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An Extension of Akash Distribution by Using Biweight Kernel function

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In this paper, we introduce new continuous distribution called Biweight-Akash distribution. This distribution derived by using the Akash distribution and the Biweight kernel function. Several properties of Biweight-Akash distribution are studied such as the reliability and hazard rate functions, the moments, Re'nyi entropy. The maximum likelihood estimators of the unknown parameter is derived. A simulation study is introduced for different values of the parameter distribution. A comparison between the performance of the new distribution and well-known distributions using a real data sets are illustrated. The new distribution fits more than Exponential, Lindley and Akash distributions.

keywords: Akash Distribution, Biweight kernel function, Moments and entropy.

1 Introduction

The modeling and analyzing lifetime data are needed in many applied sciences including social science, medicine, engineering, insurance, biology and many others. Many generalization of new probability models were suggested for analyzing lifetime data. One of them that used for modelling medical science and engineering data is Akash distribution Shanker (2016). The Akash distribution is a mixture of an exponential distribution with scale parameter $[\theta]$ and gamma distribution with shape parameter 3 and scale parameter with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$ respectively. The probability density function pdf and cumulative distribution function CDF of Akash distribution are defined as

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respectively,

$$f(x) = \frac{\theta^3}{\theta^2 + 2}(1 + x^2)e^{-\theta x} \tag{1}$$

and

$$F(x) = 1 - \left[1 + \frac{\theta x(\theta x + 2)}{\theta^2 + 2}\right] e^{-\theta x} \tag{2}$$

One of the symmetric kernels is the Biweight kernel function Hansen (2009). It is defined as:

$$K(u) = \begin{cases} \frac{15}{16}(1 - u^2)^2 : |u| < 1 \\ 0 : |u| \geq 1 \end{cases} \tag{3}$$

In this article, a new continuous probability distribution is proposed for fitting real data using Biweight kernel function and the Akash distribution. The suggested distribution is named the Biweight-Akash distribution. The main motivation of this paper is to generalize new probability distribution from the base distribution without adding any new parameter that do more flexible in fitting the original one.

Many researchers generalized some distributions that have been studied and proposed over years. Such as, the transmuted Janardan distribution Al-Omari et al. (2017). The transmuted Lindley distribution by using the quadratic transmutation map developed by Shaw and Buckley (2007) Umar et al. (2019). M.Al-khazaleh, Al-Omari, and A. Al-khazaleh suggested the Transmuted Tow parameter Lindley Distribution Al-khazaleh et al. (2016). AL- khazaleh (2016) used the transmutation map to generalize Burr type XII distribution Al-Khazaleh (2016). Alkhazaalh and Al-Zoubi (2021) proposed Epanechnikov-Exponential distribution using kernel functions . Al-Khazaleh and Al-zoubi (2021) proposed Biweight-Exponential distribution using kernel function .

In this paper, we use the Biwight kerenel function to generate new distribution called the Biweight- Akash distribution. The repose of this article is organized as follows: in Section 2, we defined the pdf and cdf of the Biweight-Akash Distribution. Section 3 defines the Reliability Analysis of the new distribution. The Moment and generated function is defined in Section 4. The Maximum likelihood Estimation are discussed in Section 5. We have defined the Re'nyi entropy in Section 6. The Simulation Study is stated in Sections 7. Finally in Section 8, Real data applications are mentioned.

2 Biweight-Akash Distribution

A random variable X is said to have Biweight Akash probability distribution (BWAD) with its CDF of G(x) and the pdf of g(x) are obtained as follows respectively :

$$G(x) = 2\int_{0}^{F(x)} K(u)du \tag{4}$$

Where F(x) is the CDF of the Akash distribution and K(u) is the Biweight kernel function. Now by substituting equation (2) and equation (3) in equation (4), we can compute the CDF of BWAD as follows:

$$G(x) = 2\int_{0}^{1-\left[1+\frac{\theta x(\theta x+2)}{\theta^2+2}\right]e^{-\theta x}} \frac{15}{16}\left(1-u^2\right)^2 du$$

$$G(x) = \begin{bmatrix} \frac{15}{8}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right) \\ -\frac{2}{3}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^3 \\ +\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^5 \end{bmatrix} \tag{5}$$

The derivative of the CDF of BWAD Equation (5) we can derive the pdf of BWAD which is as follows:

$$g(x) = \left[\frac{-\theta^3 e^{-\theta x}(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]}{8(2+\theta^2)}\right] \tag{6}$$

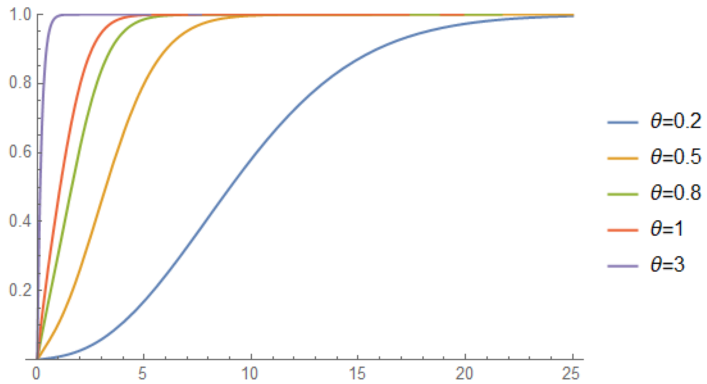

Figure 1: The CDF of BWAD

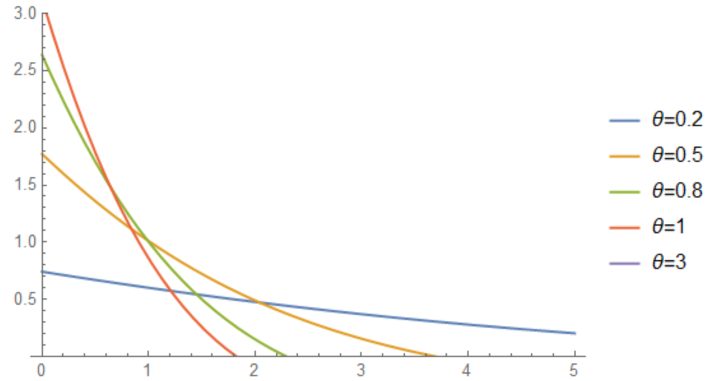

Figure 2: The Pdf of BWAD

Figures (1) and (2) show the plot of the CDF and pdf of BWAD for different values of $\theta = 0.2,\ 0.5,\ 0.8,\ 1,\ 3$ respectively.

3 Reliability Analysis

The Reliability Function is a central component of reliability investigation, and can be defined as the complement of the CDF as follows Forbes et al. (2011).

$$R(x) = 1 - G(x) =$$

$$1 - \left[\begin{array}{l} \frac{15}{8}\left(1 - \left[1 + \frac{\theta^2x^2+2\theta x}{\theta^2+2}\right] e^{-\theta x}\right) - \frac{2}{3}\left(1 - \left[1 + \frac{\theta^2x^2+2\theta x}{\theta^2+2}\right] e^{-\theta x}\right)^3 \\ +\frac{1}{5}\left(1 - \left[1 + \frac{\theta^2x^2+2\theta x}{\theta^2+2}\right] e^{-\theta x}\right)^5 \end{array} \right]$$

The survival function of the BWAD with for different values of the hazard rate function h(x) of the BWAD is given by:

$$h(x) = \frac{g(x)}{1-G(x)} = \frac{\left[-\theta^3 e^{-\theta x}(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]\right]}{8(2+\theta^2)\left\{1-\left[\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)\left\{\frac{15}{8}-\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\left[\frac{2}{3}+\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\right]\right\}\right]\right\}}$$

The reversed hazard rate function (RH) of BWAD can be obtained as follows:

$$RH = \frac{g(x)}{G(x)} = \frac{\left[-\theta^3(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]\right]}{8(2+\theta^2)\left[\left(e^{\theta x}-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)\left\{\frac{15}{8}-\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\left[\frac{2}{3}+\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\right]\right\}\right]}$$

The cumulative hazard function of BWAD is calculated as follows:

$$c.h.f = -\ln[1-G(x)] = -\left\{ \begin{array}{l} \ln\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right) \\ +\ln\left[\left\{ \begin{array}{l} \left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^{-1} \\ -\left[\left\{ \begin{array}{l} \frac{15}{8}-\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2 \\ \left[\frac{2}{3}+\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\right] \end{array}\right\}\right] \end{array}\right\}\right] \end{array}\right\}$$

While, The Odd function O(x) of BWAD is calculated as follows:

$$O(x) = \frac{G(x)}{1-G(x)} \frac{\left[\left[\left\{\frac{15}{8}-\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\left[\frac{2}{3}+\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\right]\right\}\right]\right]}{\left[\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^{-1}-\left[\left\{\frac{15}{8}-\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\left[\frac{2}{3}+\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^2\right]\right\}\right]\right]}$$

4 Moment and generated function

In this section, the r^{th} moment of the BWAD is derived. Also, the mean, variance, coefficient of kurtosis, coefficient of skewness, coefficient of variation, and moment generating function.

4.1 Moments

Theorem: The r moment of the BWAD random variable is given by:

$$E(X^r) =$$

$$\frac{r!}{\theta^r}\left[\begin{array}{l}
7\theta^2\left[1+\frac{(r+2)(r+1)}{(\theta)^2}\right]+8\theta^2\left[\frac{4}{(3)^{r+1}}-\frac{4}{(4)^{r+1}}+\frac{1}{(5)^{r+1}}\right] \\
+\frac{4\theta(r+1)}{(\theta^2+2)}\left[\frac{384\theta^3+67\theta^5+32\theta^7+256}{(\theta^2+2)^3(3)^{r+2}}-\frac{32\theta}{(4)^{r+2}}+\frac{71\theta^3+48\theta^5+8\theta^7+64\theta}{(\theta^2+2)^3(5)^{r+2}}\right] \\
+\frac{(r+2)(r+1)}{(\theta^2+2)^2}\left[\frac{2048\theta^4+768\theta^6+97\theta^8+512}{(\theta^2+2)^2(3)^{r+3}}-\frac{576\theta^2+128\theta^4+128}{(4)^{r+3}}+\frac{192\theta^2+40\theta^4+32}{(5)^{r+3}}\right] \\
+\frac{4(r+3)(r+2)(r+1)}{(\theta^2+2)^2}\left[\frac{64\theta^2+64}{(3)^{r+4}}-\frac{352\theta^2+77\theta^4+128}{(4)^{r+4}}+\frac{128\theta^2+40\theta^4+32}{(\theta^2+2)(5)^{r+4}}\right] \\
+\frac{4(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^2(\theta)^2}\left[\frac{24\theta^4+64\theta^2+16}{(3)^{r+5}}-\frac{304\theta^2+48\theta^4+224}{(4)^{r+5}}+\frac{400\theta^4+184\theta^6+20\theta^8+135\theta^2}{(\theta^2+2)^2(5)^{r+5}}\right] \\
+\frac{32(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^2}\left[\frac{4}{(3)^{r+6}}-\frac{16\theta^2+28}{(\theta^2+2)(4)^{r+6}}+\frac{38\theta^2+9\theta^4+24}{(\theta^2+2)^2(5)^{r+6}}\right] \\
+\frac{16(r+6)(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^2}\left[\frac{8}{(3)^{r+7}}-\frac{8\theta^2+36}{(\theta^2+2)(4)^{r+7}}+\frac{44\theta^2+5\theta^4+52}{(\theta^2+2)^2(5)^{r+7}}\right] \\
+\frac{2(r+7)(r+6)(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^4(\theta)^5}\left[-\frac{192\theta^5+96\theta^7+384}{(4)^{r+8}}+\frac{112\theta^7+263\theta^5}{(5)^{r+8}}\right] \\
+\frac{8(r+8)(r+7)(r+6)(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^3}\left[-\frac{4}{(4)^{r+9}}+\frac{5\theta^2+32}{(\theta^2+2)(5)^{r+9}}\right] \\
+\frac{64(r+9)(r+8)(r+7)(r+6)(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^4(5)^{r+10}} \\
+\frac{8(r+10)(r+9)(r+8)(r+7)(r+6)(r+5)(r+4)(r+3)(r+2)(r+1)}{(\theta^2+2)^2(5)^{r+11}}
\end{array}\right] \tag{7}$$

Proof: by using the definition of the r^{th} expectation $E(x^r)=\int_0^\infty x^r g(x)dx$ and by using gamma function $\int_0^\infty x^{r-1}e^{-\tau x}dx=\frac{\Gamma(r)}{\tau^r}$ and $\Gamma(r)=(r-1)!$,we have:

$$\begin{array}{l}
=\frac{15\theta\,\Gamma(r+1)}{4(\theta^2+2)(\theta)^{r+1}}\left[2-\frac{\theta^2(r+2)(r+1)}{(\theta)^2}\right]+ \\
\frac{5\Gamma(r+1)}{2(\theta^2+2)^3(3)^r(\theta)^{r-2}}\left[\begin{array}{l}(\theta^7+4\theta^5+4\theta^3)\Gamma(r+1)+\frac{(4\theta^5+8\theta^3)\Gamma(r+2)}{3}+\frac{(3\theta^4+4\theta^3+12\theta^2+4)\Gamma(r+3)}{(3\theta)^2}+ \\ \frac{(4\theta^2+8)\Gamma(r+4)}{(3)^3(\theta)^2}+\frac{(3\theta^2+4\theta+8)\Gamma(r+5)}{(3)^4(\theta)^2}+\frac{\Gamma(r+7)}{(3)^5\theta}\end{array}\right] \\
-\frac{15}{(\theta^2+2)^4 2(4)^{r+1}(\theta)^r}\left[\begin{array}{l}(\theta^8+6\theta^6+12\theta^4+8\theta^2)\Gamma(r+1)+\frac{(6\theta^6+24\theta^4+24\theta^2)\Gamma(r+2)}{(4)} \\ +\frac{(4\theta^6+30\theta^4+48\theta^2+8)\Gamma(r+3)}{(4)^2}+\frac{(18\theta^4+56\theta^2+24)\Gamma(r+4)}{(4)^3} \\ +\frac{(6\theta^4+42\theta^2+36)\Gamma(r+5)}{(4)^4}+\frac{(19\theta^2+32)\Gamma(r+6)}{(4)^5}+\frac{(3\theta^2+18)\Gamma(r+7)}{(4)^6}+\frac{7\Gamma(r+8)}{(4)^7}\end{array}\right]
\end{array}$$

$$\begin{array}{l}
+\frac{(15\theta^{11}+120\theta^9+360\theta^7+480\theta^5+240\theta^3)\Gamma(r+1)}{8(\theta^2+2)^5(5\theta)^{r+1}}+\frac{(120\theta^{10}+720\theta^8+1440\theta^6+960\theta^4)\Gamma(r+2)}{8(\theta^2+2)^5(5\theta)^{r+2}} \\
+\frac{(75\theta^{11}+840\theta^9+2520\theta^7+2400\theta^5+240\theta^3)\Gamma(r+3)}{8(\theta^2+2)^5(5\theta)^{r+3}}+\frac{(480\theta^{10}+2640\theta^8+3840\theta^6+960\theta^4)\Gamma(r+4)}{8(\theta^2+2)^5(5\theta)^{r+4}} \\
+\frac{(150\theta^{11}+1800\theta^9+4200\theta^7+1920\theta^5)\Gamma(r+5)}{8(\theta^2+2)^5(5\theta)^{r+5}}+\frac{(720\theta^{10}+3120\theta^8+2400\theta^6)\Gamma(r+6)}{8(\theta^2+2)^5(5\theta)^{r+6}} \\
+\frac{(150\theta^{11}+1560\theta^9+2040\theta^7)\Gamma(r+7)}{8(\theta^2+2)^5(5\theta)^{r+7}}+\frac{(480\theta^{10}+1200\theta^8)\Gamma(r+8)}{8(\theta^2+2)^5(5\theta)^{r+8}}+\frac{(75\theta^{11}+480\theta^9)\Gamma(r+9)}{8(\theta^2+2)^5(5\theta)^{r+9}} \\
+\frac{120\theta^{10}\Gamma(r+10)}{8(\theta^2+2)^5(5\theta)^{r+10}}+\frac{15\theta^{11}\Gamma(r+11)}{8(\theta^2+2)^5(5\theta)^{r+11}}
\end{array}$$

Then, by some simplification, the proof is done.
Remarks: For r=1 in (7), we have the mean of BWAD:

$$E(X) = \frac{1}{\theta}\left[\begin{array}{l} 7\theta^2 + 42 + \frac{121150\theta^{10}+784888\theta^8+4613040\theta^6+3697680\theta^4+552960\theta^4+552960\theta^3+4554240\theta^2+30720}{7290\theta^2(\theta^2+2)^4} \\ -\frac{20280\theta^{15}+305600\theta^{13}+1957920\theta^{11}+3104460\theta^9+15800020\theta^7+10909560\theta^5+1689600\theta^3+506520}{10240\theta^5(\theta^2+2)^4} \\ +\frac{81250000\theta^{15}+687500000\theta^{13}+2303750000\theta^{11}+4090990000\theta^9+61394360000\theta^7+6384395200\theta^5}{10240\theta^5(\theta^2+2)^4} \end{array}\right]$$

The second moment can be computed by substituting r=2 in (7), we get:

$$E(x^2) = \frac{1}{11664 \times 10^{10}(2+\theta^2)^5\theta^2}$$

$$\left[\begin{array}{l} -132415187887692806 + 4822990848 \times 10^6\theta - 263403141062127575\theta^2 \\ +59904 \times 10^{11}\theta^3 - 197597687013586250\theta^4 + 25344 \times 10^{11}\theta^5 \\ -658725923587500000\theta^6 + 3456 \times 10^{14}\theta^7 - 808174341 \times 10^7\theta^8 + 409617 \times 10^8\theta^{10} \end{array}\right]$$

The variance of BWAD is defined as follows:

$$\mathrm{V}ar(X) = E(X^2) - (E(X))^2$$

$$\mathrm{v}ar(x) = \frac{-1}{\left(15116544\times 10^{18}\theta^2(2+\theta^2)^{10}\right)}$$

$$\left(\begin{array}{l} \left(\begin{array}{l} 334061077492352 + 768 \times 10^{11}\theta + 871945955085350\theta^2+ \\ 11136 \times 10^{10}\theta^3 + 827772780801875\theta^4 + 5376 \times 10^{10}\theta^5+ \\ 337691280150000\theta^6 + 864 \times 10^{10}\theta^7 + 5373459 \times 10^7\theta^8 + 17091 \times 10^8\theta^{10} \end{array}\right)^2 \\ -1296 \times 10^8\left(2+\theta^2\right)^5 \left(\begin{array}{l} -132415187887692806 + 4822990848 \times 10^6\theta- \\ 263403141062127575\theta^2 + 59904 \times 10^{11}\theta^3- \\ 197597687013586250\theta^4 + 25344 \times 10^{11}\theta^5- \\ 658725923587500000\theta^6 + 3456 \times 10^{11}\theta^7 - 808174341 \times 10^7\theta^8 \\ +409617 \times 10^8\theta^{10} \end{array}\right) \end{array}\right)$$

4.2 Coefficient of Variation

The coefficient of variation (CV) of the BWAD is defined as:

$$CV = \frac{\sigma}{\mu}$$

$$cv = \frac{\left(\begin{array}{l} \left[\begin{array}{l} 7\theta^2 + 42 + \frac{121500\theta^{10}+784888\theta^8+4613040\theta^6+3697680\theta^4+552960\theta^4+552960\theta^3+4554240\theta^2+30720}{729\theta^2(\theta^2+2)^4} \\ -\frac{2028\theta^{15}+30560\theta^{13}+195792\theta^{11}+310446\theta^9+1580002\theta^7+1090956\theta^5+168960\theta^3+506520}{1024\theta^5(\theta^2+2)^4} \\ +\frac{81250000\theta^{15}+687500000\theta^{13}+2303750000\theta^{11}+4090990000\theta^9+6139436000\theta^7+6384395200\theta^5}{1024\theta^5(\theta^2+2)^4} \end{array} \right] \\ -\frac{4}{\theta^3} \left[\begin{array}{l} 7\theta^2 + 12 + \frac{[7776\theta^{12}+830520\theta^{10}+3719160\theta^8+1449472\theta^6+2599936\theta^4+2277376\theta^2+92160]}{2187\theta^2(\theta^2+2)^4} \\ -\frac{\left[\begin{array}{l} 4963840\theta^{13} + 42690560\theta^{11} + 148336160\theta^9 + 261642640\theta^7 + 258733600\theta^5 \\ +5160960\theta^3 + 11017472 \end{array} \right]}{65536\theta^5(\theta^2+2)^4} \\ +\frac{\left[\begin{array}{l} 371875000\theta^{10} + 2017500000\theta^8 + 7770927560\theta^6 + 14435930240\theta^4 \\ +60000000\theta^3 + 28444501440\theta^2 + 253212800 \end{array} \right]}{488281250\theta^2(\theta^2+2)^4} \end{array} \right]^2 \end{array} \right)}{\left[\begin{array}{l} 7\theta^2 + 42 + \frac{\begin{array}{l} 121500\theta^{10} + 784888\theta^8 + 4613040\theta^6 + 3697680\theta^4 + 552960\theta^4 \\ +552960\theta^3 + 4554240\theta^2 + 30720 \end{array}}{729\theta^2(\theta^2+2)^4} \\ -\frac{\begin{array}{l} 2028\theta^{15} + 30560\theta^{13} + 195792\theta^{11} + 310446\theta^9 + 1580002\theta^7 \\ +1090956\theta^5 + 168960\theta^3 + 506520 \end{array}}{1024\theta^5(\theta^2+2)^4} \\ +\frac{\begin{array}{l} 81250000\theta^{15} + 687500000\theta^{13} + 2303750000\theta^{11} + 4090990000\theta^9 \\ +6139436000\theta^7 + 6384395200\theta^5 \end{array}}{1024\theta^5(\theta^2+2)^4} \end{array} \right]}$$

4.3 Moment Generating function

The Moment Generating function (MGF) of a BWAD can be defined as:

$$M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} g(x) dx$$

$$= \int_0^\infty e^{tx} \frac{\theta^3 e^{-\theta x}\left(1+x^2\right) \left[\begin{array}{l} 15 - 16\left(-1 + e^{-\theta x}\left(1 + \frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2 \\ +8\left(-1 + e^{-\theta x}\left(1 + \frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4 \end{array} \right]}{8\left(2+\theta^2\right)} dx$$

By computing the integral using gamma function and make some manipulation algebra we have the MGF of BWAD as follows:

$$
\begin{aligned}
M_x(t) &= \tfrac{7\theta^3}{(\theta-t)}\left[1+\tfrac{2!}{(\theta-t)^2}\right] \\
&+\tfrac{2\theta^3}{(\theta^2+2)^2}\left[\begin{array}{l}
\tfrac{1}{(3\theta-t)}\left[\begin{array}{l}
16\theta^7+64\theta^5+64\theta^3+\tfrac{768\theta^3+134\theta^5+64\theta^7+512}{(\theta^2+2)^2(3\theta-t)} \\
+\tfrac{(2048\theta^4+768\theta^6+97\theta^8+512)}{(\theta^2+2)^2(3\theta-t)^2}+\tfrac{\left(128\theta^3+128\theta\right)3!}{(3\theta-t)^3} \\
+\tfrac{\left(48\theta^4+128\theta^2+32\right)4!}{(3\theta-t)^4}+\tfrac{\left(64\theta^3\right)5!}{(3\theta-t)^5}+\tfrac{\left(64\theta^4\right)6!}{(3\theta-t)^6}
\end{array}\right] \\
-\tfrac{2}{(4\theta-t)}\left[\begin{array}{l}
8\theta^7+32\theta^5+32\theta^3+\tfrac{32\theta^3+32\theta}{(4\theta-t)}+\tfrac{\left(288\theta^2+64\theta^4+64\right)}{(4\theta-t)^2} \\
+\tfrac{\left(352\theta^3+77\theta^5+128\theta\right)3!}{(4\theta-t)^3}+\tfrac{\left(304\theta^2+48\theta^4+224\right)4!}{(4\theta-t)^5} \\
+\tfrac{\left(128\theta^5+224\theta^3\right)5!}{(\theta^2+2)(4\theta-t)^5}+\tfrac{\left(32\theta^6+144\theta^4\right)6!}{(\theta^2+2)(4\theta-t)^6}+\tfrac{\left(96\theta^5+48\theta^7+192\right)7!}{(\theta^2+2)^2(4\theta-t)^7} \\
+\tfrac{\left(8\theta^6\right)8!}{(\theta^2+2)(4\theta-t)^8}
\end{array}\right] \\
+\tfrac{1}{(5\theta-t)}\left[\begin{array}{l}
4\theta^7+16\theta^5+16\theta^3+\tfrac{\left(142\theta^3+96\theta^5+16\theta^7+128\theta\right)}{(\theta^2+2)^3(5\theta-t)}+\tfrac{\left(192\theta^2+40\theta^4+32\right)}{(5\theta-t)^2} \\
+\tfrac{\left(256\theta^3+80\theta^5+64\theta\right)3!}{(\theta^2+2)(5\theta-t)^3}+\tfrac{(800\theta^4+368\theta^6+40\theta^8+270\theta^2)4!}{(\theta^2+2)^2(5\theta-t)^4} \\
+\tfrac{(608\theta^5+144\theta^7+384\theta^3)5!}{(\theta^2+2)^2(5\theta-t)^5}+\tfrac{(352\theta^6+40\theta^8+416\theta^4)6!}{(\theta^2+2)^2(5\theta-t)^6}+\tfrac{(112\theta^7+263\theta^5)7!}{(\theta^2+2)^2(5\theta-t)^7} \\
+\tfrac{(20\theta^8+128\theta^6)8!}{(\theta^2+2)^2(5\theta-t)^8}+\tfrac{\left(32\theta^7\right)9!}{(\theta^2+2)^2(5\theta-t)^9}+\tfrac{(4\theta^8)10!}{(\theta^2+2)^2(5\theta-t)^{10}}
\end{array}\right]
\end{array}\right]
\end{aligned}
$$

5 Maximum likelihood Estimation

The Maximum likelihood Estimation is one of the main methods to estimate the parameters of the distribution. We have $x_1, x_2, \ldots, x_n$ are independent random variables distributed symmetrically and the pdf function is $g(x;\theta)$ and the distribution parameter is unknown θ, we can write the density function as a product of $g(x;\theta)$ number of n times Millar (2011). It can be written as follows:

$L(\theta \backslash x_1, x_2, ...x_n) = \prod_{i=1}^{n} g(x_i;\theta)$

$$
= \prod_{i=1}^{n} \tfrac{15\theta e^{-\theta x_i}}{2(\theta^2+2)}
\left(\begin{array}{l}
1-\tfrac{\theta^2 {x_i}^2}{2}+\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^2\left(\theta^2+\theta^2 {x_i}^2\right) \\
-\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^3\left(\theta^2+\theta^2 {x_i}^2\right) \\
+\tfrac{1}{4}\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^4\left(\theta^2+\theta^2 {x_i}^2\right)
\end{array}\right)
$$

$$
L = \left(\tfrac{15\theta}{2(\theta^2+2)}\right)^n e^{-\theta \sum\limits_{i=1}^{n} x_i}
\left(\begin{array}{l}
1-\tfrac{\theta^2 {x_i}^2}{2}+\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^2\left(\theta^2+\theta^2 {x_i}^2\right) \\
-\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^3\left(\theta^2+\theta^2 {x_i}^2\right) \\
+\tfrac{1}{4}\left(e^{-\theta x_i}+\tfrac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^4\left(\theta^2+\theta^2 {x_i}^2\right)
\end{array}\right)^n
$$

This equation is called the probability function MLE that must be maximized, the probability function $L(\theta)$ with respect to an indeterminate parameter that is difficult for us to maximize, so we resort to the natural logarithm of both sides of the previous equation, so become:

$$l = \ln(L(\theta)) = \ln \prod_{i=1}^{n} g(x_i;\theta) = \sum_{i=1}^{n} \ln\left(g(x_i;\theta)\right)$$

It is necessary to note that lnx is a monotonic incrementing function, so the maximum value of the log probability occurs at the same point as the original probability function, so we resort to using the simpler log probability instead of the original probability.

$$l = \ln(L) = n\ln(\tfrac{15}{2}) + n\ln(\theta) - n\ln(\theta^2+2)$$
$$-\theta\sum_{i=1}^{n} x_i + n\sum_{i=1}^{n}\ln\left[\begin{array}{l} 1-\frac{\theta^2 {x_i}^2}{2}+\left(e^{-\theta x_i}+\frac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^2\left(\theta^2+\theta^2 {x_i}^2\right) \\ -\left(e^{-\theta x_i}+\frac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^3\left(\theta^2+\theta^2 {x_i}^2\right) \\ +\frac{1}{4}\left(e^{-\theta x_i}+\frac{\theta^2 {x_i}^2 e^{-\theta x_i}+2\theta x_i e^{-\theta x_i}}{\theta^2+2}\right)^4\left(\theta^2+\theta^2 {x_i}^2\right) \end{array}\right]$$

By taking the partial derivative with respect to θ, we have

$$\frac{2\theta n}{\theta^2+2} - \frac{3n}{\theta} =$$
$$\left[\sum_{i=1}^{n} x_i - n\sum_{i=1}^{n}\frac{\begin{array}{c} 32e^{-\theta x_i}\left[({x_i}^2+1)\theta^5+2({x_i}^2+2)\theta^3-4({x_i}^2-1)\theta+2x_i\theta^4+6x_i\theta^2-4x_i\right] \\ \times\left[\begin{array}{l} 2e^{-\theta x_i}-\theta^2+\theta^2 e^{-\theta x_i}+{x_i}^2\theta^2 e^{-\theta x_i}+2x_i\theta e^{-\theta x_i}- \\ 2-\left(2e^{-\theta x}-\theta^2+\theta^2 e^{-\theta x_i}+{x_i}^2\theta^2 e^{-\theta x}+2x_i\theta e^{-\theta x_i}-2\right)^3 \end{array}\right] \end{array}}{(\theta^2+2)^3\left[15-16\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i(2+\theta x_i)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x_i}\left(1+\frac{\theta x_i(2+\theta x_i)}{2+\theta^2}\right)\right)^4\right]}\right]$$

This equation can be solved numerically by equating it with zero.

6 Distribution of Order Statistics

Ranking statistics are one of the important things in the fields of statistical theory, demand statistics for a random sample $x_1, x_2, \ldots.., x_n$ and the pdf g(x) and CDF G(x), The values of the placed sample are in ascending order and are as follows
The pdf of the K^{th} Order statistic, for $1 \le k \le n, x_k$:

$$g_k(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k}$$

$$g_k(x) = \left(\begin{array}{l} \frac{n!}{(k-1)!} \left[\frac{-\theta^3 e^{-\theta x}(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]}{8(2+\theta^2)} \right] \\ \times \left[\begin{array}{l} \frac{15}{8}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)-\frac{2}{3}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^3 \\ +\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^5 \end{array} \right]^{k-1} \\ \times \left[\begin{array}{l} 1-\frac{15}{8}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)+\frac{2}{3}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^3 \\ -\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^5 \end{array} \right]^{n-k} \end{array} \right)$$

The PDF of the minimum of BWAD is:

$$g_1(x) = \left(\begin{array}{l} n \left[\frac{-\theta^3 e^{-\theta x}(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]}{8(2+\theta^2)} \right] \\ \times \left[\begin{array}{l} 1-\frac{15}{8}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)+\frac{2}{3}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^3 \\ -\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^5 \end{array} \right]^{n-1} \end{array} \right)$$

The PDF of the maximum of BWAD is:

$$g_n(x) = \left(\begin{array}{l} n \left[\frac{-\theta^3 e^{-\theta x}(1+x^2)\left[15-16\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^2+8\left(1-e^{-\theta x}\left(1+\frac{\theta x(2+\theta x)}{2+\theta^2}\right)\right)^4\right]}{8(2+\theta^2)} \right] \\ \times \left[\begin{array}{l} \frac{15}{8}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)-\frac{2}{3}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^3 \\ +\frac{1}{5}\left(1-\left[1+\frac{\theta^2x^2+2\theta x}{\theta^2+2}\right]e^{-\theta x}\right)^5 \end{array} \right]^{n-1} \end{array} \right)$$

7 Rényi Entropy

Entropy is an important and fundamental concept in information theory because it is a measure of randomness and uncertainty in the probability distribution. There is also more than one definition of entropy, but it is often not suitable for all applications equally.

By Rényi (1961) Define what he called R‘enyi Entropy, which is the most general concept of measures of information to be preserved and combined with the axioms of probability, the R‘enyi entropy is given as follows:

$$E_R = \frac{1}{1-p} \ln \int_x [g(x)]^p], \qquad p > 0,\, p \neq 1$$

$$E_R =$$
$$\left(\begin{array}{l} \frac{1}{1-p}\left[3p\ln(\theta) + p\ln(8(2+\theta^2))\right] \\ \sum_{i=1}^{p}\sum_{k=0}^{p}\sum_{\sum a_i=2p-2k}\sum_{\sum b_i=4p}\sum_{\sum c_i=p}\binom{p}{c_1c_2}\binom{4p}{b_1b_2b_3b_4b_5b_6}\binom{2p-2k}{a_1a_2a_3a_4a_4a_6}\binom{p}{k}\binom{p}{i}\left(\frac{1}{2+\theta^2}\right)^{6p-2k} \\ (2)^{4p-k+a_1+a_4+a_5}(-1)^{2p+a_5+a_6+b_5+b_6}(15)^{p-i}(\theta)^{2a_2+2a_3+a_4+2a_6+2b_2+2b_3+b_4+2b_6} \\ \times\frac{\Gamma(2a_3+a_4+2b_3+b_4+2c_2+1)}{((p+a_1+a_2+a_3+a_4+b_1+b_2+b_3+b_4)\theta)^{(2a_3+a_4+2b_3+b_4+2c_2+1)}} \end{array} \right)$$

8 Simulation Study

In this section, we studied the behavior of the MLE of the BWAD parameter using the R software program R Core Team et al. (2013) to run a simulation study, in which we generate 1000 samples from BAD distribution with different simple sizes, which are = 30, 80, 100, 200, 500, with = 0.5, 1, 2, 4, 6, 9 We have also calculated the mean absolute space θ, which we define according to the following relationship Bias, the mean squared error of the MSE estimates MSE, and the average relative estimates MRE by Davis (1975). The result is shown in the Table 1 below.

By studying the results, we note that the values of Bias, MSE, MRE, decrease when the sample size increases, and therefore, the MLE property maintains the consistency property of all parameter values. As the values of increase, each of the following values increases Bias, MSE, MRE

9 Application

In this section, we used three different data set to compare the goodness of fit of the following distributions [using program mathematical]:

- Exponential distribution
- Lindley distribution
- Akash distribution
- Biweight-Akash distribution

One of the most widely used techniques in statistical modeling is the Akaike Information Criterion (AIC). The first criterion for choosing a model for global approval is defined as follows Cavanaugh and Neath (2019):

$$AIC = -2lnL + 2k \tag{8}$$

where k is the number of parameters.

The Akaike information criterion (AIC)was introduced by Akaike (1973, 1974). The AIC

Table 1: Simulation results of BWAD for different sample size and different parameters values

N	$\breve{\theta}$	$\theta = 0.5$	$\theta = 1$	$\theta = 2$	$\theta = 4$	$\theta = 6$	$\theta = 9$
$\breve{\theta}$	1.95592	2.35480	2.75303	3.55169	6.44541	9.60678	
30	Bias	0.04408	0.14520	0.24881	0.47765	0.87614	1.25212
30	MSE	0.02718	0.11126	0.22446	0.53397	1.36632	2.64611
30	MRE	0.02204	0.05808	0.08294	0.11941	0.12516	0.12521
80	$\breve{\theta}$	1.99325	2.45054	2.87090	3.68429	6.48620	9.52736
80	Bias	0.00675	0.04953	0.12970	0.32057	0.64728	0.95339
80	MSE	0.00423	0.03886	0.11891	0.35255	0.85176	1.62386
80	MRE	0.00338	0.01981	0.04323	0.08014	0.09247	0.09534
100	$\breve{\theta}$	2.00000	2.49684	2.98083	3.88899	6.60261	9.56284
100	Biea	0.00000	0.00316	0.01917	0.11101	0.41367	0.58617
100	MSE	0.00000	0.00250	0.01660	0.11459	0.48264	0.77250
100	MRE	0.00000	0.00126	0.00639	0.02775	0.05910	0.05862
200	$\breve{\theta}$	2.00000	4.50000	2.99830	3.98679	6.81655	9.69780
200	Bias	0.00000	0.00000	0.00170	0.01321	0.18384	0.34227
200	MSE	0.00000	0.00000	0.00145	0.01330	0.19259	0.36306
200	MRE	0.00000	0.00000	0.00057	0.00330	0.02626	0.03423
500	$\breve{\theta}$	2.00000	2.50000	3.00000	4.00000	6.98912	9.89421
500	Bias	0.00000	0.00000	0.00000	0.00000	0.01088	0.10608
500	MSE	0.00000	0.00000	0.00000	0.00000	0.00891	0.10122
500	MRE	0.00000	0.00000	0.00000	0.00000	0.00155	0.01061

is unbiased estimator when the sample size is big and the dimension of the candidate model is small. Hurvich and Tsai (1989) suggested the corrected Akaike information criterion (AICC) for the cases when there is a negative bias, which is Cavanaugh (1997):

$$AICC = AIC + \frac{2k(k+1)}{n-k-1} \tag{9}$$

where k is the number of parameters and n is the sample size.
The Bayesian Information Criterion (BIC) is used for selecting statistical models, which is given by Neath and Cavanaugh (2012):

$$BIC = -2ln(L) + kln(n) \tag{10}$$

Based on parameter estimate, -2logLik, Akaike Information Standard (AIC), Akaike Information Criterion Corrected (AICC), Bayesian Information Criterion (BIC). The results are shown in Table 2 for first dataset, Table 3 for second dataset and Table 4 for third dataset.

The first dataset: This data set is obtained from Smith & Naylor and represents the strength of 1.5 cm of fiberglass tested at the National Physical Laboratory in England Smith and Naylor (1987)
0.55 0.93 1.25 1.36 1.49 1.52 1.58 1.61 1.64 1.68 1.73 1.81 2 0.74 1.04 1.27 1.39 1.49 1.53 1.59 1.61 1.66 1.68 1.76 1.82 2.01 0.77 1.11 1.28 1.42 1.5 1.54 1.6 1.62 1.66 1.69 1.76 1.84 2.24 0.81 1.13 1.29 1.48 1.5 1.55 1.61 1.62 1.66 1.7 1.77 1.84 0.84 1.24 1.3 1.48 1.51 1.55 1.61 1.63 1.67 1.7 1.78 1.89

Table 2: Exponential, Lindley, Akash, BWAD goodness of fit test results for the first data

Distribution	$\breve{\theta}$	-2loglik	AIC	AICC	BIC
ED	0.663647	177.66	179.66	179.73	181.80
LD	0.996116	162.56	164.56	164.62	166.70
AD	1.355444	163.72	165.73	165.79	167.87
BWAD	0.439288	41.17	43.17	43.24	45.31

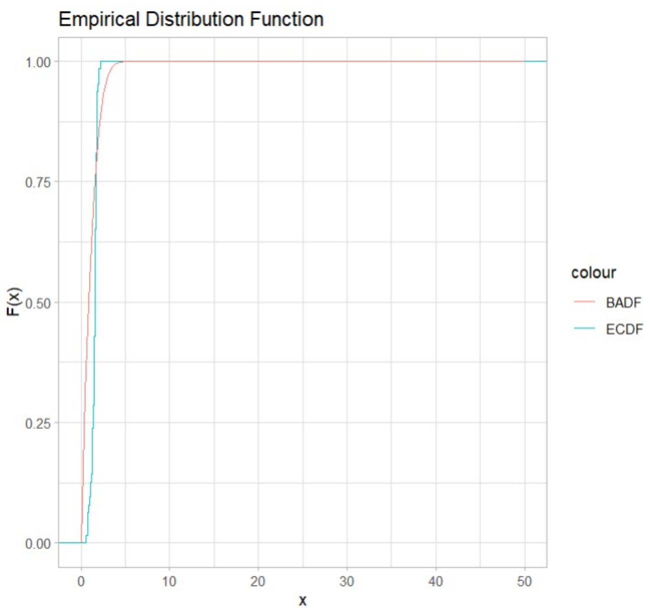

Figure 3: The Empirical CDF against the CDF for BWAD

The second dataset: Birnbaum and Saunders Birnbaum and Saunders (1969) reported data for 6061 fatigue life with T6 aluminum stamps cut parallel to the rolling direction. It revolves at a speed of 18 revolutions per second. There were 101 observations in total, with a maximum pressure per cycle of 31,000 psi. After removing 65, the data (*10-3) is shown below.
5 25 31 32 34 35 38 39 39 40 42 43 43 43 44 44 47 47 48 49 49 49 51 54 55 55 55 56 56 56 58 59 59 59 59 59 63 63 64 64 65 65 65 66 66 66 66 66 67 67 67 68 69 69 69 69 71 71 72 73 73 73 74 74 76 76 77 77 77 77 77 77 79 79 80 81 83 83 84 86 86 87 90 91 92 92 92 92 93 94 97 98 98 99 101 103 105 109 136 14.

Table 3: Exponential, Lindley, Akash, BAD goodness of fit test results for the second data

Distribution	$\breve{\theta}$	-2logLik	AIC	AICC	BIC
ED	0.014635	1046.87	1046.87	1046.91	1049.48
LD	0.028859	985.11	985.11	985.15	987.71
AD	0.043876	950.97	952.97	953.01	955.58
BWAD	0.009849	822.49	824.49	824.53	827.09

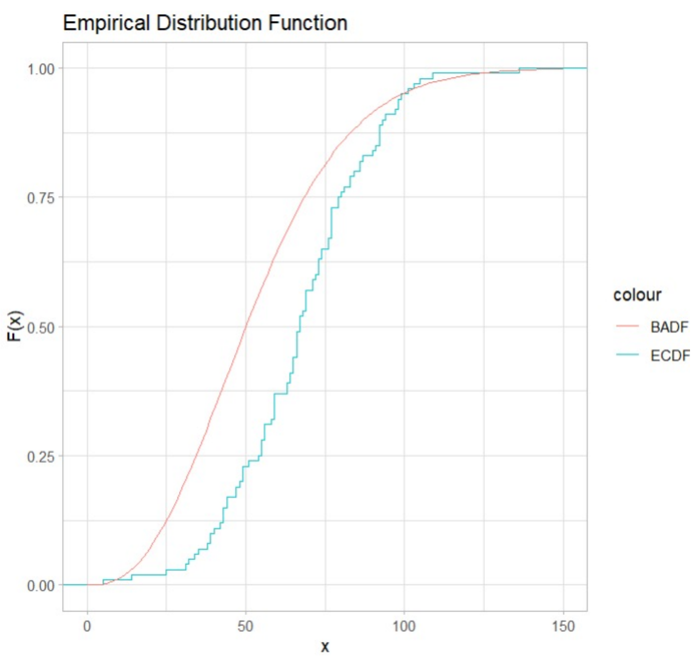

Figure 4: The Empirical CDF against the CDF for BAD

The third dataset: Lawless (1982) provided the data set. The information was gathered during deep groove ball bearing endurance tests. For each of the 23 ball bearings in the life tests, the data is the number of million revolutions before failure.
17.88 28.92 33 41.52 42.12 45.6 48.8 51.84 51.96 54.12 55.56 67.8 68.44 68.64 68.88 84.12 93.12 98.64 105.12 105.84 127.92 128.04 173.4

Table 4: Exponential, Lindley, Akash, BWAD goodness of fit test results for the third data

Distribution	$\breve{\theta}$	-2loglik	AIC	AICC	BIC
ED	0.01	242.87	244.87	245.06	246.01
LD	0.27	231.47	233.47	233.66	234.61
AD	0.04	227.06	229.06	229.25	230.19
BWAD	0.01	193.34	195.34	195.53	196.47

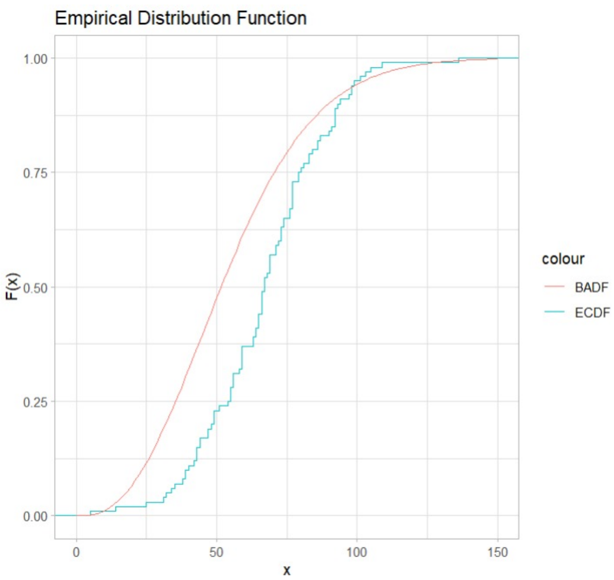

Figure 5: The Empirical CDF against the CDF for BWAD

It points out that the BAD has the lowest values of the parameter estimate, -2logLik, AIC, AICC, BIC statistic, comparing with the alternative outfitted distributions. The BAD is more flexible than the Lindley, Exponential, Akash distributions.

10 Conclusion

In this paper, we use the Biweight Kernel function and Akash distributions to suggest a new distribution called the Biweight Akash distribution. The new distribution is introduced without adding any novel parameters to the base distributions. The statistical properties are considered including the estimation of model parameters and its application validated using three real datasets. The applications show that the new distribution fits better than the Exponential, Lindley and Akash distribution.

References

Al-Khazaleh, A. M. (2016). Transmuted burr type xii distribution: A generalization of the burr type xii distribution. In *International Mathematical Forum*, volume 11, pages 547–556.

Al-Khazaleh, A. M. and Alzoubi, L. (2021). New compound probability distribution using biweight kernel function and exponential distribution. *J. Math. Comput. Sci.*, 11(5):5878–5896.

Al-khazaleh, M., Al-Omari, A. I., and Al-khazaleh, A. (2016). Transmuted two-parameter lindley distribution. *Journal of Statistics Applications and Probability*, 5(3):1–11.

Al-Omari, A. I., Al-khazaleh, A., and Alzoubi, L. M. (2017). Transmuted janardan distribution: A generalization of the janardan distribution. *Journal of Statistics Applications and Probability*, 5(2):1–11.

Alkhazaalh, A. and Al-Zoubi, L. (2021). Epanechnikov-exponential distribution: properties and applications. *General Mathematics*, 29(1):13–29.

Birnbaum, Z. W. and Saunders, S. C. (1969). Estimation for a family of life distributions with applications to fatigue. *Journal of applied probability*, 6(2):328–347.

Cavanaugh, J. E. (1997). Unifying the derivations for the akaike and corrected akaike information criteria. *Statistics & Probability Letters*, 33(2):201–208.

Cavanaugh, J. E. and Neath, A. A. (2019). The akaike information criterion: Background, derivation, properties, application, interpretation, and refinements. *Wiley Interdisciplinary Reviews: Computational Statistics*, 11(3):e1460.

Davis, K. B. (1975). Mean square error properties of density estimates. *The Annals of Statistics*, 3(4):1025–1030.

Forbes, C., Evans, M., Hastings, N., and Peacock, B. (2011). *Statistical distributions.* John Wiley & Sons.

Hansen, B. E. (2009). Lecture notes on nonparametrics. *Lecture notes.*

Lawless, J. (1982). Statistical models and methods for lifetime data, john wiley & sons. *Inc., United States of America.*

Millar, R. B. (2011). *Maximum likelihood estimation and inference: with examples in R, SAS and ADMB.* John Wiley & Sons.

Neath, A. A. and Cavanaugh, J. E. (2012). The bayesian information criterion: background, derivation, and applications. *Wiley Interdisciplinary Reviews: Computational Statistics*, 4(2):199–203.

R Core Team, R. et al. (2013). R: A language and environment for statistical computing.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*, volume 4, pages 547–562. University of California Press.

Shanker, R. (2016). A quasi akash distribution and its applications. *ASR*, 30(1):135–160.

Smith, R. L. and Naylor, J. (1987). A comparison of maximum likelihood and bayesian estimators for the three-parameter weibull distribution. *Journal of the Royal Statistical Society Series C: Applied Statistics*, 36(3):358–369.

Umar, A. A., Eraikhuemen, I. B., Koleoso, P. O., Joel, J., and Ieren, T. G. (2019). On the properties and applications of a transmuted lindley-exponential distribution. *Asian Journal of Probability and Statistics*, 5(3):1–13.