model which starts with shortages.
In Model I, t_1* represents the optimal point of time when stock vanishes due to continuous depletion as a result of demand and deterioration while shortages start occurring. However, in Model II, the inventory starts with shortages and t_1* represents the optimal point of time when replenishment takes place. Then the shortage amount is met from the replenished items.
Applying the procedure developed in the previous section, the optimal solutions for Model I and Model II are those given in Table1. It is numerically verified that these solutions satisfy the convexity condition.

Table 1. Comparison of optimal solutions of the inventory systems of Example 1(μ = 0.8)

Optimal solution	*Model I*	*Model II*	
		Situation I: $\mu < t_1$	*Situation II: $\mu > t_1$*
t_1*	3.170827	0.98707	....
T*	3.812628	3.896163	....
S*	222.516	234.058	....
Q*	273.86	281.024	....
C*	770.162 (C_1*)	709.88(C_2*)	....

From Table1 we observe that the ordering strategy for Model II (Situation I: $\mu < t_1$) is more economical than that for Model I. Though the optimal order quantity Q*, the maximum inventory level S* and the optimal cycle length T* are greater in case of Model II, the average total cost of the system C_2* of Model II is $\left(\frac{770.162-709.88}{770.162}\cdot 100\right)$ ≈ 7.83 % less than the average total cost of the system C_1* of Model I.
Therefore the percent benefit of Model II over Model I is 7.83 % when $\mu < t_1$. Hence, Model II is a better optimal policy. The graphs showing variation of the inventory levels for Model I, Model II with time as obtained in Table1 are shown below by using MATLAB.

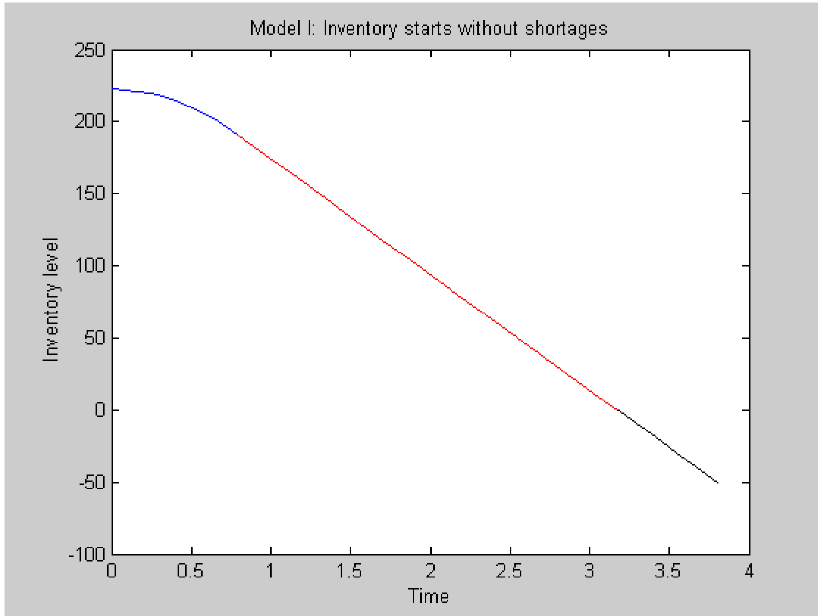

Figure 1. Model I ($\mu < t_1$) Inventory starts without shortages.

Figure1 shows that the inventory starts without shortages. Replenishment is done at time t = 0 and the optimal order quantity Q^* for each ordering cycle is 273.86 units. The maximum inventory level after replenishment and clearing all the backlogs is S^* = 222.516 units. During the period $[0, \mu)$ the demand rate of the items is D_0t and the items follow Weibull distribution deterioration. After $t = \mu = 0.8$ the demand rate becomes constant and equal to $D_0\mu$. The stock vanishes at $t_1^* = 3.170827$ units and then shortages start. The optimal cycle length of the model is $T^* = 3.812628$ units. The minimum average total cost of the inventory model per unit time is $C_1^* = 770.162$ units.

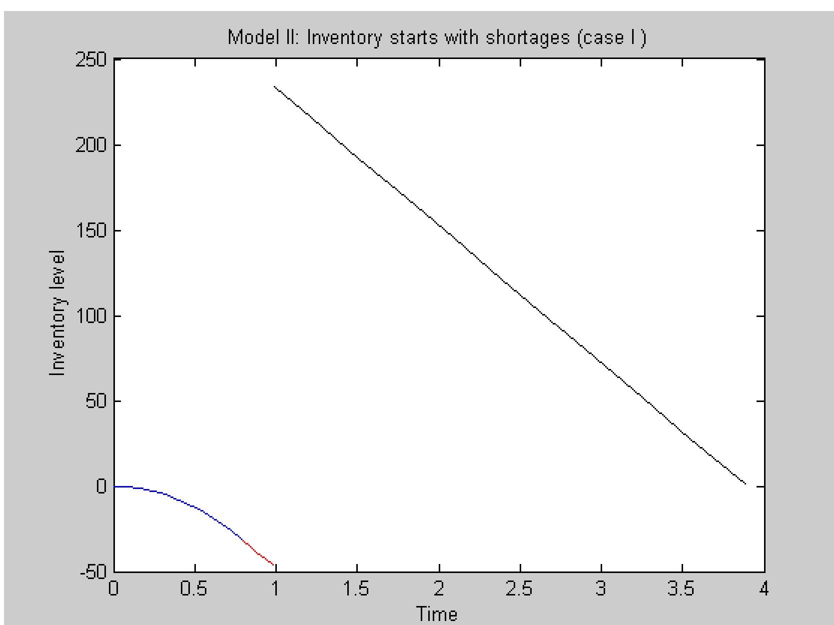

Figure 2. Model II ($\mu < t_1$) Inventory starts with shortages.

From Figure 2 we see that the inventory starts with shortages. During the period [0, μ) the demand rate of the items is D_0t. After t= μ= 0.8 the demand rate becomes constant and equal to $D_0\mu$. The replenishment is done at time t_1* = 0.98707 units and the optimal order quantity Q* for each ordering cycle is 281.024 units. The shortage amount is met from the replenished stock and the maximum inventory level S* after replenishment is 234.058 units. During the period [t_1, T) the stock decreases due to constant demand rate and Weibull distribution deterioration of the items. At t= T*= 3.896163 units the inventory becomes zero. The minimum average total cost per unit time of the model is C_2*= 709.88 units.

We now study the sensitivity of the optimal solution to changes in the values of the different parameters of Model I and Model II when μ< t_1. The sensitivity analysis is performed by changing the value of each of the parameters by -50 %, -25 %, 25 %, 50 %, taking one parameter at a time and keeping the remaining parameters unchanged. Here we have assumed that insensitive, moderately sensitive, and highly sensitive imply % changes are – 3 to + 3, - 20 to +20 and more respectively.

Table 2 (a). Sensitivity Analysis of Model I (μ = 0.8)

Parameters	***% change***	***% change in***				
		S*	***Q****	***C_1****	***t_1****	***T****
A	50	25.858	25.314	22.956	22.402	22.495
	25	13.604	13.314	12.056	11.8	11.843
	-25	-15.513	-15.173	-13.696	-13.489	-13.524
	-50	-34.039	-33.279	-29.987	-29.64	-29.698
C_h	50	-24.826	-16.985	16.997	-21.603	-15.106
	25	-14.211	-9.819	9.214	-12.356	-8.725
	-25	20.463	14.526	-11.204	17.737	12.865
	-50	53.825	38.963	-25.444	46.49	34.381
C_s	50	3.311	-3.193	2.93	2.874	-2.892
	25	1.952	-1.905	1.727	1.695	-1.724
	-25	-3.045	3.102	-2.692	-2.645	2.807
	-50	-8.465	9.065	-7.48	-7.357	8.197
C_d	50	-0.214	-0.16	0.072	-0.185	-0.142
	25	-0.107	-0.08	0.036	-0.093	-0.071
	-25	0.108	0.081	-0.036	0.094	0.072
	-50	0.216	0.162	-0.072	0.187	0.144
D_0	50	18.095	18.798	21.85	-18.504	-18.548
	25	9.728	10.062	11.511	-10.621	-10.649
	-25	-11.635	-11.919	-13.149	15.45	15.509
	-50	-26.24	-26.728	-28.84	41.074	41.284

Table 2 (b). Sensitivity Analysis of Model I (μ = 0.8)

Parameters	*% change*	*% change in*				
		S*	***Q****	***C_1****	***t_1****	***T****
μ	50	4.999	7.778	19.823	-19.842	-19.889
	25	4.451	5.645	10.822	-11.165	-11.195
	-25	-8.66	-9.455	-12.901	15.771	15.832
	-50	-22.386	-23.553	-28.609	41.517	41.73
α	50	-0.32	-0.234	0.141	-0.443	-0.344
	25	-0.161	-0.118	0.071	-0.222	-0.173
	-25	0.163	0.119	-0.071	0.225	0.175
	-50	0.327	0.239	-0.142	0.452	0.352
β	50	-2.009	-1.55	0.439	-2.163	-1.725
	25	-0.731	-0.562	0.171	-0.806	-0.641
	-25	0.393	0.299	-0.107	0.453	0.358
	-50	0.594	0.45	-0.174	0.699	0.552

A careful study of Table2 reveals the following points:

i. It is seen that the maximum inventory level *S**, the optimal order quantity *Q**, the optimal total cost C_1* and optimal time periods t_1* and *T** are insensitive to changes in the values of the parameters C_d, α, β. These are moderately sensitive to change in the value of the parameter C_s and highly sensitive to changes in the values of the parameters *A*, C_h, D_0 and μ.

Table 3 (a). Sensitivity Analysis of Model II (μ= 0.8 and $\mu < t_1$)

Parameters	*% change*	*% change in*				
		S*	***Q****	***C_2****	***t_1****	***T****
A	50	24.106	24.116	24.491	14.372	21.39
	25	12.656	12.661	12.836	7.544	11.247
	-25	-14.337	-14.341	-14.483	-8.541	-12.781
	-50	-31.257	-31.264	-31.503	-18.615	-27.913
C_h	50	-22.746	-16.308	15.378	9.384	-14.515
	25	-13.07	-9.448	8.373	5.118	-8.402
	-25	18.999	14.04	-10.324	-6.35	12.445
	-50	50.282	37.753	-23.704	-14.682	33.341
C_s	50	3.213	-2.544	3.186	-18.576	-2.286
	25	1.841	-1.568	1.824	-11.037	-1.409
	-25	-2.711	2.721	-2.682	17.72	2.446
	-50	-7.335	8.203	-7.251	50.935	7.376

Table 3 (b). Sensitivity Analysis of Model II (μ= 0.8 and $\mu < t_1$)

Parameters	*% change*	*% change in*				
		S^*	Q^*	C_2^*	t_1^*	T^*
C_d	50	-0.278	-0.251	0.12	-0.068	-0.223
	25	-0.139	-0.126	0.06	-0.034	-0.112
	-25	0.14	0.126	-0.06	0.035	0.112
	-50	0.281	0.254	-0.12	0.07	0.226
D_0	50	20.564	20.557	20.287	-11.69	-17.504
	25	10.873	10.869	10.723	-6.733	-10.072
	-25	-12.557	-12.553	-12.373	9.89	14.735
	-50	-27.784	-27.774	-27.354	26.501	39.311
μ	50	....	....	....	....	....
	25	8.664	8.666	8.531	2.363	-9.094
	-25	-10.924	-10.924	-10.74	1.034	14.123
	-50	-25.42	-25.417	-24.972	9.002	38.41
α	50	-0.325	-0.298	0.207	-0.096	-0.473
	25	-0.163	-0.15	0.104	-0.049	-0.238
	-25	0.165	0.151	-0.104	0.049	0.241
	-50	0.332	0.305	-0.209	0.1	0.485
β	50	-2.777	-2.753	1.006	-1.569	-3.296
	25	-0.943	-0.93	0.369	-0.516	-1.162
	-25	0.43	0.416	-0.207	0.208	0.574
	-50	0.604	0.577	-0.322	0.263	0.836

A careful study of Table 3 reveals the following points:

i. It is seen that the maximum inventory level S^*, the optimal order quantity Q^*, the optimal total cost C_2^* and optimal time period T^* are insensitive to changes in the values of the parameters C_d, α, β. These are moderately sensitive to change in the value of the parameter C_s and highly sensitive to changes in the values of the parameters A, C_h, D_0 and μ.

ii. It is observed that t_1^* is insensitive to changes in the values of the parameters C_d, α, β. It is moderately sensitive to changes in the values of the parameters A, C_h, and μ and highly sensitive to changes in the values of parameters C_s and D_0.

We now further investigate the effects of the following key parameters of Model I and Model II with respect to whom Q^*, S^*, C_1^*, C_2^*, T^* and t_1^* are highly or moderately sensitive as observed from the results of the sensitivity analysis.

(1) The ordering cost per order A.
(2) The unit inventory carrying cost per unit of time C_h.
(3) The demand parameter D_0.

(4) The unit shortage cost per unit time C_s.
(5) The parameter μ (the point of time when demand rate becomes constant).

Effect of ordering cost per order A:
The Table 2 shows that as A increases, Q^* increases. Thus on the occasion of higher ordering cost the purchaser would go for placing higher order as it would be more economical. Hence as A increases then S^*, t_1^*, T^* and C_1^* would increase.
The Table 3 shows similar behavior of Q^*, S^*, T^* and C_2^* like that of Table2 although, t_1^* is comparatively less sensitive to A than in Model I shown in Table 2.

Effect of the unit inventory carrying cost per unit of time C_h:
From Table 2 we find that as C_h increases, Q^* decreases so that the total inventory carrying cost of Model I is reduced. As Q^* decreases, we notice that S^*, t_1^* and T^* decrease. However, as C_h increases, the average total cost C_1^* increases since the inventory carrying cost forms an important component of the cost function.
The Table3 shows that as C_h increases Q^*, S^*, T^* and C_2^* decrease but t_1^* increases. This is expected, because high carrying cost causes the shortage period to increase so that the total cost of the system is minimized.

Effect of the demand parameter D_0:
The Table 2 shows that as D_0 increases, the demand rate:

$$R(t) = D_0[t-(t-\mu)H(t-\mu)]$$

increases which leads to larger order quantity Q^*, higher maximum level of inventory S^* and higher cost C_1^*. Moreover, as D_0 increases, T^* and t_1^* decrease since high demand rate depletes the on-hand inventory faster than before.
The Table 3 depicts the similar results. However, t_1^* decreases as D_0 increases since the shortages build up quickly due to high demand rate.

Effect of the unit shortage cost per unit time C_s:
From Table 2 we see that as C_s increases, t_1^* increases but Q^* and T^* decrease. This is justified, because the shortage period (T^*- t_1^*) in Model I plays an important role in the cost function C_1^* and should decrease as C_s increases. Hence S^* and C_1^* increase as C_s increases.
The Table 3 reveals that t_1^* is highly sensitive to C_s. When C_s increases, t_1^* decreases significantly to reduce the total cost. Also T^* and Q^* decrease as C_s increases. We note that S^* and C_2^* increase with C_s which is expected.

Effect of the parameter μ:
From Table 2, Q^* and S^* increase as μ increases. The demand rate D_0t continuously increases and acquires the constant value $D_0\mu$ at $t = \mu$ for some larger value of μ. As a result t_1^* and T^* decrease as μ increases. Total cost of the system C_1^* increases as μ increases which is expected.
From Table 3 we see that when μ increases +50%, an infeasible solution is obtained. We observe that Q^*, S^* and C_2^* increase with μ. But T^* decreases with μ, since the after replenishment stock exhausts quickly due to increased demand rate. Also it is found that t_1^* increases as μ changes in any direction.

Example 2: The parameters are similar to those as given in Example 1, except that μ is changed to 1.5. Table 4 shows the corresponding results. It is numerically verified that these solutions satisfy the convexity condition.

Table 4. Comparison of optimal solutions of the inventory systems of Example 2 (μ= 1.5)

Optimal solution	Model I	Model II Situation I: $\mu < t_1$	 Situation II: $\mu > t_1$
t_1*	2.20868	....	1.064556
T*	2.653486	....	3.003456
S*	219.299	....	282.318
Q*	286.02	....	338.983
C*	1000.81 (C_1*)	....	882.443 (C_3*)

From Table 4 we observe that the ordering strategy is more economical in case of Model II (Situation II: $\mu > t_1$) than that for Model I. Though the optimal order quantity Q*, the maximum inventory level S* and the optimal cycle length T* are greater in case of Model II, the average total cost of the system C_3* of Model II is $\left(\frac{1000.81-882.443}{1000.81}\cdot 100\right) \approx 11.83$ % less than the average total cost of the system C_1* of Model I. Therefore, the percent benefit of Model II over Model I is 11.83 % when $\mu > t_1$. So Model II is a better optimal ordering strategy compared to Model I. The graph showing variations of the inventory level of Model II ($\mu > t_1$) with time is shown below by using MATLAB.

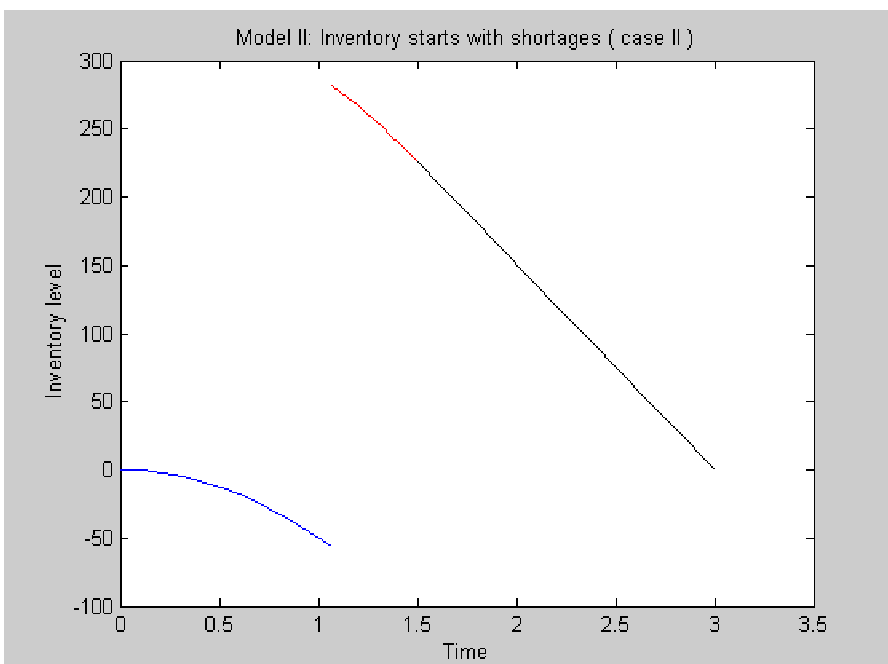

Figure 3. Model II ($\mu > t_1$) Inventory starts with shortages.

From Figure 3 we note that the inventory starts with shortages at t= 0. At t= t_1*= 1.064556 time units replenishment is done and the optimal order quantity Q* is 338.983 units. After replenishment the backorders are cleared and the maximum inventory level S* at t_1* becomes 282.318 units. The demand rate of the items varies with time up to t= μ= 1.5 time units. After that, the demand rate becomes constant and equal to $D_0\mu$. The minimum average total cost per unit time of the model is C_3*= 882.443 units. At t= T*= 3.003456 time units the stock becomes zero.

We are now studying the sensitivity of the optimal solution to changes in the values of the different parameters for Model I and Model II. The sensitivity analyses are performed by changing the value of each of the parameters by -50 %, -25 %, 25 %, 50 %, taking one parameter at a time, and keeping the remaining parameters unchanged.

Here we have assumed that insensitive, moderately sensitive, and highly sensitive imply % changes are –3 to +3, -20 to +20 and more respectively.

Table 5 (a). Sensitivity Analysis of Model I (μ= 1.5)

Parameters	***% change***	***% change in***				
		S*	***Q****	***C_1****	***t_1****	***T****
A	50	37.668	34.74	25.12	24.78	24.836
	25	19.892	18.343	13.251	13.094	13.121
	-25	-22.991	-21.192	-15.28	-15.155	-15.176
	-50	-51.203	-47.185	-33.98	-33.777	-33.811
C_h	50	-36.253	-24.582	13.78	-23.906	-17.589
	25	-20.698	-14.065	7.736	-13.642	-10.059
	-25	29.639	20.38	-10.052	19.504	14.549
	-50	77.774	54.13	-23.582	51.081	38.565
C_s	50	4.383	-3.962	2.918	2.887	-2.859
	25	2.584	-2.363	1.72	1.702	-1.705
	-25	-4.031	3.851	-2.681	-2.655	2.778
	-50	-11.209	11.255	-7.454	-7.386	8.116
C_d	50	-0.204	-0.146	0.047	-0.134	-0.104
	25	-0.102	-0.073	0.024	-0.067	-0.052
	-25	0.103	0.073	-0.023	0.067	0.052
	-50	0.206	0.147	-0.047	0.135	0.105
D_0	50	2.53	6.248	18.467	-20.866	-20.892
	25	2.441	4.178	9.984	-11.91	-11.928
	-25	-5.486	-7.004	-11.995	17.124	17.16
	-50	-15.562	-18.228	-26.99	45.254	20.905

Table 5 (b). Sensitivity Analysis of Model I (μ= 1.5)

Parameters	*% change*	*% change in*				
		S*	***Q****	***C_1****	***t_1****	***T****
μ	50	-60.496	-45.861	2.242	-31.641	-31.674
	25	-21.89	-15.666	4.792	-16.258	-16.28
	-25	6.788	2.857	-10.061	19.142	19.183
	-50	-0.309	-6.154	-25.365	48.424	48.566
α	50	-0.285	-0.201	0.079	-0.262	-0.205
	25	-0.143	-0.1	0.04	-0.132	-0.103
	-25	0.144	0.101	-0.039	0.132	0.103
	-50	0.289	0.203	-0.079	0.266	0.208
β	50	-0.994	-0.732	0.132	-0.759	-0.609
	25	-0.399	-0.293	0.056	-0.307	-0.247
	-25	0.263	0.192	-0.041	0.206	0.165
	-50	0.435	0.316	-0.072	0.344	0.274

A careful study of Table 5 reveals the following:

i. It is seen that the maximum inventory level *S** and the optimal order quantity *Q** are insensitive to changes in the values of the parameters C_d, α, β. These are moderately sensitive to change in the values of parameters C_s and D_0 and highly sensitive to changes in the values of parameters *A*, C_h and μ.

ii. The optimal total cost C_1* and optimal time periods t_1* and *T** are insensitive to changes in values of parameters C_d, α, β. These are moderately sensitive to changes in value of parameter C_s and highly sensitive to changes in the values of parameters *A*, C_h, D_0 and μ.

Table 6 (a). Sensitivity Analysis of Model II (μ= 1.5 and $\mu > t_1$)

Parameters	*% change*	*% change in*				
		S*	***Q****	***C_3****	***t_1****	***T****
A	50	28.021	28.031	25.602	13.173	20.909
	25	14.808	14.813	13.404	7.162	11.059
	-25	-17.162	-17.166	-15.113	-8.999	-12.839
	-50	-38.354	-38.361	-32.955	-21.515	-28.72
C_h	50	-24.344	-18.04	12.893	6.474	-13.476
	25	-14.049	-10.468	7.114	3.621	-7.816
	-25	20.662	15.634	-9.083	-4.825	11.165
	-50	55.165	42.231	-21.37	-11.8	31.404
C_s	50	0.856	-4.771	4.285	-18.028	-3.585
	25	0.643	-2.726	2.44	-10.286	-2.049
	-25	-1.613	3.88	-3.433	14.564	2.918
	-50	-5.794	9.995	-8.757	37.354	7.52

Table 6 (b). Sensitivity Analysis of Model II (μ= 1.5 and μ> t_1)

Parameters	*% change*	*% change in*				
		S^*	Q^*	C_3^*	t_1^*	T^*
C_d	50	-0.258	-0.228	0.091	-0.041	-0.17
	25	-0.129	-0.114	0.045	-0.02	-0.086
	-25	0.13	0.115	-0.045	0.021	0.086
	-50	0.26	0.231	-0.091	0.042	0.172
D_0	50	14.537	14.528	18.981	-12.636	-17.692
	25	8.146	8.141	10.103	-6.997	-10.085
	-25	-10.476	-10.472	-11.809	9.272	14.458
	-50	-24.406	-24.396	-26.278	23.008	38.135
μ	50	-15.339	-15.343	9.415	-8.002	-20.162
	25	-2.082	-2.083	6.359	-1.048	-10.06
	-25	-6.086	-6.088	-8.87	-3.096	12.279
	-50	....	....	....	....	....
α	50	-0.279	-0.25	0.133	-0.052	-0.292
	25	-0.14	-0.125	0.066	-0.026	-0.146
	-25	0.141	0.126	-0.067	0.026	0.148
	-50	0.283	0.253	-0.134	0.053	0.297
β	50	-1.733	-1.7	0.489	-0.77	-1.618
	25	-0.621	-0.604	0.19	-0.261	-0.592
	-25	0.324	0.308	-0.119	0.113	0.323
	-50	0.481	0.449	-0.194	0.146	0.488

A careful study of Table6 reveals the following:

i. It is seen that the maximum inventory level S^*, the optimal order quantity Q^* and the optimal total cost C_3^* are insensitive to changes in the values of the parameters C_d, α, β. These are moderately sensitive to changes in the values of parameters C_s and μ and highly sensitive to changes in the values of parameters A, C_h and D_0.

ii. T^* is insensitive to changes in the values of parameters C_d, α, β. It is moderately sensitive to changes in value of parameter C_s and highly sensitive to changes in the values of parameters A, C_h, D_0 and μ.

iii. t_1^* is insensitive to changes in the values of parameters C_d, α and β. It is moderately sensitive to changes in the values of parameters C_h and μ and highly sensitive to changes in the values of parameters A, C_s and D_0.

Effect of μ:

The results of Table 5 show that in Model I, during the shortage period (T^*-t_1^*) as μ increases, the demand rate R(t) given by

$$R(t)= D_0[t- (t-\mu)H(t-\mu)]$$

increases, which leads to accumulation of larger shortage quantity. Hence S^* decreases and C_1^* increases as μ increases. Consequently, Q^*, t_1^*, and T^* decrease as μ increases.
Table 6 depicts similar results for Model II. However, in this case T^* and t_1^* decrease as μ increases because of increased demand rate.
We observe that the effect of the other key parameters on Model I, and Model II as shown in Tables 5 and 6 are more or less similar to Tables 2 and 3 except that the decision variables are slightly more sensitive towards the key parameters.
Finally, from Tables 1 and 4 and the sensitivity analysis Tables 2, 3, 5 and 6 we observe that the minimum average total cost per unit of time C^* is smaller in case of Model II when the inventory starts with shortages. Therefore, we conclude that the proposed inventory Model II is more economical and preferable for items with Weibull distribution deterioration and ramp type demand rate. In case of Ramp type demand rate, the demand of an item starts with a nonnegative value and then gradually increases and after some point of time (μ), the demand stabilizes and becomes constant. It explains and justifies the results obtained and the conclusion reached by us.

5. Conclusions

Many supermarket managers have observed that the time-varying demand patterns are usually used to reflect sales in different phases of the product life cycle in the market. For example, the demand for inventory items increases over time in the growth phase and decreases in the decline phase. Here we have analyzed three continuous order-level inventory models for deteriorating items with shortages. In all these models, the demand rate is taken as a ramp type function of time and deterioration rate is assumed to follow a two-parameter Weibull distribution. Analytical solutions of the model are discussed and are illustrated with the help of numerical examples. Sensitivity of the optimal solutions with respect to changes in different parameter values is also examined.
However, success depends on the correctness of the estimation of the input parameters. In reality, however, management is most likely to be uncertain of the true values of these parameters. Moreover, their values may be changed over time due to their complex structures. Therefore, it is more reasonable to assume that these parameters are known only within some given ranges. A direction for future research may be to consider stochastic demand rate in the problem.

References

[1]. Aggarwal, S.P. (1978). A note on an order-level model for a system with constant rate of Deterioration. *Opsearch*, 15, 184-187.
[2]. Aggarwal, V., Bahari-Kashani H. (1991). Synchronized production policies for deteriorating items in a declining market. *IIE Transactions*, 23(2), 658-662.
[3]. Bahari-Kashani, H. (1989). Replenishment schedule for deteriorating with time-proportional Demand. *Journal of the Operational Research Society*, 40(1), 75-81.

[4]. Benkherouf, L. (1995). On an inventory model with deteriorating items and decreasing time-varying demand and shortages. *European Journal of Operational Research*, 86, 293-299

[5]. Berrotoni, J.N.(1962). Practical Applications of Weibull distribution. *ASQC Tech. Conf. Trans.*, 303-323.

[6]. Bhowmick, J., Samanta, G. P. (2007). A continuous deterministic inventory system for deteriorating items with inventory-level-dependent time varying demand rate. *Tamsui Oxford Journal of Mathematical Sciences*, 23, 173-184.

[7]. Bose, S., Goswami A., Chaudhuri, K.S. (1995). An EOQ model for deteriorating items with linear time-dependent demand rate and shortages under inflation and time discounting. *Journal of the Operational Research Society*, 46, pp. 771-782.

[8]. Chung, K.J., Ting, P.S (1994). On replenishment schedule for deteriorating items with time proportional demand. *Production Planning and Control*, 5(4), 392-396.

[9]. Covert, R.P., Philip, G.C. (1973). An EOQ model for items with Weibull distribution Deterioration. *AIIE Transaction*, 5, 323-326.

[10]. Dave, U., Patel, L.K. (1981). (T, Si) policy inventory model for deteriorating items with time-proportional demand. *Journal of the Operational Research Society*, 32, 137-142.

[11]. Donaldson, W.A. (1977). Inventory replenishment policy for a linear trend in demand: an analytic solution. *Operational Research Quarterly*, 28, 663-670.

[12]. Ghare, P.M., Schrader, G.P. (1963). A model for exponentially decaying inventory. *Journal of Industrial Engineering*, 14, 238-243.

[13]. Ghosh , S.K., Chaudhuri, K.S. (2004). An order-level inventory model for a deteriorating item with Weibull distribution deterioration, time quadratic demand and shortages. *Advanced Modeling and Optimization*, 6, 21-35.

[14]. Goswami, A., Chaudhuri, K.S. (1992). Variations of order-level inventory models for deteriorating items. *International Journal of Production Economics*, 27, 111-117.

[15]. Goyal, S.K. (1987). Economic ordering policy for deteriorating items over an infinite time horizon. *European Journal of Operational Research*, 28, 298-301.

[16]. Hariga, M.A. (1994). Economic analysis of dynamic inventory models with non-stationary costs and demand. *International Journal of Production Economics*, 36, 255 - 266.

[17]. Hariga, M.A. (1995). An EOQ model for deteriorating items with shortages and time-varying demand. *Journal of the Operational Research Society*, 46, 398-404.

[18]. Hariga, M.A., Benkherouf, L. (1994). Optimal and heuristic inventory models for deteriorating items with exponential time-varying demand. *European Journal of Operational Research*, 79, 123-137.

[19]. Hollier, R.H., Mak, K.L. (1983). Inventory replenishment policies for deteriorating items in a declining market. *International Journal of Production Research*, 21(6), 813 - 826.

[20]. Jain, K., Silver, E.A. (1994). Lot sizing for a product subject to obsolescence or perishability. *European Journal of Operational Research*, 75, 287 - 295.

[21]. Mak, K.L. (1982). A production lot size inventory model for deteriorating items. *Computers and Industrial Engineering*, 6, 309 - 317.

[22]. Mandal, B., Pal, A.K. (1998). Order level inventory system with ramp type demand rate for deteriorating items. *Journal of Interdisciplinary Mathematics*, 1, 49-66.

[23]. Misra, R.B. (1975). Optimum production lot-size model for a system with deteriorating Inventory. *International Journal of Production Research*, 13, 495-505.

[24]. Nahmias, S. (1982). Perishable inventory theory: A review. *Operations Research*, 30, 680 - 708.
[25]. Philip, G.C. (1974). A generalized EOQ model for items with Weibull Distribution Deterioration. *AIIE Transaction*, 6, 159 - 162.
[26]. Rafaat, F. (1991). Survey of literature on continuously deteriorating inventory model. *Journal of the Operational Research Society*, 42, 27-37.
[27]. Roy, A., Samanta, G.P. (2004). An EOQ model for deteriorating items with stock - dependent time varying demand. *Tamsui Oxford Journal of Mathematical Sciences*, 20, 37-46.
[28]. Sachan, R.S. (1984). On (T, Si) policy inventory model for deteriorating items with time-proportional demand. *Journal of the Operational Research Society*, 35(11), 1013 - 1019.
[29]. Samanta, G.P., Gangopadhyaya, N. (2007). A deterministic inventory system with deterioration and stock dependent time varying demand rate. *Tamsui Oxford Journal of Management Sciences*, 23, 35-48.
[30]. Shah, Y.K., Jaiswal, M.C. (1977). An order-level inventory model for a system with constant rate of deterioration. *Opsearch*, 14, 174-184.
[31]. Wee, H.M. (1995). A deterministic lot-size inventory model for deteriorating items with shortages and a declining market. *Computers and Operations Research*, 22(3), 345 - 356.
[32]. Wu, K.S., Ouyang, L.Y. (2000). A replenishment policy for deteriorating items with ramp type demand rate. *Proc. Natl. Sci. Counc. ROC* (A), 24, 279-286.
[33]. Xu, H., Wang, H.P. (1992). Optimal inventory policy for perishable items with time proportional demand. *IIE Transaction*, 24(5) , 105-110.