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# Statistical Inference for the Unit Inverse Weibull Distribution Using Ranked Set Sampling with COVID-19 Application

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This study compares parameter estimation methods for the unit inverse Weibull distribution under ranked set sampling (RSS) and simple random sampling (SRS) techniques. We examine Maximum Product Spacing Estimation, Ordinary Least Squares Estimation, Maximum Likelihood Estimation, Weighted Least Squares Estimation, Anderson-Darling Estimation, Left-Tail Anderson-Darling Estimation, Right-Tail Anderson-Darling Estimation, Cramér-von Mises Estimation, Minimum Spacing Absolute Distance Estimation, Minimum Spacing Square Distance Estimation, Minimum Spacing Absolute-Log Distance Estimation, and Minimum Spacing Square Log Distance Estimation. Monte Carlo simulations evaluate estimator performance using mean squared error (MSE), bias (AB), and mean absolute relative error (MARE). A COVID-19 dataset validates the practical applicability of the methods. Results show RSS-based estimators consistently outperform SRS counterparts across all metrics and estimation techniques. RSS demonstrates superior accuracy, reduced AB and lower MSE, particularly in small sample scenarios. These findings establish RSS as the preferred approach for unit inverse Weibull parameter estimation, providing significant improvements in statistical efficiency and reliability for practical applications.

**keywords:** Anderson-Darling, Cramer-von-Mises, minimum spacing, mean absolute relative error, ranked set sampling.

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# 1 Introduction

Modeling lifetime data has been challenging due to the complex development of data across many fields such as economics, engineering, and others. The nature of the data plays a vital role in selecting the appropriate model to describe it. Recently, There has been increasing research interest in bounded data, particularly unit data that falls between 0 and 1. This type of data is commonly used to describe fractions, rates, and percentages, which has numerous applications in machine learning, artificial intelligence, health and environmental studies, and risk analysis. The beta distribution is the most widely used distribution for analyzing this type of data. In the recent literature, several authors have developed new distributions on the unit interval. Mazucheli et al. (2018a) proposed the unit-Weibull distribution, while Mazucheli et al. (2018b) developed the unit-Birnbaum-Saunders distribution. Subsequently, Mazucheli et al. (2019) established the unit-Gompertz distribution, and Ghitany et al. (2019) presented the unit-inverse Gaussian distribution. Furthermore, Korkmaz and Chesneau (2021) contributed the unit Burr-XII distribution, followed by Korkmaz et al. (2022) who formulated the unit-Chen distribution. Additionally, Krishna et al. (2022) proposed the unit Teissier distribution, Hashmi et al. (2022) developed the unit Xgamma distribution, and Korkmaz and Korkmaz (2023) established the unit log-log distribution.

Given $T$ a random variable from the two-parameter Weibull distribution with probability density function (PDF)

$$g(t;\alpha,\beta) = \alpha\beta t^{\beta-1}\mathrm{e}^{-\alpha t^{\beta}}, \quad t>0, \quad \alpha,\beta>0, \tag{1}$$

Let $V$ be a random variable with Weibull distribution. Then the random variable $V = \frac{1}{T}$ will have an inverse Weibull (IW) distribution, and its cumulative distribution function (CDF) is given by:

$$F_{\mathrm{Iw}}(v;\rho,\tau) = \exp\left[-\rho v^{-\tau}\right], \quad v>0 \tag{2}$$

Using the transformation $Y = \mathrm{e}^{-V}$, Ribeiro-Reis (2022) introduced the unit inverse Weibull distribution (U-IWD) . Its CDF is given by

$$H(y;\rho,\tau) = 1-\exp\left[-\rho(-\log y)^{-\tau}\right], \quad 0<y<1 \tag{3}$$

and its PDF is

$$h(y;\rho,\tau) = \frac{\rho\tau}{y}(-\log y)^{-\tau-1}\exp\left[-\rho(-\log y)^{-\tau}\right], \quad 0<y<1 \tag{4}$$

The corresponding hazard rate function (HRF) is given by;

$$HRF(y;\rho,\tau) = \frac{\rho\tau}{y}(-\log y)^{-\tau-1}, \quad 0<y<1 \tag{5}$$

Figure 1 displays PDF and HRF of the U-IWD across selected parameter configurations. The figure demonstrates the distribution's remarkable flexibility, with the PDF exhibiting diverse forms including decreasing, U-shaped, unimodal, and both right- and left-skewed patterns. This versatility makes the distribution well-suited for modeling various data types. Similarly, the hazard function displays both monotonically increasing and bathtub-shaped behaviors, further illustrating the distribution's adaptability to different patterns.

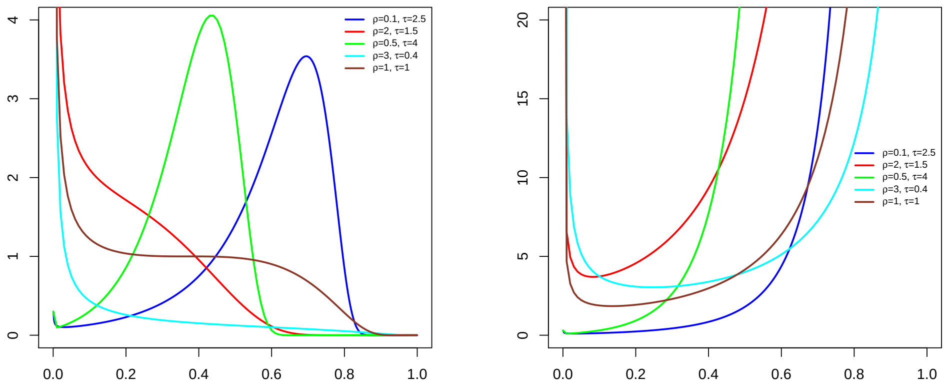


Figure 1: Plots of the PDF and HRF for the U-IWD for some values of $\rho$ and $\tau$.

A fundamental challenge for researchers is obtaining representative samples of the population under study. When measurements are expensive, invasive, or time-consuming, efficient sampling methodologies become essential. RSS provides a valuable approach for maximizing accuracy per sampling unit and achieving observational efficiency. This technique was introduced by McIntyre (1952) as an alternative to traditional SRS methods to enhance the effectiveness of sample mean estimators.

The theoretical foundation of RSS was established by Takahasi and Wakimoto (1968), who demonstrated mathematically that RSS mean estimators remain unbiased while delivering superior precision compared to SRS estimates under perfect ranking conditions. Dell and Clutter (1972) further extended this work by proving that RSS maintains its advantages even when ranking is imperfect. Their research confirmed the method's robustness and practical applicability across diverse real-world scenarios where perfect ranking may be unattainable. For comprehensive coverage of this topic, readers are directed to Patil et al. (1994) and Al-Omari and Bouza (2014).

The RSS methodology involves systematically partitioning the population into multiple equal-sized sets. Specifically, $s^2$ units are randomly selected from the population and divided into $s$ sets, each containing $s$ units. Within each set, units are ranked according to the variable of interest using visual inspection or auxiliary information, without conducting actual measurements.

For measurement purposes, only specific units are quantified: the lowest-ranked unit from the first set, the second-lowest ranked unit from the second set, and so forth, continuing this pattern until the highest-ranked unit from the final set is measured. To construct a complete RSS sample of size $M = sc$, this entire procedure is repeated $c$ times.

Given its efficiency, RSS methodologies have been widely adopted by researchers for estimating unknown distribution parameters. Abu-Dayyeh et al. (2004) developed various estimators for logistic distribution parameters using RSS. Yousef and Al-Subh (2014) compared parameter estimation for the Gumbel distribution using both SRS and RSS approaches. Pedroso et al. (2021) explored RSS applications for parameter estimation in the two-parameter Birnbaum-Saunders distribution, while Al-Omari et al. (2022) demonstrated RSS efficiency for the two-parameter Xgamma distribution.Further methodological advancements include variations of RSS proposed by Jemain et al. (2008), new sampling procedures for estimating population mean and variance developed by Zamanzade and Al-Omari (2016), and double-stage RSS introduced by Hanandeh et al. (2022). Moreover, Irshad et al. (2021) highlighted the practical utility of RSS in modeling concomitants of order statistics. Additional applications of RSS in parameter estimation are documented in Al-Omari et al. (2021); Esemen and Gürler (2018); Hassan et al. (2022); Samuh et al. (2020); Nagy et al. (2022); Qian et al. (2021); Benchiha et al. (2025).

To our knowledge, no prior research has investigated parameter estimation for the U-IWD using RSS methodology. This study is motivated by three key factors: the documented efficiency advantages of RSS over SRS, the widespread adoption of RSS across various disciplines, and the U-IWD's superior capability in modeling unit data. Our investigation encompasses a comprehensive comparison of twelve parameter estimation methods applied to the U-IWD. We examine both RSS-based and conventional SRS-based approaches, evaluating the following methodologies: Maximum Product Spacing Estimation (MPSE), Ordinary Least Squares Estimation (OLSE), Maximum Likelihood Estimation (MLE), Weighted Least Squares Estimation (WLSE), Anderson-Darling Estimation (ADE), Left-Tail Anderson-Darling Estimation (LTADE), Right-Tail Anderson-Darling Estimation (RTADE), Cramér-von Mises Estimation (CVME), Minimum Spacing Absolute Distance Estimation (MSADE), Minimum Spacing Square Distance Estimation (MSSDE), Minimum Spacing Absolute-Log Distance Estimation (MSALDE), and Minimum Spacing Square Log Distance Estimation (MSSLDE).

Due to the theoretical complexity inherent in comparing estimator performance across RSS and SRS frameworks, we employed extensive Monte Carlo simulations to provide empirical evaluation of their relative effectiveness using established performance metrics. The simulation framework assumes perfect ranking conditions to establish reliable baseline performance standards.

For practical validation, we applied these methods to real-world datasets, where goodness-of-fit assessments consistently demonstrated RSS superiority over SRS across all estimation techniques examined.

The paper organization follows this structure: Section 2 details the estimation methodologies examined in this investigation. Section 3 presents numerical simulation results across varying sample sizes. Section 4 discusses real data applications, while Section 5 provides concluding remarks.

## 2 Estimation Methods

This section outlines the RSS methodology and develops parameter estimators for the U-IWD through multiple estimation approaches under the RSS design.

### 2.1 Maximum Product of Spacing

Following the established methodology introduced by Cheng and Amin (1979, 1983), the Maximum Product Spacing Estimation (MPSE) for the unknown parameters $\rho$ and $\tau$ of the U-IWD is derived through analysis of CDF differences at consecutive data points. This methodology exhibits performance comparable to MLE while demonstrating enhanced consistency across diverse scenarios.

Consider $Y_{(1:M)}, Y_{(2:M)}, \ldots, Y_{(M:M)}$ as ordered observations constituting a RSS of size $M = cs$, where $s$ denotes the set size and $c$ represents the number of cycles, drawn from the U-IWD population.

The uniform spacings are subsequently computed as follows:

$$\omega_l(\rho,\tau) = H(y_{(l:M)}|\rho,\tau) - H(y_{(l-1:M)}|\rho,\tau), \quad l = 1, 2, \ldots, M+1 \tag{6}$$

where $H(y_{(0:M)}|\rho,\tau) = 0$ and $H(y_{(M+1:M)}|\rho,\tau) = 1$. These spacings satisfy the constraint $\sum_{l=1}^{M+1} \omega_l(\rho,\tau) = 1$.

The geometric mean of these spacings is expressed as:

$$G(\rho,\tau) = \left[\prod_{l=1}^{M+1} \omega_l(\rho,\tau)\right]^{\frac{1}{M+1}} \tag{7}$$

Taking the natural logarithm yields the log-geometric mean:

$$\mathcal{G}(\rho,\tau) = \frac{1}{M+1}\sum_{l=1}^{M+1} \log \omega_l(\rho,\tau) \tag{8}$$

The MPS estimators, $\hat{\rho}_{RSS}^{MPS}$ and $\hat{\tau}_{RSS}^{MPS}$, are the parameter values that maximize this log-geometric mean. These estimators are obtained by solving the following system of nonlinear equations:

$$\frac{\partial}{\partial \rho}\mathcal{G}(\rho,\tau) = \frac{1}{M+1}\sum_{l=1}^{M+1} \frac{1}{\omega_l(\rho,\tau)}\left[\frac{\partial H}{\partial \rho}(y_{(l:M)}|\rho,\tau) - \frac{\partial H}{\partial \rho}(y_{(l-1:M)}|\rho,\tau)\right] = 0 \tag{9}$$

$$\frac{\partial}{\partial \tau}\mathcal{G}(\rho,\tau) = \frac{1}{M+1}\sum_{l=1}^{M+1} \frac{1}{\omega_l(\rho,\tau)}\left[\frac{\partial H}{\partial \tau}(y_{(l:M)}|\rho,\tau) - \frac{\partial H}{\partial \tau}(y_{(l-1:M)}|\rho,\tau)\right] = 0 \tag{10}$$

where $\frac{\partial H}{\partial \rho}(.|\rho,\tau)$ and $\frac{\partial H}{\partial \tau}(.|\rho,\tau)$ represent the partial derivatives of the U-IWD cumulative distribution function with respect to parameters $\rho$ and $\tau$, respectively. These equations typically require numerical optimization methods for solution.

## 2.2 Least Squares

A cornerstone result in probability theory demonstrates that when evaluating the CDF at the $l^{th}$ order statistic from a sample of size $M$, denoted as $H(X_{(l:M)})$, the resulting random variable follows a Beta distribution with specific shape parameters: This fundamental relationship yields explicit expressions for the distribution's moments. The expected value and variance of this Beta-distributed random variable are:

$$E[H(X_{(l:M)})] = \frac{l}{M+1}$$

$$\mathrm{Var}[H(X_{(l:M)})] = \frac{l(M-l+1)}{(M+1)^2(M+2)}$$

These distributional properties form the theoretical foundation for two complementary least squares estimation approaches: Ordinary Least Squares (OLS) and Weighted Least Squares (WLS). The pioneering work of Swain et al. (1988) established this methodology specifically for beta distribution parameter estimation.

The OLS estimators, denoted as $\hat{\rho}^{RSS}_{OLS}$ and $\hat{\tau}^{RSS}_{OLS}$ for $\rho$ and $\tau$, respectively, are derived by minimizing the sum of squared deviations between the empirical CDF values and their theoretical expectations:

$$\begin{aligned} \psi(\rho,\tau) &= \sum_{l=1}^{M} \left[ H(y_{(l:M)}|\rho,\tau) - \frac{l}{M+1} \right]^2 \\ &= \sum_{l=1}^{M} \left[ 1 - \mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}} - \frac{l}{M+1} \right]^2. \end{aligned}$$

Similarly, the WLS estimators, denoted as $\hat{\tau}^{RSS}_{WLS}$ and $\hat{\lambda}^{RSS}_{WLS}$, are obtained by minimizing the following weighted function:

$$\begin{aligned} \delta(\rho,\tau) &= \sum_{l=1}^{M} \frac{(M+1)^2(M+2)}{l(M-l+1)} \left[ H(y_{(l:M)}|\rho,\tau) - \frac{l}{M+1} \right]^2 \\ &= \sum_{l=1}^{M} \frac{(M+1)^2(M+2)}{l(M-l+1)} \left[ 1 - \mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}} - \frac{l}{M+1} \right]^2. \end{aligned}$$

## 2.3 Maximum Likelihood Estimation

For a RSS of size $M = cs$, where $s$ represents the set size and $c$ denotes the number of cycles, the likelihood function is constructed as:

$$L_{\mathrm{RSS}}(\rho,\tau) = \prod_{i=1}^{c}\prod_{j=1}^{s} f_{(j:s)}(y_{(j:s)i};\rho,\tau), \tag{11}$$

where the ordered density function is given by:

$$\begin{aligned} f_{(j:s)}(y_{(j:s)i};\rho,\tau) &= \frac{s!}{(j-1)!(s-j)!}[H(y_{(j:s)i})]^{j-1}[1-H(y_{(j:s)i})]^{s-j}h(y_{(j:s)i}) \\ &= \frac{s!}{(j-1)!(s-j)!}\left[1-\exp\left(-\frac{\rho}{(-\log y_{(j:s)i})^{\tau}}\right)\right]^{j-1} \\ &\quad \times \left[\exp\left(-\frac{\rho}{(-\log y_{(j:s)i})^{\tau}}\right)\right]^{s-j} \\ &\quad \times \frac{\rho\tau}{y_{(j:s)i}}(-\log y_{(j:s)i})^{-\tau-1}\exp\left(-\frac{\rho}{(-\log y_{(j:s)i})^{\tau}}\right) \end{aligned} \tag{12}$$

The corresponding log-likelihood function $\mathfrak{L}_{\text{RSS}}(\rho,\tau) = \log L_{\text{RSS}}(\rho,\tau)$ is expressed as:

$$\begin{aligned} \mathfrak{L}_{\text{RSS}}(\rho,\tau) &= \sum_{i=1}^{c}\sum_{j=1}^{s}\log\left\{f_{(j:s)}(y_{(j:s)i};\rho,\tau)\right\} \\ &= \sum_{i=1}^{c}\sum_{j=1}^{s}\log\left(\frac{s!}{(j-1)!(s-j)!}\right) + \sum_{i=1}^{c}\sum_{j=1}^{s}(j-1)\log\left(1-\exp\left(-\frac{\rho}{(-\log y_{(j:s)i})^{\tau}}\right)\right) \\ &\quad - \sum_{i=1}^{c}\sum_{j=1}^{s}\frac{(s-j)\rho}{(-\log y_{(j:s)i})^{\tau}} + cs\log\rho + cs\log\tau - \sum_{i=1}^{c}\sum_{j=1}^{s}\log y_{(j:s)i} \\ &\quad - \sum_{i=1}^{c}\sum_{j=1}^{s}(\tau+1)\log(-\log y_{(j:s)i}) - \sum_{i=1}^{c}\sum_{j=1}^{s}\frac{\rho}{(-\log y_{(j:s)i})^{\tau}} \end{aligned} \tag{13}$$

The Maximum Likelihood Estimators (MLEs) are obtained by simultaneously solving the system of equations:

$$\frac{\partial\mathfrak{L}_{\text{RSS}}(\rho,\tau)}{\partial\rho} = 0 \quad \text{and} \quad \frac{\partial\mathfrak{L}_{\text{RSS}}(\rho,\tau)}{\partial\tau} = 0 \tag{14}$$

Since these partial derivatives do not yield closed-form solutions, numerical optimization techniques must be employed to determine the MLEs, denoted as $\hat{\rho}_{\text{RSS}}^{\text{MLE}}$ and $\hat{\tau}_{\text{RSS}}^{\text{MLE}}$ for parameters $\rho$ and $\tau$, respectively.

## 2.4 Methods of minimum distances

Parameter estimation can be accomplished through various distance-based approaches that minimize specific measures of discrepancy between the empirical cumulative distribution function (ECDF) and the theoretical CDF. These methodologies prove particularly advantageous when maximum likelihood estimation encounters computational difficulties or when specific distributional characteristics, such as tail behavior, require focused attention.

### 2.4.1 Anderson-Darling

The Anderson-Darling method introduced by Anderson and Darling (1952) enhances the Cramér-von Mises approach by incorporating a sophisticated weighting scheme that amplifies deviations in the distribution tails. The estimates $\hat{\rho}_{AD}^{RSS}$ and $\hat{\tau}_{AD}^{RSS}$ are determined by minimizing:

$$
\begin{aligned}
A_{AD}(\rho,\tau) &= -M - \frac{1}{M}\sum_{l=1}^{M}(2l-1)\{\log H(y_{(l:M)}|\rho,\tau) + \log \overline{H}(y_{(M-l+1:M)}|\rho,\tau)\} \\
&= -M - \frac{1}{M}\sum_{l=1}^{M}(2l-1)\left\{\log\left(1-\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}}\right) + \log\left(\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(M-l+1:M)}\right)\right)^{\tau}}}\right)\right\}
\end{aligned}
$$

### 2.4.2 Right-tail Anderson-Darling

The Right-tail Anderson-Darling method represents a specialized variant that concentrates estimation efforts on the upper tail of the distribution. This approach proves valuable when accurate modeling of extreme high values is paramount, such as in reliability analysis or extreme value modeling. The estimates $\hat{\rho}_{RATDE}^{RSS}$ and $\hat{\tau}_{RTADE}^{RSS}$ are obtained by minimizing:

$$
\begin{aligned}
A_{RAD}(\rho,\tau) &= \frac{M}{2} - 2\sum_{l=1}^{M} H(y_{(l:M)}|\rho,\tau) - \frac{1}{M}\sum_{l=1}^{M}(2l-1)\log[1-H(y_{(l:M)}|\rho,\tau)] \\
&= \frac{M}{2} - 2\sum_{l=1}^{M}\left[1-\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}}\right]\frac{1}{M}\sum_{l=1}^{M}(2l-1)\log\left[\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}}\right]
\end{aligned}
$$

### 2.4.3 Left-tail Anderson-Darling

Conversely, the Left-tail Anderson-Darling method directs attention toward the lower tail of the distribution. This specialization proves essential in applications where accurate characterization of small values or rare events in the lower tail is crucial. The estimates $\hat{\rho}_{LTADE}^{RSS}$ and $\hat{\tau}_{LTADE}^{RSS}$ are derived by minimizing:

$$
\begin{aligned}
A_{LAD}(\rho,\tau) &= \frac{-3M}{2} + 2\sum_{l=1}^{M} H\left(y_{(l:M)}|\rho,\tau\right) - \frac{1}{M}\sum_{l=1}^{M}(2l-1)\log H\left(y_{(l:M)}|\rho,\tau\right) \\
&= \frac{M}{2} - 2\sum_{l=1}^{M}\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}}\frac{1}{M}\sum_{l=1}^{M}(2l-1)\log\left\{1-\mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}}\right\}
\end{aligned}
$$

### 2.4.4 Cramér-von Mises

The Cramér-von Mises criterion, originally developed by MacDonald (1971). For the U-IWD distribution parameters $\rho$ and $\tau$, the Cramér-von Mises estimators $\hat{\rho}CV^{RSS}$ and $\hat{\tau}CV^{RSS}$ are obtained by minimizing the objective function:

$$\begin{aligned} CV(\rho,\tau) &= \frac{1}{12M} + \sum_{l=1}^{M} \frac{(M+1)^2(M+2)}{l(M-l+1)} \left[ H(y_{(l:M)};\rho,\tau) - \frac{2l-1}{2M} \right]^2 \\ &= \frac{1}{12M} + \sum_{l=1}^{M} \frac{(M+1)^2(M+2)}{l(M-l+1)} \left[ 1 - \mathrm{e}^{-\frac{\rho}{\left(-\log\left(y_{(l:M)}\right)\right)^{\tau}}} - \frac{2l-1}{2M} \right]^2 \end{aligned}$$

## 2.5 Additional Parameter Estimation Methods

This section presents a comprehensive framework of parameter estimation methods for the U-IWD using RSS. Given an ordered sample $Y_{(1:M)}, Y_{(2:M)}, \ldots, Y_{(M:M)}$ representing an RSS of size $M = sc$ drawn from the U-IWD, we develop four distinct estimation approaches.

1. Maximum Spacing Absolute Deviation Estimator (MSADE): The MSADE method leverages the absolute deviations between empirical and theoretical spacings to achieve robust parameter estimation. This approach is particularly effective when the data contains outliers or when a robust estimation procedure is required. Its objective function is defined as:

$$\zeta_1(\rho,\tau) = \sum_{l=1}^{M+1} \left| \omega_l(\rho,\tau) - \frac{1}{M+1} \right|.$$

2. The MSALD Estimator: The MSALD approach incorporates logarithmic transformation to enhance estimation accuracy, particularly for skewed distributions. Its objective function is:

$$\zeta_2(\psi) = \sum_{l=1}^{M+1} \left| \log\left(\omega_l(\rho,\tau)\right) - \log\left(\frac{1}{M+1}\right) \right|.$$

3. The MSSD Estimator: The MSSD employs squared differences between spacings, making it more sensitive to larger deviations. Its objective function is:

$$\zeta_3(\psi) = \sum_{l=1}^{M+1} \left[ \omega_l(\rho,\tau) - \frac{1}{M+1} \right]^2,$$

4. The MSSLD Estimator: The MSSLD combines the benefits of logarithmic transformation with squared differences. Its objective function is:

$$\zeta_4(\psi) = \sum_{l=1}^{M+1} \left[ \log \omega_l(\rho,\tau) - \log\left(\frac{1}{M+1}\right) \right]^2$$

Table 1 presents a concise overview of the estimation methods for the U-IWD parameters $(\rho, \tau)$ under RSS, highlighting their core principles and objective functions.

Table 1: Summary of Estimation Methods for the U-IWD Parameters $(\rho, \tau)$ under RSS

| **Method** | **Principle / Objective Function** |
|---|---|
| MPSE | Maximize log-geometric mean of uniform spacings: $\mathcal{G}(\rho, \tau) = \frac{1}{M+1} \sum_{l=1}^{M+1} \log \omega_l(\rho, \tau)$ |
| MLE | Maximize log-likelihood of RSS sample: $\mathfrak{L}_{RSS}(\rho, \tau) = \sum_{i=1}^{c} \sum_{j=1}^{s} \log f_{(j:s)}(y_{(j:s)i}; \rho, \tau)$ |
| OLS | Minimize squared deviations: $\psi(\rho, \tau) = \sum_{l=1}^{M} \left[ H(y_{(l:M)} \| \rho, \tau) - \frac{l}{M+1} \right]^2$ |
| WLS | Minimize weighted squared deviations: $\delta(\rho, \tau) = \sum_{l=1}^{M} \frac{(M+1)^2(M+2)}{l(M-l+1)} \left[ H(y_{(l:M)} \| \rho, \tau) - \frac{l}{M+1} \right]^2$ |
| CVM | Minimize Cramér–von Mises distance: $CV(\rho, \tau) = \frac{1}{12M} + \sum_{l=1}^{M} \left[ H(y_{(l:M)} \| \rho, \tau) - \frac{2l-1}{2M} \right]^2$ |
| AD | Minimize Anderson–Darling criterion: $A_{AD}(\rho, \tau) = -M - \frac{1}{M} \sum_{l=1}^{M} (2l-1) \{ \log H(y_{(l:M)}) + \log \overline{H}(y_{(M-l+1:M)}) \}$ |
| RTADE | Right-tail AD: Focus on upper tail discrepancies |
| LTADE | Left-tail AD: Focus on lower tail discrepancies |
| MSADE | Minimize absolute spacing deviations: $\zeta_1(\rho, \tau) = \sum_{l=1}^{M+1} \left\| \omega_l(\rho, \tau) - \frac{1}{M+1} \right\|$ |
| MSALD | Minimize absolute log-spacing deviations: $\zeta_2(\rho, \tau) = \sum_{l=1}^{M+1} \left\| \log \omega_l(\rho, \tau) - \log \frac{1}{M+1} \right\|$ |
| MSSD | Minimize squared spacing deviations: $\zeta_3(\rho, \tau) = \sum_{l=1}^{M+1} \left[ \omega_l(\rho, \tau) - \frac{1}{M+1} \right]^2$ |
| MSSLD | Minimize squared log-spacing deviations: $\zeta_4(\rho, \tau) = \sum_{l=1}^{M+1} \left[ \log \omega_l(\rho, \tau) - \log \frac{1}{M+1} \right]^2$ |

## 3 Numerical Simulation

This section presents a comprehensive Monte Carlo simulation study designed to evaluate the performance of the proposed parameter estimation methods for the U-IWD distribution under both SRS and RSS frameworks. The study encompasses sex sample size categories: small samples ($M = 3 \times 4 = 12$, $M = 4 \times 4 = 16$), moderate samples

($M = 5\times 4 = 20$, $M = 3\times 8 = 24$), and large samples ($M = 4\times 8 = 32$, $M = 5\times 8 = 40$), where $M = s \times c$ with $s$ representing the set size and $c$ denoting the number of cycles, assuming perfect ranking across 10,000 Monte Carlo iterations with parameter configuration $(\rho, \tau) = (0.5, 0.8)$. Parameter estimation is performed using the Nelder-Mead algorithm implemented through R's optim() function, with performance evaluated using four statistical metrics:

1. Average Estimator $AE(\hat{\Phi}_h) = \frac{1}{10000} \sum_{i=1}^{10000} \hat{\Phi}_{ih}$,

2. Absolute Bias $Bias(\hat{\Phi}_h) = \frac{1}{10000} \sum_{i=1}^{10000} |\hat{\Phi}_{ih} - \Phi_h|$,

3. Mean Squared Error $MSE(\hat{\Phi}_h) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\Phi}_{ih} - \Phi_h)^2$,

4. Mean Absolute Relative Error $MRE(\hat{\Phi}_h) = \frac{1}{10000} \sum_{i=1}^{10000} \frac{|\hat{\Phi}_{ih} - \Phi_h|}{\Phi_h}$,

where $\Phi_1 = \rho$ and $\Phi_2 = \tau$ represent the parameters of the U-IWD distribution.

Additionally, the ratios of SRS to RSS statistical metrics are computed, enabling a direct comparison of sampling strategies' performance. All results are reported in Tables 2-4. Furthermore, the numerical values from Tables 2-4 are presented graphically in Figures 2-3, providing visual representations of Bias, MSE, and MRE metrics.

Table 2: AE, Bias, MSE, MRE and relative efficiency for MPSE, OLSE, MLE and WLSE for both RSS and SRS when$(\rho, \tau) = (0.5, 0.8)$.

| $(m,k)$ | Par. | **MPSE** | | | **OLSE** | | | **MLE** | | | **WLSE** | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff |
| $(3,4)$ | $AE(\rho)$ | 0.5244 | 0.5193 | | 0.52 | 0.5104 | | 0.4824 | 0.4757 | | 0.5169 | 0.5079 | |
| | $AB(\rho)$ | 0.1378 | 0.1019 | 1.3529 | 0.1542 | 0.111 | 1.39 | 0.1567 | 0.1145 | 1.3688 | 0.152 | 0.109 | 1.3942 |
| | $MSE(\rho)$ | 0.0323 | 0.0164 | 1.9706 | 0.0531 | 0.0195 | 2.7203 | 0.0408 | 0.0203 | 2.0136 | 0.0539 | 0.0188 | 2.8689 |
| | $MRE(\rho)$ | 0.2756 | 0.2037 | 1.3529 | 0.3085 | 0.2219 | 1.39 | 0.3133 | 0.2289 | 1.3688 | 0.3039 | 0.218 | 1.3942 |
| | $AE(\tau)$ | 0.7687 | 0.7487 | | 0.8219 | 0.794 | | 0.9387 | 0.9114 | | 0.8282 | 0.8 | |
| | $AB(\tau)$ | 0.1531 | 0.1423 | 1.0764 | 0.1971 | 0.1798 | 1.0962 | 0.1923 | 0.1592 | 1.2081 | 0.1862 | 0.1682 | 1.1072 |
| | $MSE(\tau)$ | 0.0372 | 0.0299 | 1.2439 | 0.081 | 0.0614 | 1.3186 | 0.0746 | 0.0476 | 1.5656 | 0.071 | 0.0536 | 1.3242 |
| | $MRE(\tau)$ | 0.1914 | 0.1778 | 1.0764 | 0.2464 | 0.2248 | 1.0962 | 0.2404 | 0.199 | 1.2081 | 0.2328 | 0.2102 | 1.1072 |
| $(3,8)$ | $AE(\rho)$ | 0.5069 | 0.5067 | | 0.5047 | 0.5014 | | 0.4787 | 0.48 | | 0.4998 | 0.4978 | |
| | $AB(\rho)$ | 0.0993 | 0.0715 | 1.3882 | 0.1082 | 0.0794 | 1.3615 | 0.1091 | 0.079 | 1.3798 | 0.1063 | 0.0774 | 1.3727 |
| | $MSE(\rho)$ | 0.0161 | 0.0081 | 2.0013 | 0.0192 | 0.01 | 1.9146 | 0.0188 | 0.0097 | 1.9446 | 0.0184 | 0.0094 | 1.9489 |
| | $MRE(\rho)$ | 0.1986 | 0.143 | 1.3882 | 0.2163 | 0.1589 | 1.3615 | 0.2181 | 0.1581 | 1.3798 | 0.2125 | 0.1548 | 1.3727 |
| | $AE(\tau)$ | 0.7832 | 0.7691 | | 0.8126 | 0.7997 | | 0.8815 | 0.8636 | | 0.8241 | 0.8089 | |
| | $AB(\tau)$ | 0.1009 | 0.0938 | 1.0757 | 0.1288 | 0.1187 | 1.0857 | 0.1228 | 0.1016 | 1.2079 | 0.1195 | 0.1085 | 1.1017 |
| | $MSE(\tau)$ | 0.0166 | 0.0137 | 1.2097 | 0.0283 | 0.0242 | 1.1725 | 0.0277 | 0.0184 | 1.51 | 0.0251 | 0.0206 | 1.2164 |
| | $MRE(\tau)$ | 0.1262 | 0.1173 | 1.0757 | 0.161 | 0.1483 | 1.0857 | 0.1535 | 0.127 | 1.2079 | 0.1494 | 0.1356 | 1.1017 |
| $(4,4)$ | $AE(\rho)$ | 0.5184 | 0.5141 | | 0.5131 | 0.5076 | | 0.4825 | 0.4795 | | 0.5091 | 0.504 | |
| | $AB(\rho)$ | 0.1215 | 0.0815 | 1.4909 | 0.1338 | 0.0886 | 1.5098 | 0.1346 | 0.09 | 1.4967 | 0.1316 | 0.0871 | 1.5098 |
| | $MSE(\rho)$ | 0.024 | 0.0104 | 2.2938 | 0.0296 | 0.0126 | 2.3465 | 0.0287 | 0.0125 | 2.2928 | 0.0285 | 0.012 | 2.3697 |
| | $MRE(\rho)$ | 0.2429 | 0.1629 | 1.4909 | 0.2676 | 0.1772 | 1.5098 | 0.2693 | 0.1799 | 1.4967 | 0.2631 | 0.1743 | 1.5098 |
| | $AE(\tau)$ | 0.7721 | 0.7499 | | 0.8173 | 0.7854 | | 0.9069 | 0.8753 | | 0.8264 | 0.7943 | |
| | $AB(\tau)$ | 0.1255 | 0.1156 | 1.0854 | 0.1596 | 0.1401 | 1.1393 | 0.1558 | 0.121 | 1.2873 | 0.15 | 0.1305 | 1.1499 |
| | $MSE(\tau)$ | 0.0251 | 0.0202 | 1.2436 | 0.0465 | 0.034 | 1.3665 | 0.0458 | 0.0263 | 1.7421 | 0.0421 | 0.03 | 1.4027 |
| | $MRE(\tau)$ | 0.1569 | 0.1445 | 1.0854 | 0.1995 | 0.1751 | 1.1393 | 0.1947 | 0.1512 | 1.2873 | 0.1875 | 0.1631 | 1.1499 |
| $(4,8)$ | $AE(\rho)$ | 0.5027 | 0.5005 | | 0.5 | 0.4978 | | 0.4793 | 0.4801 | | 0.4957 | 0.4935 | |
| | $AB(\rho)$ | 0.0864 | 0.056 | 1.5422 | 0.0949 | 0.0617 | 1.537 | 0.0935 | 0.0621 | 1.5075 | 0.0927 | 0.0602 | 1.5416 |
| | $MSE(\rho)$ | 0.0118 | 0.005 | 2.3627 | 0.0142 | 0.006 | 2.3748 | 0.0135 | 0.0059 | 2.2636 | 0.0135 | 0.0057 | 2.3778 |
| | $MRE(\rho)$ | 0.1728 | 0.112 | 1.5422 | 0.1897 | 0.1234 | 1.537 | 0.1871 | 0.1241 | 1.5075 | 0.1855 | 0.1203 | 1.5416 |
| | $AE(\tau)$ | 0.7868 | 0.7769 | | 0.8161 | 0.7986 | | 0.8655 | 0.8505 | | 0.8258 | 0.8091 | |
| | $AB(\tau)$ | 0.0844 | 0.0758 | 1.1136 | 0.1107 | 0.0928 | 1.1927 | 0.1014 | 0.0813 | 1.2475 | 0.1021 | 0.0847 | 1.2055 |
| | $MSE(\tau)$ | 0.0111 | 0.0088 | 1.2643 | 0.0207 | 0.0143 | 1.4523 | 0.0179 | 0.0114 | 1.5706 | 0.018 | 0.0122 | 1.4767 |
| | $MRE(\tau)$ | 0.1055 | 0.0948 | 1.1136 | 0.1383 | 0.116 | 1.1927 | 0.1268 | 0.1016 | 1.2475 | 0.1276 | 0.1059 | 1.2055 |
| $(5,4)$ | $AE(\rho)$ | 0.5106 | 0.5091 | | 0.5068 | 0.5043 | | 0.4791 | 0.4799 | | 0.5025 | 0.5005 | |
| | $AB(\rho)$ | 0.1085 | 0.064 | 1.6944 | 0.1195 | 0.0696 | 1.7172 | 0.12 | 0.0705 | 1.702 | 0.1175 | 0.0675 | 1.7391 |
| | $MSE(\rho)$ | 0.0193 | 0.0065 | 2.9622 | 0.0237 | 0.0078 | 3.0484 | 0.0228 | 0.0078 | 2.9187 | 0.0227 | 0.0073 | 3.1203 |
| | $MRE(\rho)$ | 0.217 | 0.128 | 1.6944 | 0.2391 | 0.1392 | 1.7172 | 0.24 | 0.141 | 1.702 | 0.2349 | 0.1351 | 1.7391 |
| | $AE(\tau)$ | 0.7812 | 0.7554 | | 0.8211 | 0.7822 | | 0.8949 | 0.8605 | | 0.8301 | 0.7913 | |
| | $AB(\tau)$ | 0.111 | 0.0973 | 1.1407 | 0.142 | 0.1147 | 1.2377 | 0.1371 | 0.0989 | 1.3858 | 0.133 | 0.1043 | 1.275 |
| | $MSE(\tau)$ | 0.0199 | 0.0141 | 1.4104 | 0.0373 | 0.0219 | 1.7058 | 0.0351 | 0.0177 | 1.9814 | 0.0335 | 0.0185 | 1.8095 |
| | $MRE(\tau)$ | 0.1388 | 0.1216 | 1.1407 | 0.1775 | 0.1434 | 1.2377 | 0.1714 | 0.1237 | 1.3858 | 0.1662 | 0.1304 | 1.275 |
| $(5,8)$ | $AE(\rho)$ | 0.4994 | 0.495 | | 0.4991 | 0.4933 | | 0.4793 | 0.4787 | | 0.4944 | 0.4894 | |
| | $AB(\rho)$ | 0.0783 | 0.0459 | 1.7077 | 0.0858 | 0.0505 | 1.699 | 0.0844 | 0.0511 | 1.6506 | 0.084 | 0.0492 | 1.7066 |
| | $MSE(\rho)$ | 0.0096 | 0.0033 | 2.8893 | 0.0116 | 0.004 | 2.8909 | 0.0109 | 0.004 | 2.6932 | 0.011 | 0.0038 | 2.8915 |
| | $MRE(\rho)$ | 0.1566 | 0.0917 | 1.7077 | 0.1717 | 0.101 | 1.699 | 0.1688 | 0.1023 | 1.6506 | 0.1679 | 0.0984 | 1.7066 |
| | $AE(\tau)$ | 0.7937 | 0.7851 | | 0.8153 | 0.8052 | | 0.86 | 0.8462 | | 0.8257 | 0.8141 | |
| | $AB(\tau)$ | 0.0759 | 0.0627 | 1.211 | 0.099 | 0.08 | 1.2368 | 0.0915 | 0.0702 | 1.303 | 0.0911 | 0.0725 | 1.2564 |
| | $MSE(\tau)$ | 0.0092 | 0.0061 | 1.5058 | 0.0164 | 0.0104 | 1.5735 | 0.0145 | 0.0082 | 1.7724 | 0.0141 | 0.0087 | 1.6184 |
| | $MRE(\tau)$ | 0.0949 | 0.0784 | 1.211 | 0.1237 | 0.1 | 1.2368 | 0.1144 | 0.0878 | 1.303 | 0.1138 | 0.0906 | 1.2564 |

Table 3: AE, Bias, MSE, MRE and relative efficiency for ADE, LTADE, RTADE and CVME for both RSS and SRS when$(\rho, \tau) = (0.5, 0.8)$.

| $(m,k)$ | Par. | ADE | | | LTADE | | | RTADE | | | CVME | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff |
| (3,4) | $AE(\rho)$ | 0.5062 | 0.498 | | 0.5239 | 0.5048 | | 0.496 | 0.4862 | | 0.4942 | 0.4796 | |
| | $AB(\rho)$ | 0.1516 | 0.1093 | 1.386 | 0.1714 | 0.1186 | 1.4443 | 0.163 | 0.1203 | 1.3545 | 0.1713 | 0.1217 | 1.4073 |
| | $MSE(\rho)$ | 0.0389 | 0.0188 | 2.0734 | 0.1313 | 0.0251 | 5.237 | 0.0437 | 0.0228 | 1.9171 | 0.078 | 0.0234 | 3.3357 |
| | $MRE(\rho)$ | 0.3031 | 0.2187 | 1.386 | 0.3427 | 0.2373 | 1.4443 | 0.326 | 0.2407 | 1.3545 | 0.3427 | 0.2435 | 1.4073 |
| | $AE(\tau)$ | 0.8591 | 0.835 | | 0.9192 | 0.8905 | | 0.8916 | 0.8689 | | 0.9691 | 0.9354 | |
| | $AB(\tau)$ | 0.1682 | 0.1489 | 1.1299 | 0.2203 | 0.1992 | 1.1055 | 0.2024 | 0.1805 | 1.1215 | 0.2416 | 0.2113 | 1.1434 |
| | $MSE(\tau)$ | 0.0537 | 0.0387 | 1.3884 | 0.112 | 0.0876 | 1.2787 | 0.0921 | 0.0738 | 1.2484 | 0.1394 | 0.102 | 1.3672 |
| | $MRE(\tau)$ | 0.2102 | 0.1861 | 1.1299 | 0.2753 | 0.2491 | 1.1055 | 0.253 | 0.2256 | 1.1215 | 0.302 | 0.2641 | 1.1434 |
| (3,8) | $AE(\rho)$ | 0.4958 | 0.4936 | | 0.4998 | 0.4957 | | 0.4935 | 0.4902 | | 0.485 | 0.4818 | |
| | $AB(\rho)$ | 0.1073 | 0.0773 | 1.3887 | 0.1122 | 0.0801 | 1.4004 | 0.1139 | 0.0843 | 1.3522 | 0.1136 | 0.083 | 1.3689 |
| | $MSE(\rho)$ | 0.0186 | 0.0094 | 1.9827 | 0.021 | 0.0103 | 2.0398 | 0.0208 | 0.0112 | 1.8589 | 0.0207 | 0.0108 | 1.9199 |
| | $MRE(\rho)$ | 0.2146 | 0.1545 | 1.3887 | 0.2245 | 0.1603 | 1.4004 | 0.2278 | 0.1685 | 1.3522 | 0.2272 | 0.166 | 1.3689 |
| | $AE(\tau)$ | 0.838 | 0.8238 | | 0.8684 | 0.8495 | | 0.8442 | 0.8324 | | 0.8914 | 0.8755 | |
| | $AB(\tau)$ | 0.1148 | 0.1021 | 1.124 | 0.1404 | 0.1257 | 1.1171 | 0.1268 | 0.1152 | 1.1012 | 0.1434 | 0.1277 | 1.123 |
| | $MSE(\tau)$ | 0.0231 | 0.0181 | 1.279 | 0.0365 | 0.0289 | 1.2632 | 0.0295 | 0.0241 | 1.2208 | 0.0382 | 0.0309 | 1.2375 |
| | $MRE(\tau)$ | 0.1435 | 0.1277 | 1.124 | 0.1755 | 0.1571 | 1.1171 | 0.1586 | 0.144 | 1.1012 | 0.1792 | 0.1596 | 1.123 |
| (4,4) | $AE(\rho)$ | 0.5028 | 0.4966 | | 0.5112 | 0.4999 | | 0.4965 | 0.4902 | | 0.4901 | 0.4816 | |
| | $AB(\rho)$ | 0.1321 | 0.0868 | 1.5219 | 0.1422 | 0.0919 | 1.548 | 0.1418 | 0.0948 | 1.4949 | 0.1445 | 0.0955 | 1.5131 |
| | $MSE(\rho)$ | 0.0282 | 0.0118 | 2.3873 | 0.0369 | 0.0139 | 2.6633 | 0.0318 | 0.0144 | 2.2051 | 0.0354 | 0.0145 | 2.451 |
| | $MRE(\rho)$ | 0.2642 | 0.1736 | 1.5219 | 0.2845 | 0.1838 | 1.548 | 0.2836 | 0.1897 | 1.4949 | 0.289 | 0.191 | 1.5131 |
| | $AE(\tau)$ | 0.8473 | 0.8199 | | 0.8934 | 0.8596 | | 0.8658 | 0.8368 | | 0.93 | 0.8932 | |
| | $AB(\tau)$ | 0.1397 | 0.1165 | 1.199 | 0.1798 | 0.153 | 1.175 | 0.1631 | 0.1359 | 1.2002 | 0.1902 | 0.1564 | 1.2163 |
| | $MSE(\tau)$ | 0.0352 | 0.0237 | 1.4857 | 0.0651 | 0.0467 | 1.3958 | 0.0537 | 0.037 | 1.4505 | 0.0729 | 0.0503 | 1.4481 |
| | $MRE(\tau)$ | 0.1746 | 0.1456 | 1.199 | 0.2248 | 0.1913 | 1.175 | 0.2039 | 0.1699 | 1.2002 | 0.2377 | 0.1955 | 1.2163 |
| (4,8) | $AE(\rho)$ | 0.4935 | 0.4907 | | 0.4958 | 0.4917 | | 0.4924 | 0.4895 | | 0.4838 | 0.481 | |
| | $AB(\rho)$ | 0.093 | 0.0603 | 1.5419 | 0.0961 | 0.062 | 1.5504 | 0.0995 | 0.066 | 1.5074 | 0.0979 | 0.0641 | 1.5259 |
| | $MSE(\rho)$ | 0.0135 | 0.0057 | 2.3731 | 0.0147 | 0.006 | 2.4446 | 0.0153 | 0.0068 | 2.2467 | 0.0148 | 0.0064 | 2.3132 |
| | $MRE(\rho)$ | 0.186 | 0.1207 | 1.5419 | 0.1922 | 0.124 | 1.5504 | 0.1989 | 0.132 | 1.5074 | 0.1957 | 0.1283 | 1.5259 |
| | $AE(\tau)$ | 0.833 | 0.8197 | | 0.8572 | 0.8415 | | 0.8364 | 0.8233 | | 0.8771 | 0.8589 | |
| | $AB(\tau)$ | 0.0972 | 0.0809 | 1.2013 | 0.1183 | 0.1014 | 1.1669 | 0.1073 | 0.0914 | 1.1741 | 0.1206 | 0.0984 | 1.2262 |
| | $MSE(\tau)$ | 0.0159 | 0.0112 | 1.4193 | 0.025 | 0.0184 | 1.3626 | 0.0201 | 0.0146 | 1.3825 | 0.0261 | 0.0175 | 1.4894 |
| | $MRE(\tau)$ | 0.1215 | 0.1011 | 1.2013 | 0.1479 | 0.1268 | 1.1669 | 0.1342 | 0.1143 | 1.1741 | 0.1508 | 0.1229 | 1.2262 |
| (5,4) | $AE(\rho)$ | 0.4976 | 0.4941 | | 0.5029 | 0.4962 | | 0.4942 | 0.4902 | | 0.4856 | 0.4816 | |
| | $AB(\rho)$ | 0.1179 | 0.0679 | 1.7359 | 0.1246 | 0.0709 | 1.7575 | 0.126 | 0.0751 | 1.678 | 0.1268 | 0.0733 | 1.7298 |
| | $MSE(\rho)$ | 0.0227 | 0.0073 | 3.0976 | 0.027 | 0.0082 | 3.2947 | 0.0255 | 0.009 | 2.8251 | 0.0262 | 0.0086 | 3.056 |
| | $MRE(\rho)$ | 0.2359 | 0.1359 | 1.7359 | 0.2492 | 0.1418 | 1.7575 | 0.252 | 0.1502 | 1.678 | 0.2536 | 0.1466 | 1.7298 |
| | $AE(\tau)$ | 0.8465 | 0.8146 | | 0.8858 | 0.8429 | | 0.8561 | 0.8247 | | 0.9132 | 0.8693 | |
| | $AB(\tau)$ | 0.1252 | 0.0963 | 1.2996 | 0.1596 | 0.1219 | 1.3094 | 0.1405 | 0.1115 | 1.2598 | 0.1644 | 0.1214 | 1.354 |
| | $MSE(\tau)$ | 0.0288 | 0.0159 | 1.8124 | 0.0509 | 0.0273 | 1.8616 | 0.0385 | 0.0231 | 1.6686 | 0.0546 | 0.0285 | 1.9138 |
| | $MRE(\tau)$ | 0.1565 | 0.1204 | 1.2996 | 0.1995 | 0.1523 | 1.3094 | 0.1756 | 0.1394 | 1.2598 | 0.2055 | 0.1517 | 1.354 |
| (5,8) | $AE(\rho)$ | 0.4927 | 0.4871 | | 0.4945 | 0.4874 | | 0.4929 | 0.4876 | | 0.4846 | 0.479 | |
| | $AB(\rho)$ | 0.0841 | 0.0495 | 1.697 | 0.0865 | 0.0508 | 1.7023 | 0.0893 | 0.054 | 1.6539 | 0.0878 | 0.0528 | 1.6639 |
| | $MSE(\rho)$ | 0.011 | 0.0038 | 2.8633 | 0.0118 | 0.0041 | 2.9119 | 0.0124 | 0.0046 | 2.7251 | 0.0119 | 0.0043 | 2.7377 |
| | $MRE(\rho)$ | 0.1681 | 0.0991 | 1.697 | 0.1731 | 0.1017 | 1.7023 | 0.1787 | 0.108 | 1.6539 | 0.1756 | 0.1055 | 1.6639 |
| | $AE(\tau)$ | 0.8311 | 0.8222 | | 0.8533 | 0.8423 | | 0.8314 | 0.822 | | 0.8668 | 0.8547 | |
| | $AB(\tau)$ | 0.088 | 0.0702 | 1.2532 | 0.1069 | 0.0895 | 1.1944 | 0.0956 | 0.0772 | 1.2388 | 0.1055 | 0.0854 | 1.2356 |
| | $MSE(\tau)$ | 0.013 | 0.0081 | 1.6004 | 0.0202 | 0.0145 | 1.3872 | 0.0159 | 0.01 | 1.5837 | 0.0197 | 0.0127 | 1.546 |
| | $MRE(\tau)$ | 0.11 | 0.0878 | 1.2532 | 0.1336 | 0.1118 | 1.1944 | 0.1196 | 0.0965 | 1.2388 | 0.1319 | 0.1068 | 1.2356 |

Table 4: AE, Bias, MSE, MRE and relative efficiency for MSADE, MSSDE, MSALDE and MSSLDE for both RSS and SRS when$(\rho, \tau) = (0.5, 0.8)$.

| $(m, k)$ | Par. | **MSADE** | | | **MSSDE** | | | **MSALDE** | | | **MSSLDE** | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff | SRS | RSS | Eff |
| (3, 4) | $AE(\rho)$ | 0.5361 | 0.5352 | | 0.5309 | 0.5285 | | 0.5252 | 0.5235 | | 0.5183 | 0.5118 | |
| | $AB(\rho)$ | 0.1621 | 0.1356 | 1.1955 | 0.1577 | 0.1266 | 1.2463 | 0.1533 | 0.1276 | 1.201 | 0.1511 | 0.1158 | 1.3049 |
| | $MSE(\rho)$ | 0.0439 | 0.0295 | 1.4868 | 0.0427 | 0.0259 | 1.6482 | 0.039 | 0.026 | 1.5025 | 0.0386 | 0.0214 | 1.803 |
| | $MRE(\rho)$ | 0.3241 | 0.2711 | 1.1955 | 0.3155 | 0.2531 | 1.2463 | 0.3066 | 0.2552 | 1.201 | 0.3022 | 0.2316 | 1.3049 |
| | $AE(\tau)$ | 0.7929 | 0.7698 | | 0.7394 | 0.7122 | | 0.783 | 0.761 | | 0.8215 | 0.8019 | |
| | $AB(\tau)$ | 0.1829 | 0.1693 | 1.0801 | 0.2135 | 0.2052 | 1.0404 | 0.1646 | 0.1528 | 1.0768 | 0.158 | 0.1418 | 1.1148 |
| | $MSE(\tau)$ | 0.0609 | 0.0486 | 1.253 | 0.0797 | 0.068 | 1.1717 | 0.0452 | 0.036 | 1.2541 | 0.0447 | 0.0339 | 1.3176 |
| | $MRE(\tau)$ | 0.2286 | 0.2117 | 1.0801 | 0.2668 | 0.2565 | 1.0404 | 0.2057 | 0.191 | 1.0768 | 0.1975 | 0.1772 | 1.1148 |
| (3, 8) | $AE(\rho)$ | 0.5156 | 0.5159 | | 0.5148 | 0.516 | | 0.5056 | 0.5053 | | 0.5014 | 0.4999 | |
| | $AB(\rho)$ | 0.1185 | 0.0996 | 1.1893 | 0.1199 | 0.0985 | 1.2171 | 0.1118 | 0.09 | 1.2425 | 0.1112 | 0.0833 | 1.3348 |
| | $MSE(\rho)$ | 0.0229 | 0.0159 | 1.4373 | 0.0238 | 0.0157 | 1.5189 | 0.0202 | 0.0128 | 1.5774 | 0.02 | 0.0111 | 1.8039 |
| | $MRE(\rho)$ | 0.237 | 0.1993 | 1.1893 | 0.2398 | 0.197 | 1.2171 | 0.2236 | 0.1799 | 1.2425 | 0.2225 | 0.1667 | 1.3348 |
| | $AE(\tau)$ | 0.792 | 0.7808 | | 0.7586 | 0.7386 | | 0.7933 | 0.7781 | | 0.8178 | 0.8045 | |
| | $AB(\tau)$ | 0.1238 | 0.121 | 1.0232 | 0.1514 | 0.1479 | 1.0236 | 0.1105 | 0.1042 | 1.0611 | 0.1064 | 0.0951 | 1.1182 |
| | $MSE(\tau)$ | 0.0252 | 0.0238 | 1.0595 | 0.0392 | 0.0344 | 1.1384 | 0.0202 | 0.0171 | 1.1828 | 0.0196 | 0.0152 | 1.2936 |
| | $MRE(\tau)$ | 0.1547 | 0.1512 | 1.0232 | 0.1893 | 0.1849 | 1.0236 | 0.1382 | 0.1302 | 1.0611 | 0.1329 | 0.1189 | 1.1182 |
| (4, 4) | $AE(\rho)$ | 0.5265 | 0.5259 | | 0.5246 | 0.5216 | | 0.5171 | 0.5162 | | 0.5135 | 0.5068 | |
| | $AB(\rho)$ | 0.1446 | 0.1152 | 1.2558 | 0.1432 | 0.1094 | 1.3085 | 0.1361 | 0.1063 | 1.2799 | 0.1339 | 0.0946 | 1.4162 |
| | $MSE(\rho)$ | 0.0336 | 0.0213 | 1.5804 | 0.0335 | 0.0197 | 1.6959 | 0.0293 | 0.0179 | 1.6405 | 0.0295 | 0.0145 | 2.0384 |
| | $MRE(\rho)$ | 0.2893 | 0.2304 | 1.2558 | 0.2863 | 0.2188 | 1.3085 | 0.2722 | 0.2127 | 1.2799 | 0.2678 | 0.1891 | 1.4162 |
| | $AE(\tau)$ | 0.7906 | 0.7665 | | 0.7456 | 0.7209 | | 0.7828 | 0.7602 | | 0.8166 | 0.7946 | |
| | $AB(\tau)$ | 0.1551 | 0.1443 | 1.0749 | 0.1825 | 0.1807 | 1.0102 | 0.1388 | 0.1282 | 1.0831 | 0.1306 | 0.1132 | 1.1537 |
| | $MSE(\tau)$ | 0.0412 | 0.0337 | 1.2214 | 0.0552 | 0.0531 | 1.0383 | 0.031 | 0.025 | 1.2372 | 0.0295 | 0.0216 | 1.3679 |
| | $MRE(\tau)$ | 0.1939 | 0.1804 | 1.0749 | 0.2282 | 0.2259 | 1.0102 | 0.1735 | 0.1602 | 1.0831 | 0.1632 | 0.1415 | 1.1537 |
| (4, 8) | $AE(\rho)$ | 0.5099 | 0.5089 | | 0.5118 | 0.5098 | | 0.5012 | 0.5 | | 0.4969 | 0.4941 | |
| | $AB(\rho)$ | 0.1048 | 0.0819 | 1.2793 | 0.1073 | 0.0844 | 1.2711 | 0.0983 | 0.0736 | 1.335 | 0.0961 | 0.0689 | 1.3941 |
| | $MSE(\rho)$ | 0.0176 | 0.0108 | 1.6368 | 0.0186 | 0.0116 | 1.612 | 0.0154 | 0.0085 | 1.8131 | 0.0146 | 0.0074 | 1.9781 |
| | $MRE(\rho)$ | 0.2096 | 0.1638 | 1.2793 | 0.2146 | 0.1688 | 1.2711 | 0.1966 | 0.1472 | 1.335 | 0.1921 | 0.1378 | 1.3941 |
| | $AE(\tau)$ | 0.7947 | 0.781 | | 0.7638 | 0.7482 | | 0.7944 | 0.784 | | 0.8163 | 0.807 | |
| | $AB(\tau)$ | 0.1056 | 0.0982 | 1.076 | 0.1293 | 0.1218 | 1.0615 | 0.093 | 0.0852 | 1.0914 | 0.0905 | 0.079 | 1.1451 |
| | $MSE(\tau)$ | 0.0184 | 0.0154 | 1.1896 | 0.0277 | 0.024 | 1.1542 | 0.0138 | 0.0114 | 1.2139 | 0.0136 | 0.0103 | 1.3222 |
| | $MRE(\tau)$ | 0.132 | 0.1227 | 1.076 | 0.1616 | 0.1522 | 1.0615 | 0.1162 | 0.1065 | 1.0914 | 0.1131 | 0.0988 | 1.1451 |
| (5, 4) | $AE(\rho)$ | 0.5185 | 0.5204 | | 0.5183 | 0.5193 | | 0.5097 | 0.5098 | | 0.5055 | 0.502 | |
| | $AB(\rho)$ | 0.1286 | 0.0999 | 1.2874 | 0.1303 | 0.0935 | 1.394 | 0.1222 | 0.0894 | 1.3671 | 0.1204 | 0.0788 | 1.5284 |
| | $MSE(\rho)$ | 0.0271 | 0.0161 | 1.6821 | 0.0282 | 0.0145 | 1.9493 | 0.0242 | 0.0127 | 1.912 | 0.0236 | 0.01 | 2.3665 |
| | $MRE(\rho)$ | 0.2572 | 0.1998 | 1.2874 | 0.2606 | 0.187 | 1.394 | 0.2443 | 0.1787 | 1.3671 | 0.2407 | 0.1575 | 1.5284 |
| | $AE(\tau)$ | 0.7936 | 0.7647 | | 0.7533 | 0.7194 | | 0.7901 | 0.7651 | | 0.8199 | 0.7959 | |
| | $AB(\tau)$ | 0.1364 | 0.1234 | 1.1058 | 0.1661 | 0.1518 | 1.0948 | 0.1215 | 0.1096 | 1.1089 | 0.1158 | 0.0957 | 1.2093 |
| | $MSE(\tau)$ | 0.0324 | 0.0235 | 1.3751 | 0.0468 | 0.0349 | 1.3407 | 0.024 | 0.018 | 1.3325 | 0.0232 | 0.0151 | 1.5365 |
| | $MRE(\tau)$ | 0.1705 | 0.1542 | 1.1058 | 0.2077 | 0.1897 | 1.0948 | 0.1519 | 0.137 | 1.1089 | 0.1447 | 0.1197 | 1.2093 |
| (5, 8) | $AE(\rho)$ | 0.5023 | 0.499 | | 0.5082 | 0.5057 | | 0.4969 | 0.4905 | | 0.494 | 0.4903 | |
| | $AB(\rho)$ | 0.0922 | 0.0688 | 1.3394 | 0.0974 | 0.0729 | 1.3368 | 0.0874 | 0.0633 | 1.3797 | 0.0876 | 0.0593 | 1.4775 |
| | $MSE(\rho)$ | 0.0137 | 0.0076 | 1.7929 | 0.0152 | 0.0087 | 1.7623 | 0.0121 | 0.0063 | 1.9311 | 0.012 | 0.0055 | 2.1611 |
| | $MRE(\rho)$ | 0.1844 | 0.1376 | 1.3394 | 0.1949 | 0.1458 | 1.3368 | 0.1747 | 0.1267 | 1.3797 | 0.1751 | 0.1185 | 1.4775 |
| | $AE(\tau)$ | 0.7998 | 0.7915 | | 0.7696 | 0.7575 | | 0.8017 | 0.795 | | 0.8191 | 0.8104 | |
| | $AB(\tau)$ | 0.0934 | 0.0834 | 1.1198 | 0.1128 | 0.1036 | 1.0891 | 0.0837 | 0.0722 | 1.1593 | 0.0831 | 0.0675 | 1.2314 |
| | $MSE(\tau)$ | 0.0148 | 0.0111 | 1.3347 | 0.0213 | 0.0169 | 1.2592 | 0.0115 | 0.0082 | 1.3934 | 0.0115 | 0.0075 | 1.5295 |
| | $MRE(\tau)$ | 0.1167 | 0.1042 | 1.1198 | 0.141 | 0.1295 | 1.0891 | 0.1046 | 0.0902 | 1.1593 | 0.1038 | 0.0843 | 1.2314 |

Following a thorough analysis of the simulation study, we obtained the following results:

- Both SRS and RSS methodologies demonstrate robust parameter estimation capabilities, with estimates consistently approaching true values as sample sizes increase. This convergence behavior confirms the asymptotic properties of both sampling approaches.
- Method effectiveness varies by sampling approach, with MPSE estimation showing superior performance for both SRS and RSS datasets
- RSS demonstrates clear superiority over SRS across all evaluation criteria, overall estimation accuracy. This efficiency advantage makes RSS the recommended sampling strategy when operationally viable.

# 4 Application

To demonstrate the practical applicability of our proposed estimation methods, this section presents a comprehensive analysis of real-world dataset. Through detailed empirical examination, we illustrate how these methodologies translate from theoretical frameworks to actionable analytical tools, providing valuable insights for both research applications and evidence-based decision-making processes. The dataset under investigation comprises COVID-19 recovery rates for patients in Spain, spanning the period from March 3 to May 7, 2020. This data, originally presented by Afify et al. (2022). The complete dataset is presented in Table 5.

Table 5: Recovery rates of COVID-19 patients in Spain.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.667 | 0.5 | 0.5 | 0.4286 | 0.75 | 0.6531 | 0.5161 | 0.7895 | 0.7689 | 0.6873 | 0.52 |
| 0.7251 | 0.6375 | 0.6078 | 0.6289 | 0.5712 | 0.5923 | 0.6061 | 0.5924 | 0.5921 | 0.5592 | 0.5954 |
| 0.6164 | 0.6455 | 0.6725 | 0.6838 | 0.685 | 0.6947 | 0.721 | 0.7315 | 0.7412 | 0.7508 | 0.7519 |
| 0.7547 | 0.7645 | 0.7715 | 0.7759 | 0.7807 | 0.7838 | 0.7847 | 0.7871 | 0.7902 | 0.7934 | 0.7913 |
| 0.7962 | 0.7971 | 0.7977 | 0.8007 | 0.8038 | 0.8289 | 0.8322 | 0.8354 | 0.8371 | 0.8387 | 0.8456 |
| 0.849 | 0.8535 | 0.8547 | 0.8564 | 0.858 | 0.8604 | 0.8628 | 0.6586 | 0.707 | 0.7963 | 0.8516 |

Table 6 presents the descriptive statistics for the considered dataset. The data is further illustrated graphically through TTT plots, density plots, and violin plots in Figure 5.

Table 6: Descriptive Statistics Summary

| $n$ | Mean | Median | SD | Skewness | Kurtosis | Min | Max | Q1 | Q3 | IQR |
|---|---|---|---|---|---|---|---|---|---|---|
| 66.000 | 0.724 | 0.753 | 0.109 | -0.705 | 2.602 | 0.429 | 0.863 | 0.647 | 0.798 | 0.150 |

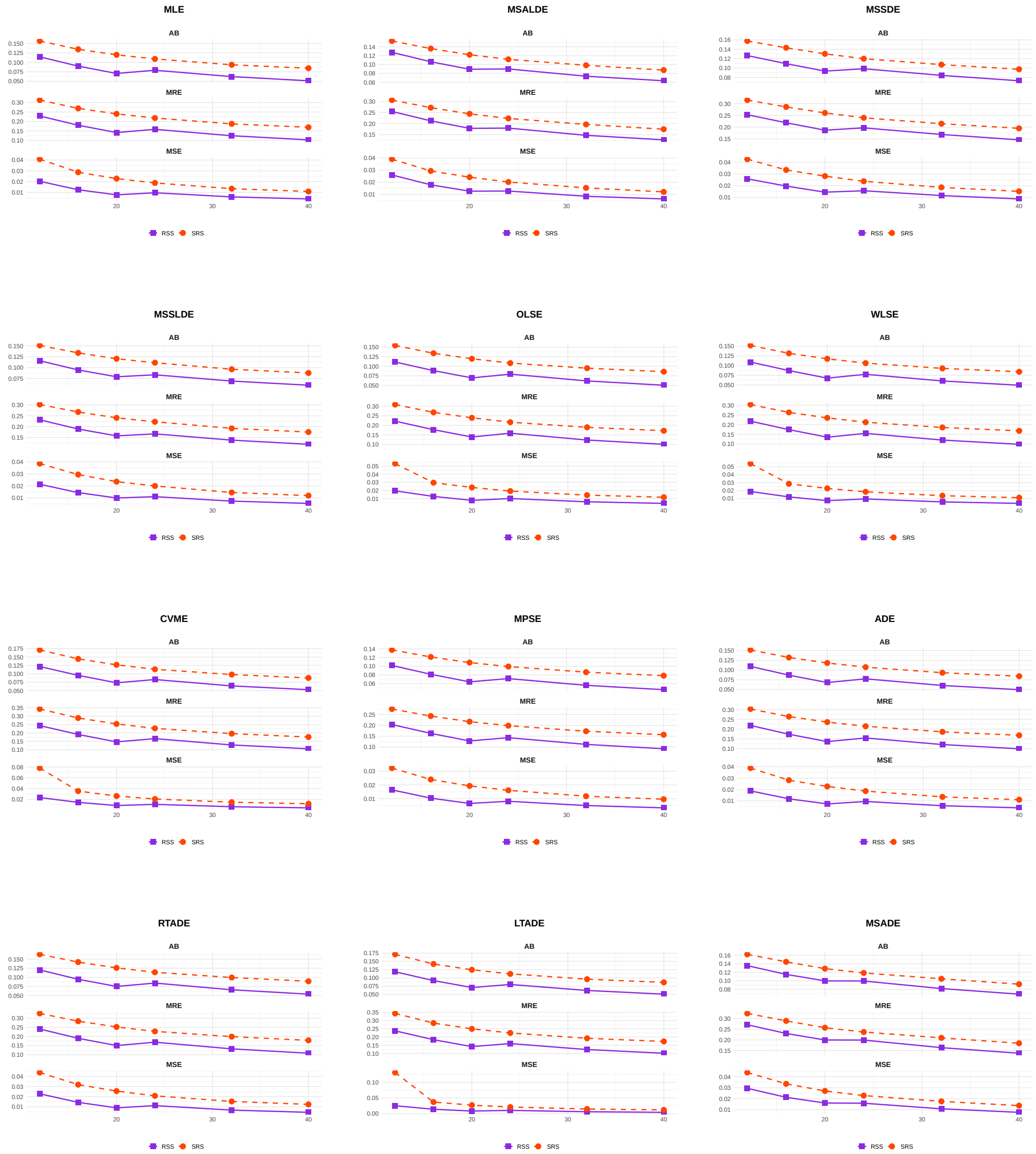


Figure 2: Comparison between bias, MSE and MRE based on SRS and RSS for estimating $\rho$.

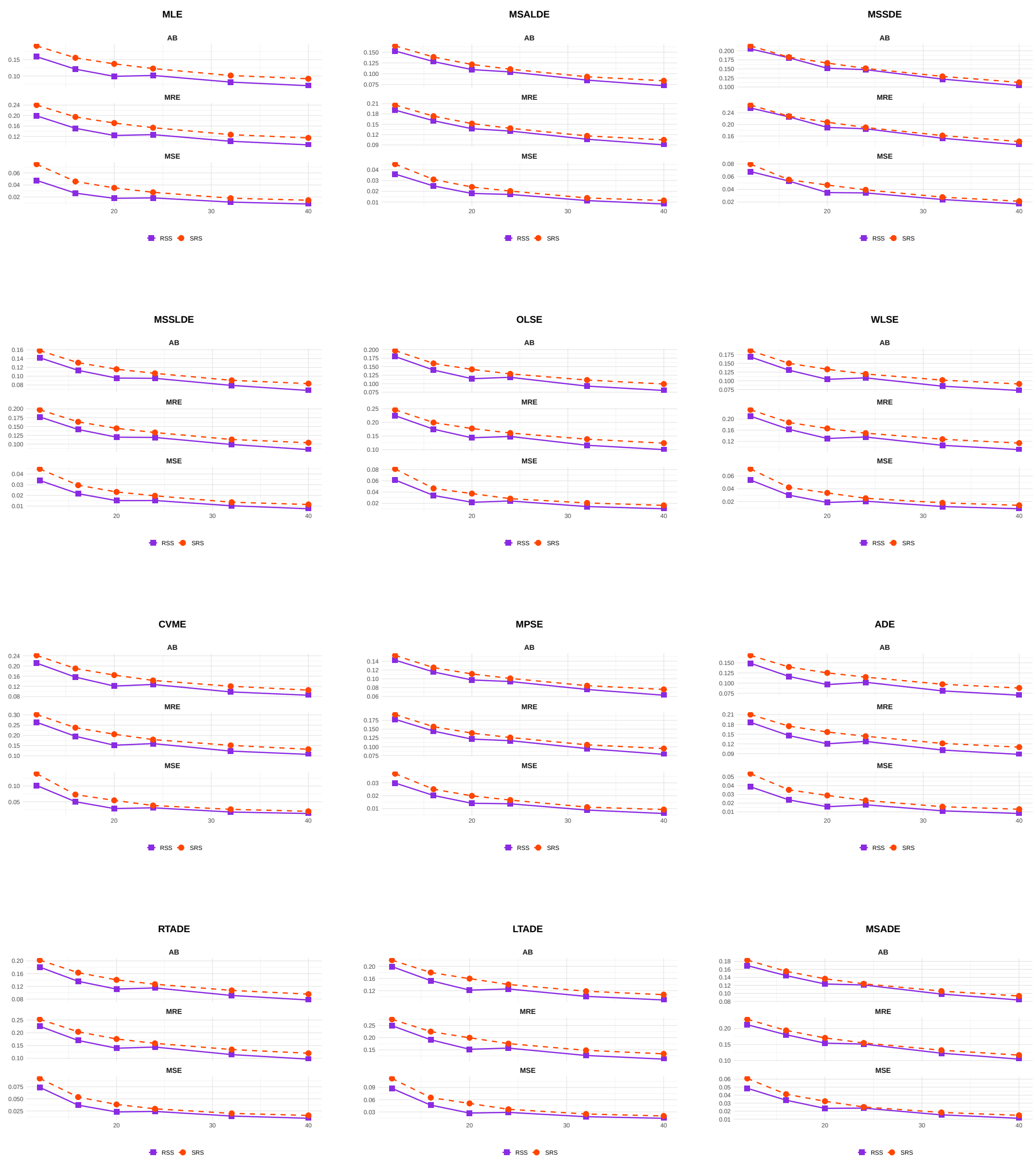


Figure 3: Comparison between bias, MSE and MRE based on SRS and RSS for estimating $\tau$.

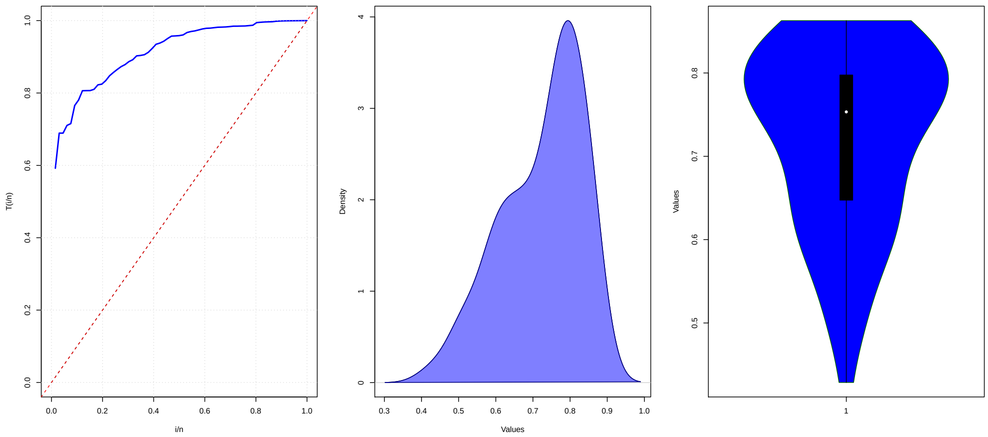

Figure 4: Graphical representation for recovery rates of COVID-19 patients in Spain.

Model adequacy was evaluated using the Kolmogorov-Smirnov (KS) goodness-of-fit test to assess the suitability of the U-IWD distribution for the observed data. Maximum likelihood estimation yielded parameter estimates of $\hat{\rho} = 0.0853$ and $\hat{\tau} = 2.5483$, with corresponding standard errors of 0.0108 and 0.2423, respectively. The KS test produced a test statistic of 0.085341 and a p-value of 0.7223, indicating no significant departure from the hypothesized U-IWD distribution at conventional significance levels. For comparison, the Kumaraswamy distribution yielded a KS distance of 0.099698 with a p-value of 0.528, while the Beta distribution produced a KS distance of 0.11472 with a p-value of 0.3501. These comparative results demonstrate that the U-IWD distribution provides the best overall fit among the considered models. Figure 5 provides complementary visual evidence through multiple diagnostic plots, including the histogram overlaid with the estimated PDF, the ECDF compared with the theoretical CDF, and the probability-probability plot. The close alignment between observed and theoretical patterns across all diagnostic displays reinforces the statistical conclusion that the U-IWD distribution provides an adequate fit to the data.

The estimation methods outlined above—including MPSE, OLSE, MLE, WLSE, ADE, LTADE, RTADE, CVME, MSADE, MSSDE, MSALDE, and MSSLDE were implemented and compared using both sampling strategies. For the comparative analysis, we employed a sample size of 15 units under SRS. For RSS, we maintained equivalence by setting the set size $s = 3$ and the number of cycles $c = 5$, ensuring identical sample sizes across both sampling methodologies while assuming perfect ranking capability. Model performance was evaluated using multiple goodness-of-fit criteria: the Anderson-Darling statistic ($T_1$), the Cramér-von Mises statistic ($T_2$), the Kolmogorov-Smirnov distance ($T_3$), and the corresponding Kolmogorov-Smirnov p-values ($T_3^p$). These comprehensive assessment metrics were applied to both SRS and RSS estimation results, with findings summarized in Table 7. Visual comparisons are presented in Figures 6 and 7.

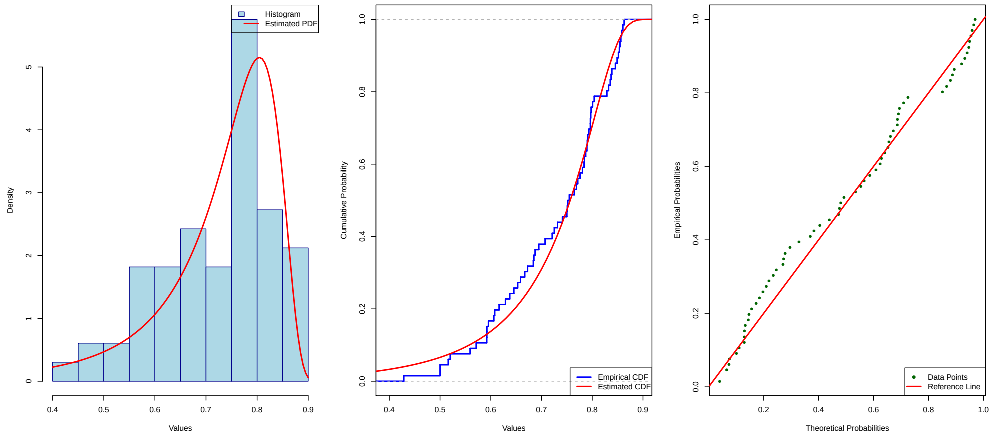


Figure 5: The plots of estimated pdf and cdf and PP plot for the dataset.

| Method | $\hat{\rho}$ | | $\hat{\tau}$ | | $T_1$ | |
|---|---|---|---|---|---|---|
| | SRS | RSS | SRS | RSS | SRS | RSS |
| MPSE | 0.0364 | 0.0588 | 2.1599 | 1.9643 | 2.2286 | 1.1132 |
| OLSE | 0.0543 | 0.073 | 1.8826 | 1.8273 | 2.6725 | 1.531 |
| MLE | 0.0221 | 0.0377 | 2.5028 | 2.3038 | 2.4262 | 0.5911 |
| WLSE | 0.0433 | 0.0642 | 2.0486 | 1.9105 | 2.1986 | 1.2491 |
| CVME | 0.0325 | 0.0502 | 2.2522 | 2.1073 | 1.9718 | 0.7717 |
| ADE | 0.0383 | 0.0538 | 2.1271 | 2.0473 | 2.2302 | 0.8842 |
| LTADE | 0.037 | 0.0424 | 2.1587 | 2.2745 | 2.0588 | 0.7259 |
| RTADE | 0.0392 | 0.0639 | 2.1101 | 1.9336 | 2.2435 | 1.1712 |
| MSADE | 0.0261 | 0.0756 | 2.4181 | 1.8825 | 1.8039 | 1.6634 |
| MSSDE | 0.0581 | 0.0799 | 1.9136 | 1.8166 | 1.5893 | 1.7273 |
| MSALDE | 0.0221 | 0.0452 | 2.5239 | 2.1516 | 2.0874 | 0.7482 |
| MSSLDE | 0.0264 | 0.0397 | 2.3054 | 2.1463 | 3.8969 | 1.5195 |

| Method | $T_2$ | | $T_3$ | | $T_3^p$ | |
|---|---|---|---|---|---|---|
| | SRS | RSS | SRS | RSS | SRS | RSS |
| MPSE | 0.3738 | 0.1157 | 0.1654 | 0.0992 | 0.0541 | 0.5347 |
| OLSE | 0.4084 | 0.1705 | 0.1739 | 0.1178 | 0.037 | 0.3193 |
| MLE | 0.4443 | 0.0655 | 0.1587 | 0.0729 | 0.0721 | 0.8743 |
| WLSE | 0.3477 | 0.132 | 0.1642 | 0.1062 | 0.057 | 0.4463 |
| CVME | 0.336 | 0.0768 | 0.1558 | 0.0879 | 0.0811 | 0.6878 |
| ADE | 0.3686 | 0.0885 | 0.1656 | 0.0926 | 0.0535 | 0.6237 |
| LTADE | 0.3395 | 0.0804 | 0.1601 | 0.0779 | 0.0677 | 0.818 |
| RTADE | 0.3684 | 0.1238 | 0.1661 | 0.1065 | 0.0525 | 0.4422 |
| MSADE | 0.3198 | 0.2227 | 0.1439 | 0.1248 | 0.1301 | 0.2557 |
| MSSDE | 0.1947 | 0.2189 | 0.1318 | 0.1289 | 0.2019 | 0.2231 |
| MSALDE | 0.3794 | 0.0791 | 0.1472 | 0.0913 | 0.1144 | 0.6417 |
| MSSLDE | 0.7231 | 0.2299 | 0.2027 | 0.141 | 0.0088 | 0.1451 |

Table 7: Estimates and test statistics for each method under the two sampling schemes SRS and RSS.

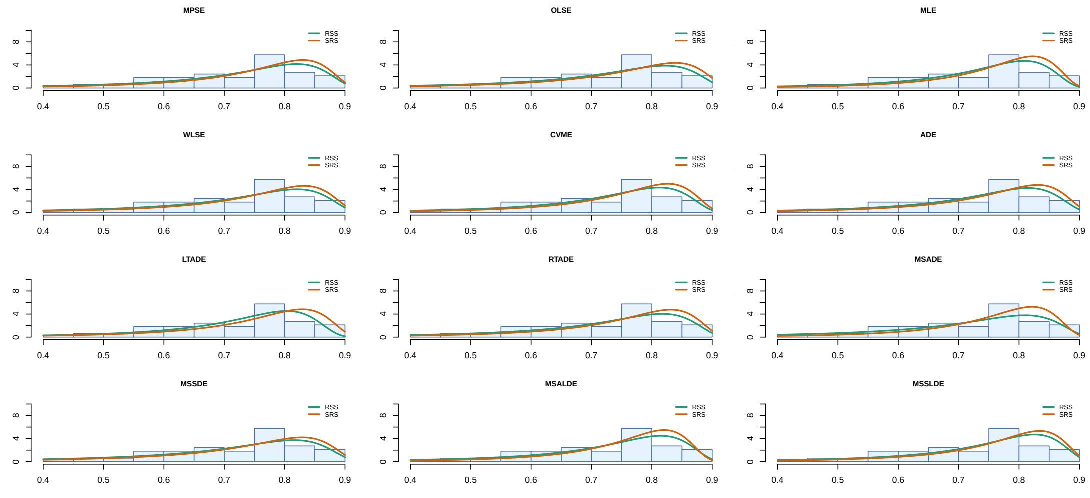


Figure 6: Plots of the estimated PDFs of the U-IWD for $s = 3$ and $c = 5$.

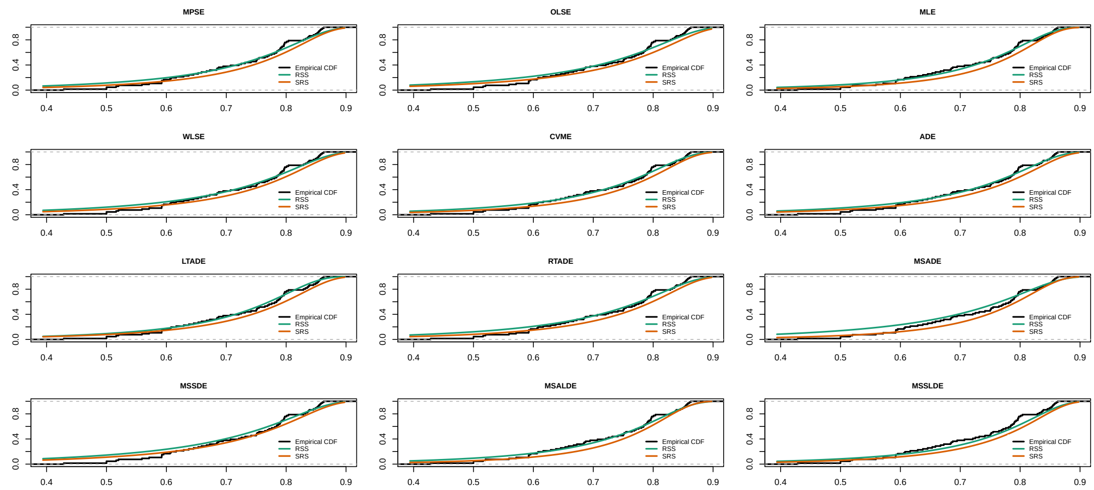


Figure 7: Plots of the estimated CDFs of the U-IWD for $s = 3$ and $c = 5$.

The results presented in Table 7 demonstrate the consistent superiority of RSS over SRS across all estimation methods examined. This superior performance is evidenced by systematically lower values for the Anderson-Darling ($T_1$), Cramér-von Mises ($T_2$), and Kolmogorov-Smirnov distance ($T_3$) statistics, coupled with higher Kolmogorov-Smirnov p-values ($T_3^p$) for RSS-based estimates. These findings corroborate the theoretical expectations regarding ranked sampling's advantages, where the structured data collection process yields more precise parameter estimates and improved distributional fit. The visual evidence presented in Figures 6 and 7 provides compelling support for these statistical conclusions. The graphical comparisons clearly illustrate that RSS-based estimates achieve closer alignment with the observed data patterns compared to their SRS counterparts, confirming the enhanced modeling accuracy achieved through ranked sampling methodology.

## 5 Conclusions

This research examines the effectiveness of different parameter estimation techniques for the unit inverse Weibull distribution ($U-IWD$) within both RSS and SRS methodologies. We implement a comprehensive suite of estimation approaches, includingMPSE, OLSE, MLE, WLSE, ADE, LTADE, RTADE, CVME, MSADE, MSSDE, MSALDE, and MSSLDE.

Our simulation analysis reveals that RSS methodology systematically outperforms SRS across all estimation procedures, yielding substantial improvements in statistical precision through reduced MSE, decreased bias, and lower MARE. The superiority of RSS is further validated through empirical analysis using COVID-19 data, which confirms the practical utility of this sampling approach in real-world statistical applications. The results establish RSS as the optimal framework for parameter estimation in U-IWD modeling, offering enhanced accuracy and reliability that prove especially valuable in applications demanding high statistical precision and robust inferential outcomes.

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