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By Obeidat, Al-Masri

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Exact Bahadur Slope for Combining Independent Tests In The Case of Conditional Laplace Distribution

Mohammed Obeidat*[a] and Abdel-Qadir Al-Masri[a]

[a] *Yarmouk University, Department of Statistics, Irbid, Jordan*

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This paper compares four methods of combining n independent tests. The methods are Fisher, logistic, sum of p-values and inverse normal. It is assumed that n independent test statistics $\{T^{(i)}, i = 1, ..., n\}$ are available to combine the n independent tests. The four methods are compared, as $n \to \infty$, via exact Bahadur slope under the assumption that the test statistics follow Conditional Laplace Distribution $T^{(i)}|\xi_i \sim \mathcal{L}(\tau\xi_i, 1)$, $\xi_i \in [a, \infty)$, $a \geq 0$ where $\xi_1, \xi_2, ...$ are distributed according to the distribution function (DF) H_ξ . It is shown that Fisher's method performs the best as the evidence against the null hypothesis

keywords: Meta-analysis , Conditional Laplace distribution , combining independent tests , Bahadur efficiency , Exact Bahadur Slope.

1 Introduction

Rapid developments in many fields, such as molecular biology and bioinformatics, require more statistical testing to be performed simultaneously. This is because large amount of information are available for researchers in such fields. The more statistical tests to be performed, the higher Type I error rate is expected. A natural question that arises is whether there is a global method that can combine evidences from these sources while keeping error rates at their acceptable levels. In order to control the Type I error, p-values from multiple tests can be adjusted through false discovery rate and its extended

*Corresponding authors: mohammad.obidat@yu.edu.jo

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methods (Cheng and Pounds, 2007). However, large sample sizes are required in large-scale hypothesis testing.

The omnibus methods of combining p-values from numerous individual tests may combine evidence from multiple sources, and reduce the high dimensionality of p-values into rich information. Several omnibus methods for combining p-values are available in literature. These methods include, but not limited to, Fisher's (Fisher, 1932), inverse normal (Stouffer et al., 1949), Lancaster's (Lancaster, 1961), Tippett's (Tippett, 1931), logistic (Mudholkar and George, 1979) and sum of p-values methods (Edgington, 1972). These methods rely on the assumption that the individual tests from which p-values calculated are independent. Recently Dai et al. (2014) and DJ (2019) proposed new approaches for combining p-values from dependent tests.

These methods of combining p-values can be judged using several criteria, such as consistency, admissibility, minimaxity, local optimality, Bahadur exact slope and Pitman efficiency. Marden (1991) introduced recently a new criteria for evaluating test statistics based on p-values. Combining p-values methods are studied and compared by many researchers. A key result was that no single combining method is uniformly the best, see for example Birnbaum (1955) and more recently Loughin (2004) and Kocak (2017).(Littell and Folks, 1971) and Littell and Folks (1973) showed under mild conditions that the Fisher's method is optimal among all methods for combining a finite number of independent tests. A recent study by Heard and Rubin-Delanchy (2018), who compared between several combining methods, provide guidance about how a powerful combiner might be chosen in practice.

Most of the work cited above compare between methods assuming finite number of independent tests. Limited work can be found in literature that assume infinite number of independent tests. Abu-Dayyeh and El-Masri (1994), Al-Masri (2010), and Al-Talib et al. (2019) compare between different combiners when the number of independent test approaches infinity via exact Bahadur slope. They assumed that the distributions under the alternative hypothesis are triangular, exponential, and normal respectively.

Different from the above, we consider combining n independent tests under conditional Laplace distribution. Limiting behavior, as $n \to \infty$, is studied for four methods, namely, Fisher, inverse normal, logistic, and sum of p-values. These methods are chosen for their ease of implementation as well as their spanning the range of comparing criteria. The four combination methods are compared via exact Bahadur slop (EBS).

The remainder of this paper is organized as follows. The specific problem is given in Section 2. Section 3 reviews necessary definitions and preliminaries that are available in literature. Section 4 provides a derivation of the EBS under the Laplace distribution.

2 The specific problem

Consider n-hypotheses of the form

$$H_0^{(i)} : \eta_i = 0 \quad vs \quad H_1^{(i)} : \eta_i \in \Omega_i - \{0\} \tag{1}$$

such that each $H_0^{(i)}$ is rejected for large values of some random variable $T^{(i)}, i = 1, 2, ..., n$. Then, the n-hypotheses are combined into one as

$$H_0^{(i)} : (\eta_1, ..., \eta_n) = (0, ..., 0) \quad vs \quad H_1^{(i)} : (\eta_1, ..., \eta_n) \in \Omega_1 \times \Omega_2 \times \ldots \times \Omega_n - \{(0, ..., 0)\}. \quad (2)$$

Additionally, the p-value of the i-th test is given by

$$P_i = P_{H_0^{(i)}} \left(T^{(i)} > t \right) = 1 - F_{H_0^{(i)}} (t) , \quad (3)$$

where, $F_{H_0^{(i)}} (t)$ is the cumulative distribution function (CDF) of $T^{(i)}$ under $H_0^{(i)}$. Note that $P_i \sim U(0, 1)$ under $H_0^{(i)}$.

This study considers the case when $T^{(i)}|\xi_i \sim \mathcal{L}(\tau\xi_i, 1)$ with $\eta_i = \tau\xi_i,\ i = 1, \ldots, n$. Where, $\xi_1, \xi_2, \ldots \xi_n$ are independent identically distributed with undetermined CDF H_ξ with support defined on $[a, \infty)$, $a \geq 0$.

Therefore (1) reduces to

$$H_0 : \tau = 0 \quad vs \quad H_1 : \tau > 0, \quad (4)$$

We will also consider four combining methods, namely, Fisher, logistic, sum of P-values and inverse normal. These procedures reject H_0 in (4) for large values of $-2\sum_{i=1}^{n} \log(P_i)$, $-\sum_{i=1}^{n} \log\left(\frac{P_i}{1-P_i}\right)$, $-\sum_{i=1}^{n} \Phi^{-1}(P_i)$, and $-\sum_{i=1}^{n} P_i$ respectively. Where, Φ is the cdf of standard normal distribution. These methods will be compared via EBS as $n \to \infty$.

3 Definitions and preliminaries

In this section we list necessary definitions and theorems that are available in literature. These theorems are necessary for our development in the next section.

Definition *(Bahadur efficiency and exact Bahadur slope (EBS), (Bahadur, 1959))* Let $X_1, \ldots, X_n$ be i.i.d. from a distribution with a probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \in \theta - \{\theta_0\}$. Let $\left\{T_n^{(1)}\right\}$ and $\left\{T_n^{(2)}\right\}$ be two sequences of test statistics for testing H_0. Let the significance attained by $T_n^{(i)}$ be $L_n^{(i)} = 1 - F_i\left(T_n^{(i)}\right)$, where $F_i\left(T_n^{(i)}\right) = P_{H_0}\left(T_n^{(i)} \leq t_i\right)$, $i = 1, 2$. Then there exists a positive valued function $C_i(\theta)$ called the exact Bahadur slope of the sequence $\{T_n^{(i)}\}$ such that

$$C_i(\theta) = \lim_{\theta \to \infty} -2n^{-1} \ln\left(L_n^i\right)$$

with probability 1 (w.p.1) under θ and the Bahadur efficiency of $\left\{T_n^{(1)}\right\}$ relative to $\left\{T_n^{(2)}\right\}$ is given by $e_B\left(T_1, T_2\right) = C_1(\theta)/C_2(\theta)$.

The following two theorems can be found in Serfling (1980). They are used to give the EBS for tests based on sums of iid random variables.

Theorem 3.1. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. with distribution function F and put $S_n = \sum_{i=1}^{n} X_i$. Assume existence of the moment generating function $M(t) = E_F\left(e^{tX}\right)$ in the neighbourhood of zero. Put $m(z) = \inf_t e^{-zt} M(t)$. Then $\lim_{n\to\infty} -2n^{-1} \ln P_F\left(S_n \geq nz\right) = -2\ln m(z)$.*

Theorem 3.2. *Let $\{T_n\}$ be a sequence of test statistics which satisfies the following:*

1. *Under H_1: $n^{-\frac{1}{2}} T_n \to b(\theta)$ a.s. under θ, where $b(\theta)$ is a real function.*
2. *There exists an open interval I containing $\{b(\theta) : \theta \in \Theta\}$, and a function g continuous on I, such that*

$$\lim_{n\to\infty} -2n^{-1} \log\left[1 - F_n(n^{\frac{1}{2}} t)\right] = g(t),$$

where F_n is the distribution function of T_n under H_0.

Then the EBS of $\{T_n\}$ is $C(\theta) = g(b(\theta))$.

Theorem 3.3. *Let $X_1, \cdots, X_n$ be i.i.d. with probability density function $f(x, \theta)$, and we want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. For $j = 1, 2$, let $T_{n,j} = \sum_{i=1}^{n} f_i(x_i)/\sqrt{n}$ be a sequence of statistics such that H_0 will be rejected for large values of $T_{n,j}$ and let φ_j be the test based on $T_{n,j}$. Assume $E_\theta(f_i(x)) > 0, \forall\theta \in \theta$, $E_0(f_i(x)) = 0$, $Var(f_i(x)) > 0$ for $j = 1, 2$. Then*
1. If the derivative $b'_j(0)$ is finite for $j = 1, 2$, then
$\lim_{\theta\to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\frac{b'_1(0)}{b'_2(0)}\right]^2,$
where $b_i(\theta) = E_\theta(f_i(x))$, and $C_j(\theta)$ is the EBS of test φ_j at θ.
2. If the derivative $b'_j(0)$ is infinite for $j = 1, 2$, then

$$\lim_{\theta\to 0} \frac{C_1(\theta)}{C_2(\theta)} = \frac{Var_{\theta=0}(f_2(x))}{Var_{\theta=0}(f_1(x))} \left[\lim_{\theta\to 0} \frac{b'_1(\theta)}{b'_2(\theta)}\right]^2.$$

The following theorems can be found in many references, see for example D'Agostino (2017) and the references therein.

Theorem 3.4. *If $T_n^{(1)}$ and $T_n^{(2)}$ are two test statistics for testing $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$ with distribution functions $F_0^{(1)}$ and $F_0^{(2)}$ under H_0, respectively, and that $T_n^{(1)}$ is at least as powerful as $T_n^{(2)}$ at θ for any α, then if φ_j is the test based on $T_n^{(j)}$, $j = 1, 2$, then*

$$C_{\varphi_1}^{(1)}(\theta) \geq C_{\varphi_2}^{(2)}(\theta).$$

Corollary 1. *If T_n is the uniformly most powerful test for all α, then it is the best via EBS.*

Theorem 3.5.

$$2t \leq m_S(t) \leq et, \quad \forall : 0 \leq t \leq 0.5,$$

where

$$m_S(t) = \inf_{z>0} e^{-zt} \frac{e^z - 1}{z}.$$

Theorem 3.6.

1. $m_L(t) \geq 2te^{-t}, \quad \forall t \geq 0,$
2. $m_L(t) \leq te^{1-t}, \quad \forall t \geq 0.852,$
3. $m_L(t) \leq t\left(\frac{t^2}{1+t^2}\right)^3 e^{1-t}, \quad \forall t \geq 4,$
   where $m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\ csc(\pi z)$ *and csc is an abbreviation for cosecant function.*

Theorem 3.7. *For* $x > 0$*,*

$$\phi(x)\left[\frac{1}{x} - \frac{1}{x^3}\right] \leq 1 - \Phi(x) \leq \frac{\phi(x)}{x}.$$

Where ϕ *is the pdf of standard normal distribution.*

Theorem 3.8. *For* $x > 0$*,*

$$1 - \Phi(x) > \frac{\phi(x)}{x + \sqrt{\frac{\pi}{2}}}.$$

Lemma 3.9.

1. $m_L(t) \geq \inf_{0<z<1} e^{-zt} = e^{-t}$
2. $m_L(t) \leq \dfrac{e^{-t^2/(t+1)}\left(\frac{\pi t}{t+1}\right)}{sin\left(\frac{\pi t}{t+1}\right)}$
3. $\begin{cases} m_s(t) = \inf_{z>0} \frac{e^{-zt}(1-e^{-z})}{z} \leq \inf_{z>0} \frac{e^{-zt}}{z} \leq -et, & t < 0 \\ m_s(t) \geq -2t, & -\frac{1}{2} \leq t \leq 0. \end{cases}$
4. $\dfrac{x-1}{x} \leq \ln x \leq x - 1, \quad x > 0$

Theorem 3.10. *For any integrable function* f *and any* η *in the interior of* Θ*, the integral*

$$\int f(x)e^{\sum \eta_i T_i(x)} h(x) d\mu(x)$$

is continuous and has derivatives of all orders with respect to the $\eta's$*, and these can be obtained by differentiating under the integral sign.*

4 Derivation of the EBS with general DF H_ξ

In this section we will study testing problem (4). We will compare the four methods viz. Fisher, logistic, sum of P-values and the inverse normal method via EBS. Let $X_1, \ldots, X_n$ be i.i.d. with conditional probability density function $\mathcal{L}(\tau\xi, 1)$, and we want to test (4). The P-value in this case is given by

$$P_i = 1 - F^{H_0}(x) = 1 - F_0(x) = \tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}, x \in \mathbb{R} \tag{5}$$

The next four lemmas give the EBS for Fisher (C_F), logistic (C_L), inverse normal (C_N), and sum of p-values (C_S) methods.

Lemma 4.1. *The EBS for the methods mentioned above are as follows:*

- **A1** *Fisher method.* $C_F(\tau) = b_F(\tau) - 2\ln(b_F(\tau)) + 2\ln(2) - 2$, *where*
$$b_F(\tau) = 2\ln 2 + 2\tau\, \mathbb{E}_{H_\xi}\, \xi - 2\,(\ln 2 - 1)\, \mathbb{E}_{H_\xi}\, e^{-\tau\xi}.$$

- **A2** *Logistic method.* $C_L(\tau) = -2\ln(m(b_L(\tau)))$, *where*
$$m_L(t) = \inf_{z\in(0,1)} e^{-zt}\pi z\ csc(\pi z)$$
and
$$\begin{aligned} b_L(\tau) \ &= \ -\tfrac{1}{2} + \tfrac{1}{2}\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi} + \tau\,\mathbb{E}_{H_\xi}\,\xi + \ln 2\,\mathbb{E}_{H_\xi}\, e^{\tau\xi} + \mathbb{E}_{H_\xi}\ln\left(2 - e^{-\tau\xi}\right) \\ &- \ \mathbb{E}_{H_\xi}\, e^{\tau\xi}\ln\left(2 - e^{-\tau\xi}\right) - \tfrac{1}{4}\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi}\ln\left(32e^{\tau\xi} - 16\right). \end{aligned}$$

- **A3** *Sum of p-values method.* $C_S(\tau) = -2\ln(m(b_S(\tau)))$, *where*
$$m_S(t) = \inf_{z>0} e^{-zt}\frac{1 - e^{-z}}{z}$$
and
$$b_S(\tau) = -\tfrac{1}{2}\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi} - \frac{\tau}{4}\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi}.$$

- **A4** *Inverse Normal method.* $C_N(\tau) = -2\ln(m(b_N(\tau))) = b_N^2(\tau)$ *where*
$$b_N(\tau) = -\tfrac{1}{2}E_{H_\xi}\left(\int_{\mathbb{R}} \Phi^{-1}\left(\tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right) e^{-|x-\tau\xi|}dx\right)$$

*Proof. P*roof of A1

$$T_F = -2\sum_{i=1}^{n} \frac{\ln\left[\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right]}{\sqrt{n}}.$$

By the strong law of large number (SLLN) Theorem (2)

$$\frac{T_F}{\sqrt{n}} \xrightarrow{\text{w.p.1}} b_F(\tau) = -2\,\mathbb{E}^{H_1} \ln\left[\tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right]$$

then

$$\begin{aligned} b_F(\tau) &= -2\,\mathbb{E}_{H_\xi}\,\mathbb{E}_{\mathcal{L}(\tau\xi,1)} \ln\left[\tfrac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right] \\ &= 2\ln 2 - 2\,\mathbb{E}_{H_\xi}\int_{\mathbb{R}} \ln\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \tfrac{1}{2} e^{-|x-\tau\xi|}\,dx \\ &= 2\ln 2 - \mathbb{E}_{H_\xi}\left(I_1 + I_2 + I_3\right). \end{aligned}$$

Hence under $H_1 : \tau > 0$, then

$$I_1 = \int_{x<0} \ln\left(2 - e^x\right) e^{x-\tau\xi}\,dx = (\ln 4 - 1)\,e^{-\tau\xi}$$

$$I_2 = \int_{0<x<\tau\xi} x e^{x-\tau\xi}\,dx = 1 - \tau\xi - e^{-\tau\xi}$$

and

$$I_3 = \int_{x>\tau\xi} x e^{\tau\xi - x}\,dx = -1 - \tau\xi,$$

then

$$b_F(\tau) = 2\ln 2 - \mathbb{E}_{H_\xi}\left(-2\tau\xi + 2\left(\ln 2 - 1\right)e^{-\tau\xi}\right) = 2\ln 2 + 2\tau\,\mathbb{E}_{H_\xi}\,\xi - 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\,e^{-\tau\xi}.$$

Now under H_0, then by Theorem 1, we have $m_S(t) = \inf_{z>0} e^{-zt} M_S(z)$, where $M_S(z) = \mathbb{E}_F(e^{zX})$. Under $H_0 : -\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\} \sim U(-1, 0)$, so $M_S(z) = \frac{1-e^{-z}}{z}$, by part (2) of Theorem 2 we complete the proof, that is

$$C_F(\tau) = -2\ln(m_F(b_F(\tau))) = -2\ln\left(\frac{b_F(\tau)}{2} e^{1-\frac{b_F(\tau)}{2}}\right) = b_F(\tau) - 2\ln(b_F(\tau)) + 2\ln(2) - 2.$$

□

4.1 The Limiting ratio of the EBS for different tests when $\tau \to 0$

Corollary 2. *The limits of ratios for different tests when $\tau \to 0$ are as follows:*

B1 $e_B\left(T_S, T_F\right) \to 1.56$

B2 $e_B\left(T_L, T_F\right) \to 1.21585$

B3 $e_B\left(T_N, T_F\right) \to 1.32504$

B4 $e_B\left(T_N, T_L\right) \to 1.08981$

B5 $e_B\left(T_S, T_N\right) \to 1.1781$

B6 $e_B\left(T_S, T_L\right) \to 1.2839$

Proof of B1.

$$b_F(\tau) = 2\ln 2 + 2\tau\,\mathbb{E}_{H_\xi}\,\xi - 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\,e^{-\tau\xi}.$$

Therefore

$$b'_F(\tau) = 2\,\mathbb{E}_{H_\xi}\,\xi + 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi},$$

then

$$\lim_{\tau\to 0} b'_F(\tau) = \ln(4)\,\mathbb{E}_{H_\xi}\,\xi < \infty.$$

Also

$$b_S(\tau) = -\tfrac{1}{2}\,\mathbb{E}_{H_\xi}\,e^{-\tau\xi} - \frac{\tau}{4}\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi},$$

then

$$\lim_{\tau\to 0} b'_S(\tau) = \tfrac{1}{4}\,\mathbb{E}_{H_\xi}\,\xi < \infty.$$

Now under $H_0 : h_F(x) = -2\ln\left[\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}\right] \sim \chi^2_2$ and $h_S(x) = -\frac{1}{2}\left\{1 - sgn(x)\left(1 - e^{-|x|}\right)\right\}$ $U(-1, 0)$, so $Var_{\tau=0}(h_F(x)) = 4$ and $Var_{\tau=0}(h_S(x)) = \frac{1}{12}$, also, $\dfrac{b'_S(0)}{b'_F(0)} = \dfrac{1}{4\ln(4)}$. By applying Theorem 3 we can get $e_B\left(T_S, T_F\right) = \lim\limits_{\tau\to 0}\dfrac{C_S(\tau)}{C_F(\tau)} = 1.56103$. Similarly we can prove the other parts. □

4.2 The Limiting ratio of the EBS for different tests when $\tau \to \infty$

Corollary 3. *The limits of ratios for different tests $\tau \to \infty$ are as follows:*

D1 $e_B\left(T_L, T_F\right) \to \frac{1}{2}$

D2 $e_B\left(T_N, T_S\right) \to 0, e_B\left(T_N, T_L\right) \to 0, e_B\left(T_S, T_L\right) \to 0$

D3 *From D1 and D2,* $e_B\left(T_N, T_F\right) \to 0, e_B\left(T_S, T_F\right) \to 0$

Proof of $e_B\left(T_L, T_F\right) \to \frac{1}{2}$. By Lemma 1 part (1) $C_L(\tau) \le 2b_L(\tau)$. So

$$\lim_{\tau\to\infty}\frac{C_L(\tau)}{C_F(\tau)} \le \lim_{\tau\to\infty}\frac{2b_L(\tau)}{b_F(\tau) - 2\ln(b_F(\tau)) + 2\ln(2) - 2}$$

where

$$b_F(\tau) = 2\ln 2 + 2\tau\,\mathbb{E}_{H_\xi}\,\xi - 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\,e^{-\tau\xi}.$$

and

$$\begin{aligned} b_L(\tau) &= -\tfrac{1}{2} + \tfrac{1}{2}\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi} + \tau\,\mathbb{E}_{H_\xi}\,\xi + \ln 2\,\mathbb{E}_{H_\xi}\, e^{\tau\xi} + \mathbb{E}_{H_\xi} \ln\left(2 - e^{-\tau\xi}\right) \\ &- \mathbb{E}_{H_\xi}\, e^{\tau\xi} \ln\left(2 - e^{-\tau\xi}\right) - \tfrac{1}{4}\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi} \ln\left(32e^{\tau\xi} - 16\right). \end{aligned}$$

Now we will check the limit for $b_L(\tau)$terms when $\tau \to \infty$,

$\mathbb{E}_{H_\xi}\, e^{-\tau\xi} \to 0$, $\ln 2\,\mathbb{E}_{H_\xi}\, e^{\tau\xi} - \mathbb{E}_{H_\xi}\, e^{\tau\xi} \ln\left(2 - e^{-\tau\xi}\right) = \mathbb{E}_{H_\xi}\, e^{\tau\xi} \ln\left[\frac{2}{2-e^{-\tau\xi}}\right] \to 0$, $\mathbb{E}_{H_\xi} \ln\left(2 - e^{-\tau\xi}\right) \to \ln 2$,

$$\begin{aligned} \mathbb{E}_{H_\xi}\, e^{-\tau\xi} \ln\left(32e^{\tau\xi} - 16\right) &= \mathbb{E}_{H_\xi}\, e^{-\tau\xi} \ln\left(16e^{\tau\xi}\left[2 - e^{-\tau\xi}\right]\right) \\ &= -\tau\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi} - \ln 16\,\mathbb{E}_{H_\xi}\, e^{-\tau\xi} - \mathbb{E}_{H_\xi}\, e^{-\tau\xi} \ln\left[2 - e^{-\tau\xi}\right] \to 0. \end{aligned}$$

It is clear that it is sufficient to obtain the limit of $\lim\limits_{\tau\to\infty} \dfrac{-\frac{1}{2} + \ln 2 + \tau\,\mathbb{E}_{H_\xi}\,\xi}{b_F(\tau)}$. Then

$$\lim_{\tau\to\infty} \frac{-\frac{1}{2} + \ln 2 + \tau\,\mathbb{E}_{H_\xi}\,\xi}{2\ln 2 + 2\tau\,\mathbb{E}_{H_\xi}\,\xi - 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\, e^{-\tau\xi}} = \lim_{\tau\to\infty} \frac{\mathbb{E}_{H_\xi}\,\xi}{2\,\mathbb{E}_{H_\xi}\,\xi} = \tfrac{1}{2}.$$

Then

$$\lim_{\tau\to\infty} \frac{C_L(\tau)}{C_F(\tau)} \le \tfrac{1}{2}.$$

Also, by Theorem (6) part (2), we have

$$C_L(\tau) \ge b_L(\tau) - 2\ln\left[b_L(\tau)\right] - 1,$$

so

$$\frac{C_L(\tau)}{C_F(\tau)} \ge \frac{b_L(\tau) - 2\ln\left[b_L(\tau)\right] - 1}{b_F(\tau)}.$$

Now it is sufficient to obtain the limit of $\lim\limits_{\tau\to\infty} \dfrac{b_L(\tau)}{b_F(\tau)}$, which is reduce to $\lim\limits_{\tau\to\infty} \dfrac{-\frac{1}{2} + \ln 2 + \tau\,\mathbb{E}_{H_\xi}\,\xi}{b_F(\tau)}$.
So,

$$\lim_{\tau\to\infty} \frac{C_L(\tau)}{C_F(\tau)} \ge \frac{-\frac{1}{2} + \ln 2 + \tau\,\mathbb{E}_{H_\xi}\,\xi}{2\ln 2 + 2\tau\,\mathbb{E}_{H_\xi}\,\xi - 2\left(\ln 2 - 1\right)\mathbb{E}_{H_\xi}\, e^{-\tau\xi}} = \frac{1}{2}.$$

Then $\lim\limits_{\tau\to\infty} \dfrac{C_L(\tau)}{C_F(\tau)} \ge \dfrac{1}{2}$.
By pinching theorem, we have $\lim\limits_{\tau\to\infty} \dfrac{C_L(\tau)}{C_F(\tau)} = \dfrac{1}{2}$. □

Proof of $e_B\left(T_N, T_S\right) \to 0$. We have

$$C_N(\tau) = b_N^2(\tau)$$

where
$$b_N(\tau) = -\tfrac{1}{2} E_{H_\xi}\, \mathbb{E}_{\mathcal{L}(\tau\xi,1)} \left(\Phi^{-1} \left(\tfrac{1}{2} \left\{ 1 - sgn(x) \left(1 - e^{-|x|}\right) \right\} \right) |\xi \right).$$

By lemma 1 part (3) $C_S(\tau) \geq -2 - 2\ln(-b_S(\tau))$, where $b_S(\tau) = -\frac{1}{2}\,\mathbb{E}_{H_\xi} e^{-\tau\xi} - \frac{\tau}{4}\,\mathbb{E}_{H_\xi} \xi e^{-\tau\xi}$.
So

$$\lim_{\tau\to\infty} \frac{C_N(\tau)}{C_S(\tau)} \leq \lim_{\tau\to\infty} \frac{\left\{ -\frac{1}{2} E_{H_\xi}\, \mathbb{E}_{\mathcal{L}(\tau\xi,1)} \left(\Phi^{-1} \left(\frac{1}{2} \left\{ 1 - sgn(x) \left(1 - e^{-|x|}\right) \right\} \right) |\xi \right) \right\}^2}{-2 - 2\ln(-b_S(\tau))}.$$

Putting $u = x - \tau\xi$, we get

$$\frac{C_N(\tau)}{C_S(\tau)} \leq \frac{\left\{ -\frac{1}{2} E_{H_\xi}\, \mathbb{E}_{\mathcal{L}(0,1)} \left(\Phi^{-1} \left(\frac{1}{2} \left\{ 1 - sgn(u+\tau\xi) \left(1 - e^{-|u+\tau\xi|}\right) \right\} \right) |\xi \right) \right\}^2}{-2 - 2\ln(-b_S(\tau))}.$$

Again, let $t = \Phi^{-1}\left(\frac{1}{2} \left(1 - sgn(u+\tau\xi) \left[1 - e^{-|u+\tau\xi|} \right] \right) \right)$, then, $2\Phi(t) = 1 - sgn(u + \tau\xi) \left[1 - e^{-|u+\tau\xi|} \right]$, and $2\phi(t) \left| \frac{dt}{du} \right| = \frac{e^{-|u+\tau\xi|}}{2}$. Then,

$$\frac{C_N(\tau)}{C_S(\tau)} \leq \frac{\left\{ -\frac{1}{4} E_{H_\xi}\, \mathbb{E}_{\mathcal{N}(0,1)} (t|\xi) \right\}^2}{-2 - 2\ln(-b_S(\tau))}.$$

Now using L'Hopital's rule, we get, $b_S(\tau) \to 0$ as $\tau \to \infty$, and $\mathbb{E}_{\mathcal{N}(0,1)}(t|\xi) = 0$. Then

$$\lim_{\tau\to\infty} \frac{C_N(\tau)}{C_S(\tau)} \leq 0.$$

So

$$e_B(T_N, T_S) \to 0.$$

□

Proof of $e_B(T_N, T_L) \to 0$. From the last two proofs, it is clear that $e_B(T_N, T_L) \to 0$. □

Proof of $e_B(T_S, T_L) \to 0$. By lemma 1 part (3) $C_S(\tau) \leq -2\ln 2 - 2\ln(-b_S(\tau))$, where $b_S(\tau) = -\frac{1}{2}\,\mathbb{E}_{H_\xi} e^{-\tau\xi} - \frac{\tau}{4}\,\mathbb{E}_{H_\xi} \xi e^{-\tau\xi}$.
And by Theorem (6) part (2), we have $C_L(\tau) \geq b_L(\tau) - 2\ln[b_L(\tau)] - 1$, then

$$\frac{C_S(\tau)}{C_L(\tau)} \leq \frac{-2\ln 2 - 2\ln(-b_S(\tau))}{b_L(\tau) - 2\ln[b_L(\tau)] - 1}.$$

Now it is sufficient to obtain the limit of $\frac{C_S(\tau)}{C_L(\tau)} \leq \frac{\ln(-b_S(\tau))}{b_L(\tau)}$. Then

$$\lim_{\tau\to\infty} \frac{C_S(\tau)}{C_L(\tau)} \leq \lim_{\tau\to\infty} \frac{\ln\left(\frac{1}{2}\,\mathbb{E}_{H_\xi} e^{-\tau\xi} + \frac{\tau}{4}\,\mathbb{E}_{H_\xi} \xi e^{-\tau\xi} \right)}{-\frac{1}{2} + \ln 2 + \tau\,\mathbb{E}_{H_\xi} \xi}.$$

From the last proofs, also, it is sufficient to obtain the limit of

$$\lim_{\tau\to\infty}\frac{C_S(\tau)}{C_L(\tau)} \leq \lim_{\tau\to\infty}\frac{\ln\left(\frac{\tau}{4}\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi}\right)}{-\frac{1}{2}+\ln 2+\tau\,\mathbb{E}_{H_\xi}\,\xi} = \lim_{\tau\to\infty}\frac{\ln\left(\frac{\tau}{4}\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi}\right)}{\tau\,\mathbb{E}_{H_\xi}\,\xi}.$$

By comparing the dominated terms, then,

$$\lim_{\tau\to\infty}\frac{\ln\left(\frac{\tau}{4}\,\mathbb{E}_{H_\xi}\,\xi e^{-\tau\xi}\right)}{\tau\,\mathbb{E}_{H_\xi}\,\xi} = 0.$$

Then

$$\lim_{\tau\to\infty}\frac{C_S(\tau)}{C_L(\tau)} \leq 0.$$

So,

$$e_B\left(T_S, T_L\right) \to 0$$

□

4.3 Comparison of the EBS for the four combination procedures

From the relations in section (4.1) we conclude that locally as $\tau \to 0$, the sum of p-values procedure is better than all other procedures since it has the highest EBS, followed in decreasing order by the inverse normal and the logistic procedure. The worst is the Fisher's procedure, i.e,

$$C_S(\tau) > C_N(\tau) > C_L(\tau) > C_F(\tau).$$

Whereas, from result of Section (4.2) as $\tau \to \infty$ the Fisher's procedure is better than the other procedures, followed in decreasing order by the logistics procedure, and the sum of p-values. The worst is inverse normal procedure, i.e,

$$C_F(\tau) > C_L(\tau) > C_S(\tau) > C_N(\tau)$$

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