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Length-Biased Benrabia Distribution with Applications

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The idea of weighted distributions is widely used in many fields, for example: medicine, ecology and reliability. They are more significant for modeling data when the base distributions are not proper or fit the data with less competence. Length biased distributions are special cases of the weighted distributions. In this paper, we propose a new generalization of Benrabia distribution, called the length-biased Benrabia distribution. The recommended distribution's various characteristics are deduced and thoroughly explored. Some numerical studies are implemented, they demonstrate that the distribution is skewed to the right with heavier tail than the normal distribution. To estimate the distribution's parameters six methods of estimation are employed. A simulation study is conducted shows that the estimators are approximately unbiased and consistent. Three data sets applications are performed, they show that the suggested distribution has the best fit for these data sets comparing to some competence distributions.

keywords: Benrabia distribution, length biased, moments, reliability analysis, Rényi entropy, methods of estimation.

1 Introduction

The idea of a weighted distribution was initially introduced by Fisher (1934) and developed by Rao (1965). Recently, this idea has been employed frequently in many researches related to reliability, ecology, analysis of family data, bio-medicine, and some other fields for the improved performance of appropriate statistical models. It is defined by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \ x > 0, \tag{1.1}$$

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where $w(x)$ is a non-negative weight function.

Let X be a random variable with the probability density function $f(x)$, then the size-biased distribution can be produced by using the weight function $w(x) = X^m$. It was first studied by Patil and Ord (1976). Thus, the resulting pdf

$$f_m(x) = \frac{x^m f(x)}{E(X^m)} \tag{1.2}$$

For $m = 1$, we get the length-biased distribution, which was first introduced by Cox (1968); Zelen (1974). It is defined as

$$f_1(x) = \frac{x f(x)}{E(X)} \tag{1.3}$$

The idea of weighted distributions attracted many researchers. For example, Warren (1975) applied them in forestry and ecology in sampling wood cells. Kersey (2010) proposed the weighted inverse Weibull distribution and beta-inverse Weibull distribution. Roman (2010) provided the theoretical properties and estimation in weighted Weibull and related distributions. Shi et al. (2012) developed the theoretical properties of Weighted Generalized Rayleigh and related distributions. Ye et al. (2012) displayed a weighted generalized beta distribution of the second kind. Aleem et al. (2013) presented a class of modified weighted Weibull distribution and its properties. Rashwan (2013) gave the properties of the double-weighted Rayleigh distribution and its estimation. Badmus et al. (2014) offered a Lehman Type II weighted Weibull distribution. Bashir and Rasul (2015) established the weighted Lindley distribution. Alqallaf et al. (2015) offered a number of estimation methods to estimate the parameters of the weighted exponential distribution. Asgharzadeh et al. (2016) established a new weighted Lindley distribution with application in survival analysis. Fatima and Ahmad (2017) provided a description of the weighted inverse Rayleigh distribution, including its characteristics and applications. Jan et al. (2017) studied the weighted Ailamujia distribution and found its applications in real data sets. Saghir et al. (2017a) reviewed some work of weighted distributions and their applications. Alsmairan and Al-Omari (2020) suggested the weighted Suja distribution, which was then applied to ball bearings data for safety engineering and studied for its statistical features.
The length and area biased distributions are special types of weighted distributions as mentioned above. In recent times, many authors are interested in studying these types of distributions, such as Sharma et al. (2018) introduced length and area-biased Maxwell distribution. Al-Omari et al. (2019) suggested power length-biased Suja distribution as a new extension of the length-biased Suja distribution. Saghir et al. (2017b) studied a new class of Maxwell length-biased distribution. Shen et al. (2009) used semi-parametric transformations to model the length-biased data. Al-Omari and Alanzi (2021) suggested and studied the properties of the one parameter inverse length biased Maxwell distribution. Das and Roy (2011) suggested the Length-Biased form of weighted Weibull distribution.

2 Benrabia Distribution

Benrabia and Alzoubi (2022) suggested a new two parameters continuous distribution known as Benrabia distribution (BrD) with probability density function given by

$$g_{Br}(x;\alpha,\beta) = \frac{\beta}{\alpha+\beta}\left(\alpha+\frac{x^{\alpha-2}\beta^{\alpha-1}}{\Gamma(\alpha-1)}\right)e^{-\beta x} \quad x>0,\ \alpha>1,\beta>0 \tag{2.1}$$

and cumulative distribution function defined as

$$G_{Br}(x;\alpha,\beta) = \frac{1}{\alpha+\beta}\left[\alpha(1-e^{-\beta x})+\beta P(\alpha-1,\beta x)\right], \tag{2.2}$$

where $P(\alpha,x)=\frac{\gamma(\alpha,x)}{\Gamma(\alpha)}$ is the lower regularized gamma function, and $\gamma(\alpha,x)=\int_0^x t^{\alpha-1}e^{-t}dt$ is the lower incomplete gamma function.

The r^{th} moment and moment generating function of BrD are, respectively, given by

$$E(X^r) = \frac{1}{\beta^r(\alpha+\beta)}\left[\alpha\Gamma(\alpha+1)+\beta\frac{\Gamma(\alpha+r-1)}{\Gamma(\alpha-1)}\right];\ r=1,2,...$$

$$M_X(t) = \frac{\beta}{\alpha+\beta}\left[\frac{\alpha}{\beta-t}+\left(1-\frac{t}{\beta}\right)^{-(\alpha-1)}\right],\ t<\beta$$

The first moment of BrD is

$$E(X) = \frac{\alpha-\beta+\alpha\beta}{\beta(\alpha+\beta)} \tag{2.3}$$

3 Length-Biased Benrabia Distribution

This section displays the pdf and CDF of LBBD formulas along with their graphical representations.

Definition 3.1 *A random variable X is said to have a length biased Benrabia distribution with parameters α and β, ($X \sim LBBD$ (α, β)) if its pdf is defined using (1.1), (2.1) and (2.3) as:*

$$f_l(x;\alpha,\beta) = \frac{(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)},\ x>0,\ \alpha>1,\ \beta>0 \tag{3.1}$$

Theorem 3.1 *Let X be a random variable that follows the LBBD with parameters α and β. The cumulative distribution function of X is defined by*

$$F_l(x;\alpha,\beta) = \frac{\alpha\Gamma(\alpha-1)(1-(\beta x+1)e^{-\beta x})+\beta\gamma(\alpha,\beta x)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}, \tag{3.2}$$

$x\geq 0,\alpha>1,\beta>0,$

Proof 3.1 *The CDF of the LBBD distribution can be obtained as*

$$\begin{aligned}
F_l(x) &= p(X \leq x) \\
&= \int_0^x \frac{(\alpha\Gamma(\alpha-1)\beta^2 t + \beta^{\alpha+1}t^{\alpha-1})e^{-\beta t}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}dt \\
&= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\left[\int_0^x (\alpha\Gamma(\alpha-1)\beta^2 te^{-\beta t} + \beta^{\alpha+1}t^{\alpha-1}e^{-\beta t})dt\right] \\
&= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\left[\alpha\Gamma(\alpha-1)\beta^2\int_0^x te^{-\beta t}dt + \beta^{\alpha+1}\int_0^x t^{\alpha-1}e^{-\beta t}dt\right]
\end{aligned}$$

By using the lower incomplete gamma function $\gamma(\alpha, x) = \int_o^x t^{\alpha-1}e^{-t}dt$ *and the facts* $\Gamma(\alpha, 0) = \Gamma(\alpha)$, $\gamma(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x)$ *We get*

$$F_l(x) = \frac{\alpha\Gamma(\alpha-1)(1-(\beta x+1)e^{-\beta x}) + \beta\gamma(\alpha, \beta x)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}$$

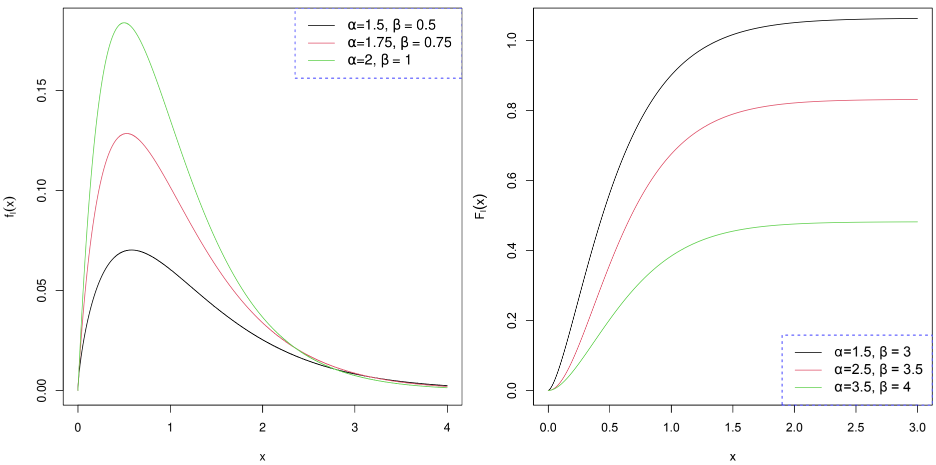

Figure 1: The pdf and CDF of LBBD for different values of α and β.

Figure 1 shows the plots of the pdf (left) of LBBD for the values of α of 1.5, 1.75, and 2 with $\beta = 0.5$, 0.75 and 1. It shows the plot of the CDF (right) of LBBD for the values of α of 1.5, 2.5, and 3.5 with $\beta = 3$, 3.5, and 4. The figure shows that the LBBD is skewed right.

4 Moments and Related Measures

This section presents some moments and related measures of the LBBD as well as tables showing the mean, standard deviation, coefficient of variation, coefficient of skewness, and excess kurtosis for a few chosen parameters.

4.1 Moments

Theorem 4.1 *Let X be a LBBD random variable with pdf defined in (3.1), then the r^{th} moment of X is*

$$E(X^r) = \frac{\alpha\Gamma(\alpha-1)\Gamma(r+2)+\beta\Gamma(r+\alpha)}{\beta^r(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}, r=1,2,... \tag{4.1}$$

Proof 4.1 *The r^{th} moment of the LBBD can be calculated as*

$$\begin{aligned} E(X^r) &= \int_0^\infty x^r f_l(x)dx = \int_0^\infty x^r \left(\frac{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right)e^{-\beta x}dx \\ &= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\int_0^\infty \left(\alpha\Gamma(\alpha-1)\beta^2 x^{r+1}e^{-\beta x}+\beta^{\alpha+1}x^{r+\alpha-1}e^{-\beta x}\right)dx \\ &= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\left[\frac{\alpha\beta^2\Gamma(\alpha-1)\Gamma(r+2)}{\beta^{r+2}}+\frac{\beta^{\alpha+1}\Gamma(r+\alpha)}{\beta^{r+\alpha}}\right] \\ &= \frac{\alpha\Gamma(\alpha-1)\Gamma(r+2)+\beta\Gamma(r+\alpha)}{\beta^r(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \end{aligned}$$

For r =1, we get the first moment (mean) of the LBBD random variable. The second, third, and fourth moments can be calculated by substituting r = 2, 3, and 4 in (4.1). Thus, we have

$$\mu = E(X) = \frac{2\alpha+\beta\alpha(\alpha-1)}{\beta(\alpha-\beta+\alpha\beta)} \tag{4.2}$$

$$E(X^2) = \frac{6\alpha+\beta\alpha(\alpha+1)(\alpha-1)}{\beta^2(\alpha-\beta+\alpha\beta)} \tag{4.3}$$

$$E(X^3) = \frac{24\alpha+\beta\alpha(\alpha+2)(\alpha+1)(\alpha-1)}{\beta^3(\alpha-\beta+\alpha\beta)} \tag{4.4}$$

$$E(X^4) = \frac{120\alpha+\beta\alpha(\alpha+3)(\alpha+2)(\alpha+1)(\alpha-1)}{\beta^4(\alpha-\beta+\alpha\beta)} \tag{4.5}$$

4.2 Related measures

The variance of the random variable X that follows LBBD is defined by (4.2) and (4.3) as

$$\begin{aligned} \sigma^2 &= \left(E(X^2)-\mu^2\right) \\ &= \left(\frac{6\alpha+\beta\alpha(\alpha+1)(\alpha-1)}{\beta^2(\alpha-\beta+\alpha\beta)}\right)-\left(\frac{2\alpha+\beta\alpha(\alpha-1)}{\beta(\alpha-\beta+\alpha\beta)}\right)^2 \\ &= \frac{\left(\alpha\left((\alpha-1)^2\beta^2+(\alpha-1)((\alpha-3)\alpha+6)\beta+2\alpha\right)\right)}{\beta^2\left(\alpha-\beta+\alpha\beta\right)^2} \end{aligned} \tag{4.6}$$

By using (4.6), the standard deviation of the LBBD is given by

$$\sigma = \sqrt{\sigma^2} = \frac{\sqrt{\left(\alpha\left((\alpha-1)^2\beta^2 + (\alpha-1)((\alpha-3)\alpha+6)\beta + 2\alpha\right) \right)}}{\beta\left(\alpha-\beta+\alpha\beta\right)} \qquad (4.7)$$

The CV is defined using (4.2) and (4.7) as

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\left(\alpha\left((\alpha-1)^2\beta^2 + (\alpha-1)((\alpha-3)\alpha+6)\beta + 2\alpha\right) \right)}}{2\alpha + \beta\alpha(\alpha-1)}$$

Using (4.2), (4.3), (4.4) and (4.7) the skewness is defined as

$$\begin{aligned} Sk(X) &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\ &= \frac{\left(\begin{array}{c} \alpha(4\alpha^2 + 2(\alpha-1)^3\beta^3 - (\alpha-1)^2((\alpha-5)(\alpha-4)\alpha \\ -24)\beta^2 + (\alpha-1)\alpha((\alpha-2)(\alpha-1)\alpha+12)\beta) \end{array} \right)}{\left(\sqrt{\left(\alpha((\alpha-1)^2\beta^2 + (\alpha-1)((\alpha-3)\alpha+6)\beta+2\alpha) \right)}\right)^3} \end{aligned}$$

The kurtosis is defined using (4.2), (4.3), (4.4), (4.5) and (4.7)as

$$Kur(X) = \frac{\left(\begin{array}{c} \alpha(-3\alpha^3((\alpha-1)\beta+2)^4 + 6\alpha^2(\alpha\beta \\ +\alpha-\beta)((\alpha^2-1)\beta+6)((\alpha-1)\beta+2)^2 \\ -4\alpha(\alpha\beta+\alpha-\beta)^2((\alpha-1)(\alpha+1)(\alpha \\ +2)\beta+24)((\alpha-1)\beta+2) + (\alpha\beta+\alpha \\ -\beta)^3((\alpha-1)(\alpha+1)(\alpha+2)(\alpha+3)\beta+120)) \end{array} \right)}{\left(\alpha((\alpha-1)^2\beta^2 + (\alpha-1)((\alpha-3)\alpha+6)\beta+2\alpha) \right)^2}$$

Table 1 displays the values of the mean, standard deviation, skewness, excess kurtosis, and the coefficient of variation of the LBBD distribution for values of α of 3.5-7(step 0.5) and values of β of 4-6(step 0.5). It demonstrates that the distribution is right-skewed despite the values of α and β. The excess kurtosis values are all positive showing that the distribution's tails are heavier than the normal distribution tails.

4.3 Moment generating function

Theorem 4.2 *The moment generating function of the LBBD is given by*

$$M_X(t) = \frac{1}{(\alpha-\beta+\alpha\beta)}\left[\frac{\alpha\beta^2}{(\beta-t)^2} + \frac{(\alpha-1)\beta^{\alpha+1}}{(\beta-t)^{\alpha}}\right], \quad \alpha>0, \beta>t \qquad (4.8)$$

Table 1: Related moments measures for LBBD for different values of α and β

α	β	μ	σ	Sk	$eKur$	CV	α	β	μ	σ	Sk	$eKur$	CV
3.5	4.0	0.778	0.471	1.081	1.705	60.504	5.5	4.0	1.170	0.656	0.658	0.543	56.024
3.5	4.5	0.699	0.419	1.074	1.692	59.938	5.5	4.5	1.056	0.580	0.650	0.565	54.911
3.5	5.0	0.634	0.377	1.069	1.683	59.455	5.5	5.0	0.963	0.519	0.646	0.588	53.970
3.5	5.5	0.581	0.343	1.066	1.678	59.039	5.5	5.5	0.884	0.470	0.644	0.610	53.164
3.5	6.0	0.536	0.315	1.063	1.673	58.676	5.5	6.0	0.818	0.429	0.644	0.632	52.466
4.0	4.0	0.875	0.515	0.970	1.349	58.902	6.0	4.0	1.269	0.703	0.564	0.352	55.422
4.0	4.5	0.787	0.458	0.963	1.344	58.176	6.0	4.5	1.146	0.621	0.555	0.382	54.207
4.0	5.0	0.716	0.412	0.959	1.342	57.560	6.0	5.0	1.045	0.556	0.551	0.413	53.180
4.0	5.5	0.656	0.374	0.956	1.343	57.030	6.0	5.5	0.961	0.502	0.549	0.442	52.300
4.0	6.0	0.606	0.343	0.954	1.345	56.569	6.0	6.0	0.889	0.458	0.549	0.470	51.539
4.5	4.0	0.973	0.561	0.862	1.038	57.697	6.5	4.0	1.368	0.752	0.475	0.191	54.926
4.5	4.5	0.877	0.498	0.855	1.042	56.829	6.5	4.5	1.236	0.663	0.466	0.231	53.617
4.5	5.0	0.798	0.447	0.851	1.048	56.092	6.5	5.0	1.128	0.592	0.461	0.268	52.513
4.5	5.5	0.732	0.406	0.848	1.056	55.459	6.5	5.5	1.037	0.535	0.459	0.304	51.568
4.5	6.0	0.676	0.371	0.847	1.064	54.910	6.5	6.0	0.960	0.487	0.459	0.337	50.749
5.0	4.0	1.071	0.608	0.757	0.770	56.765	7.0	4.0	1.468	0.800	0.393	0.058	54.509
5.0	4.5	0.966	0.539	0.750	0.783	55.767	7.0	4.5	1.327	0.705	0.382	0.105	53.118
5.0	5.0	0.880	0.483	0.746	0.798	54.923	7.0	5.0	1.211	0.629	0.376	0.150	51.943
5.0	5.5	0.808	0.438	0.744	0.813	54.199	7.0	5.5	1.114	0.567	0.374	0.191	50.939
5.0	6.0	0.747	0.400	0.743	0.828	53.571	7.0	6.0	1.031	0.516	0.374	0.230	50.069

Proof 4.2 *The moment generating function of the lbbd can be obtained by using (3.1) as follows:*

$$\begin{aligned} M_X(t) &= \int_0^\infty e^{tx} f_l(x) = \int_0^\infty e^{tx}\left(\frac{\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right)e^{-\beta x}dx \\ &= \frac{1}{(\alpha-\beta+\alpha\beta)}\left[\frac{\alpha\beta^2}{(\beta-t)^2} + \frac{(\alpha-1)\beta^{\alpha+1}}{(\beta-t)^\alpha}\right] \end{aligned}$$

5 Reliability Analysis

Reliability is the probability that a component will survive for a specific period without failure. This section will define some reliability analysis functions statistically. The reliability function is defined using (3.2) as:

$$R(t) = 1 - F_l(x) = \frac{\left(\begin{array}{c} e^{-\beta t}(-\beta e^{\beta t}\gamma(\alpha,\beta t) + \alpha\Gamma(\alpha-1) \\ +\alpha\beta t\Gamma(\alpha-1) - \beta\Gamma(\alpha-1)e^{\beta t} \\ +\alpha\beta\Gamma(\alpha-1)e^{\beta t}) \end{array}\right)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \tag{5.1}$$

The hazard rate function of LBBD is calculated using (3.1) and (5.1):

$$h(t) = \frac{f_l(t)}{1-F_l(t)} = \frac{\alpha\Gamma(\alpha-1)\beta^2 t + \beta^{\alpha+1}t^{\alpha-1}}{\left(\begin{array}{c} -\beta e^{\beta t}\gamma(\alpha,\beta t) + \alpha\Gamma(\alpha-1) + \alpha\beta t\Gamma(\alpha-1) \\ -\beta\Gamma(\alpha-1)e^{\beta t} + \alpha\beta\Gamma(\alpha-1)e^{\beta t} \end{array}\right)}$$

The cumulative hazard rate function of LBBD is defined (5.1) as:

$$\begin{aligned} H(t) &= -ln(1-F_l(t)) = -ln\begin{pmatrix} -\beta e^{\beta t}\gamma(\alpha,\beta t)+\alpha\Gamma(\alpha-1)+\alpha\beta t\Gamma(\alpha-1) \\ -\beta\Gamma(\alpha-1)e^{\beta t}+\alpha\beta\Gamma(\alpha-1)e^{\beta t} \end{pmatrix} \\ &\quad +\beta t + ln\left((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)\right) \end{aligned}$$

The reversed hazard rate function of LBBD is defined using (3.1) and (3.2) by:

$$RH(t) = \frac{f_l(t)}{F_l(t)} = \frac{(\alpha\Gamma(\alpha-1)\beta^2 t+\beta^{\alpha+1}t^{\alpha-1})e^{-\beta t}}{\left(\alpha\Gamma(\alpha-1)(1-(\beta t+1)e^{-\beta t})+\beta\gamma(\alpha,\beta t)\right)}$$

Whereas, the odds rate function of LBBD is determined by (3.2), as:

$$O(t) = \frac{F(t)}{1-F(t)} = \frac{\alpha\Gamma(\alpha-1)(1-(\beta t+1)e^{-\beta t})+\beta\gamma(\alpha,\beta t)}{\begin{pmatrix} e^{-\beta t}(-\beta e^{\beta t}\gamma(\alpha,\beta t)+\alpha\Gamma(\alpha-1)+\alpha\beta t\Gamma(\alpha-1) \\ -\beta\Gamma(\alpha-1)e^{\beta t}+\alpha\beta\Gamma(\alpha-1)e^{\beta t}) \end{pmatrix}}$$

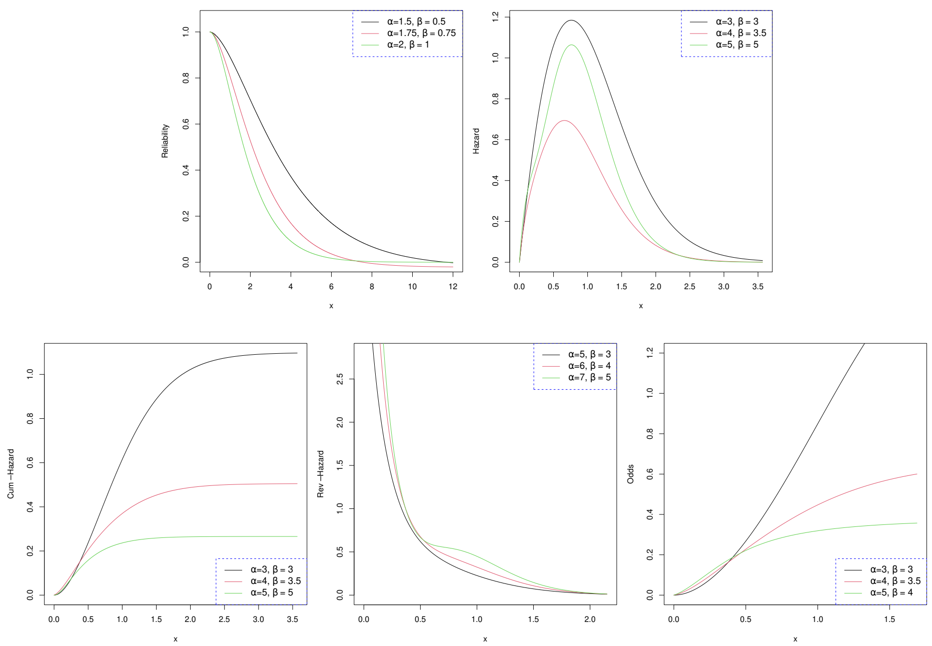

Figure 2: The reliability analysis functions plots of LBBD distribution for different values of α and β.

Figure 2 shows the plot of the reliability function (top left) of LBBD for $\alpha = 1.5$, 1.75 and 2 and $\beta = 0.5$, 0.75 and 1 and the plot of the hazard rate function (top right) of LBBD for $\alpha = 3$, 4 and 5 and $\beta = 3$, 3.5 and 5. The cumulative hazard rate function for the LBBD is presented bottom left, in this plot we have used the values of $\alpha = 3$, 4 and 5 and $\beta = 3$, 3.5 and 5. It also shows the reversed hazard rate function (bottom middle) for the LBBD with the values of α= 5, 6, 7 and $\beta = 3, 4, 5$. The odds rate plot (bottom right) for LBBD with $\alpha = 3$, 4 and 5 and $\beta = 3$, 3.5 and 4.

6 Order Statistics

In this section, we introduce the pdfs of first, n^{th} and k^{th} order statistics of LBBD. Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ be the order statistics of the random sample $X_1, X_2, ..., X_n$ obtained from LBBD. The pdf of the k^{th} order statistics is obtained by using (3.1) and (3.2). Thus, we get

$$\begin{aligned} f_{(k)}(x) &= k\binom{n}{k}\frac{\left[\alpha\Gamma(\alpha-1)(1-(\beta x+1)e^{-\beta x})+\beta\gamma(\alpha,\beta x)\right]^{k-1}}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^n}\left[\left(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}\right)e^{-\beta x}\right] \\ &\times\begin{pmatrix} e^{-\beta x}(-\beta e^{\beta x}\gamma(\alpha,\beta x)+\alpha\Gamma(\alpha-1) \\ +\alpha\beta x\Gamma(\alpha-1)-\beta\Gamma(\alpha-1)e^{\beta x} \\ +\alpha\beta\Gamma(\alpha-1)e^{\beta x}) \end{pmatrix}^{n-k} \end{aligned} \tag{6.1}$$

By substituting $k = 1$ and $k = n$ in (6.1), respectively, the minimum and maximum order statistics of LBBD can be determined. Consequently, we have

$$f_{(1)}(x) = \frac{n\left[\left(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}\right)e^{-\beta x}\right]}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^n}\begin{pmatrix} e^{-\beta x}(-\beta e^{\beta x}\gamma(\alpha,\beta x)+\alpha\Gamma(\alpha-1) \\ +\alpha\beta x\Gamma(\alpha-1)-\beta\Gamma(\alpha-1)e^{\beta x} \\ +\alpha\beta\Gamma(\alpha-1)e^{\beta x}) \end{pmatrix}^{n-1}$$

$$f_{(n)}(x) = \frac{n\left[\alpha\Gamma(\alpha-1)(1-(\beta x+1)e^{-\beta x})+\beta\gamma(\alpha,\beta x)\right]^{n-1}}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^n}\left[\left(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}\right)e^{-\beta x}\right]$$

Figure 3 shows the plot of the pdf of the k^{th} order statistics for $n = 10$ for all values of $k = 1$ to 10.

7 Stochastic Ordering

This section discusses how to compare two LBBD random variables using stochastic ordering.

Theorem 7.1 *Let $X \sim f_X(x;\alpha,\beta)$, $Y \sim f_Y(x;\theta,\eta)$, and if $\theta < \beta$ and $\eta < \alpha$, then for the LBBD distribution, we have $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ and $X \leq_{SO} Y$.*

Proof 7.1 *Consider*

$$\begin{aligned} \Upsilon &= \frac{f_X(x;\alpha,\beta)}{f_Y(x;\theta,\eta)} \\ \Upsilon &= \frac{\left[\frac{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]e^{-\beta x}}{\left[\frac{\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}}{(\theta-\eta+\theta\eta)\Gamma(\theta-1)}\right]e^{-\eta x}} \\ &= \left[\frac{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}{\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}}\right]\left[\frac{(\theta-\eta+\theta\eta)\Gamma(\theta-1)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]e^{-(\beta-\eta)x} \end{aligned}$$

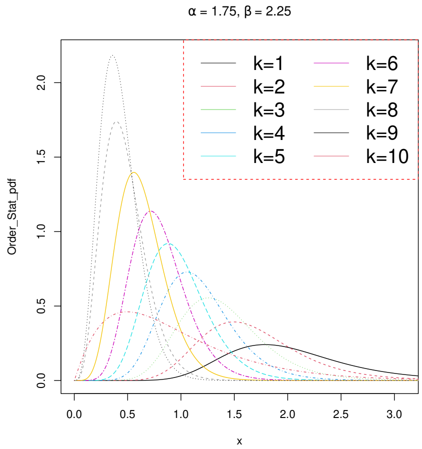

Figure 3: The plot of the pdfs of the order statistics.

Therefore,

$$\begin{aligned} ln(\Upsilon) &= ln\left(\left[\frac{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}{\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}}\right]\left[\frac{(\theta-\eta+\theta\eta)\Gamma(\theta-1)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]e^{-(\beta-\eta)x}\right) \\ &= ln\left[\frac{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}{\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}}\right]+ln\left[\frac{(\theta-\eta+\theta\eta)\Gamma(\theta-1)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]-(\beta-\eta)x \\ &= ln\left(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}\right)-ln(\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}) \\ &\quad +ln\left[\frac{(\theta-\eta+\theta\eta)\Gamma(\theta-1)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]-(\beta-\eta)x \end{aligned}$$

Deriving with respect to x, we get:

$$\frac{\partial ln(\Upsilon)}{\partial x} = \frac{\alpha\Gamma(\alpha-1)\beta^2+(\alpha-1)\beta^{\alpha+1}x^{\alpha-2}}{\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1}}-\frac{\theta\Gamma(\theta-1)\eta^2+(\theta-1)\eta^{\theta+1}x^{\theta-2}}{\theta\Gamma(\theta-1)\eta^2 x+\eta^{\theta+1}x^{\theta-1}}-(\beta-\eta)$$

$\frac{\partial ln(\Upsilon)}{\partial x} < 0$ *if* $\theta < \beta$, $\eta < \alpha$. *Thus,* $X \leq_{LRO} Y$, $X \leq_{HRO} Y$, $X \leq_{MRLO} Y$ *and* $X \leq_{SO} Y$.

8 Bonferroni and Lorenz Curves

Assume that the random variable X has a cumulative distribution function $F(x)$ that is continuous and twice differentiable, and that the random variable X is non-negative. The Bonferroni and Lorenz curves of the LBBD are provided in Theorem 8.1.

Theorem 8.1 *The Bonferroni and Lorenz curves of the LBBD are, respectively, provided by*

$$\begin{aligned} B(p) &= \frac{\left(\ \alpha\Gamma(\alpha-1)(e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2)+\beta\gamma(\alpha+1,\beta q)\ \right)}{p(2\alpha+\beta\alpha(\alpha-1))\Gamma(\alpha-1)} \\ L(p) &= \frac{\left(\ \alpha\Gamma(\alpha-1)(e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2)+\beta\gamma(\alpha+1,\beta q)\ \right)}{(2\alpha+\beta\alpha(\alpha-1))\Gamma(\alpha-1)} \end{aligned}$$

Proof 8.1 *In order to find the Bonferroni and Lorenz Curves of LBBD, we need*

$$\begin{aligned}
\int_0^q x f_l(x)dx &= \int_0^q x\left(\frac{(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right)dx \\
&= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\int_0^q(\alpha\Gamma(\alpha-1)\beta^2x^2+\beta^{\alpha+1}x^{\alpha})e^{-\beta x}dx \\
&= \frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\left[\alpha\Gamma(\alpha-1)\beta^2\int_0^q x^2e^{-\beta x}dx+\beta^{\alpha+1}\int_0^q x^{\alpha}e^{-\beta x}dx\right] \\
&= \frac{\left[\alpha\Gamma(\alpha-1)\beta^2\left(\frac{e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2}{\beta^3}\right)+(\gamma(\alpha+1,\beta q))\right]}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \\
&= \frac{\alpha\Gamma(\alpha-1)(e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2)+\beta\gamma(\alpha+1,\beta q)}{\beta(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \qquad (8.1)
\end{aligned}$$

Thus, the Bonferroni and Lorenz curves are defined using (8.1) as:

$$\begin{aligned}
B(p) &= \frac{\left(\ \alpha\Gamma(\alpha-1)(e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2)+\beta\gamma(\alpha+1,\beta q)\ \right)}{p(2\alpha+\beta\alpha(\alpha-1))\Gamma(\alpha-1)} \\
L(p) &= \frac{\left(\ \alpha\Gamma(\alpha-1)(e^{-\beta q}(-\beta^2q^2-2\beta q-2)+2)+\beta\gamma(\alpha+1,\beta q)\ \right)}{(2\alpha+\beta\alpha(\alpha-1))\Gamma(\alpha-1)}
\end{aligned}$$

9 Entropy

Entropy is the average amount of uncertainty that is essential for possible outcomes of a random variable. This section presents the Shannon, Rényi and Tsallis entropies of the LBBD.

Theorem 9.1 *Shannon, Rényi and Tsallis entropies of the random variable X such that $X \sim$ LBBD(α, β) are defined using (3.1) as:*

$$\begin{aligned}
S_\rho &= -\int_0^\infty\left[\frac{\alpha\beta^2\Gamma(\alpha-1)x+\beta^{\alpha+1}x^{\alpha-1}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]e^{-\beta x}log\left(\left[\frac{\alpha\beta^2\Gamma(\alpha-1)x+\beta^{\alpha+1}x^{\alpha-1}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]e^{-\beta x}\right)dx \\
R_\rho &= \frac{1}{1-\rho}log\left[\frac{\sum_{j=0}^{\rho}\binom{\rho}{j}(\alpha\Gamma(\alpha-1))^j\beta^{(2\rho-j-1)}\Gamma(\alpha(\rho-j)-\rho+2j+1)}{\rho^{\alpha(\rho-j)-\rho+2j+1}((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho}\right] \\
T_\rho &= \frac{1}{\rho-1}\left[1-\frac{\sum_{j=0}^{\rho}\binom{\rho}{j}(\alpha\Gamma(\alpha-1))^j\beta^{(2\rho-j-1)}\Gamma(\alpha(\rho-j)-\rho+2j+1)}{\rho^{\alpha(\rho-j)-\rho+2j+1}((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho}\right]
\end{aligned}$$

Proof 9.1 *The Shannon entropy is straightforward. To determine Rényi and Tsallis entropies of LBBD, we need to find*

$$\begin{aligned}
\int_0^\infty(f_l(x))^\rho dx &= \int_0^\infty\left(\frac{(\alpha\Gamma(\alpha-1)\beta^2x+\beta^{\alpha+1}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right)^\rho dx \\
&= \int_0^\infty\frac{\sum_{j=0}^{\rho}\binom{\rho}{j}(\alpha\Gamma(\alpha-1)\beta^2x)^j(\beta^{\alpha+1}x^{\alpha-1})^{\rho-j}}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho}e^{-\rho\beta x}dx \\
&= \frac{\sum_{j=0}^{\rho}\binom{\rho}{j}(\alpha\Gamma(\alpha-1))^j\beta^{\alpha(\rho-j)+\rho+j}}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho}\int_0^\infty(x^{\alpha(\rho-j)-\rho+2j})e^{-\rho\beta x}dx \\
&= \frac{\sum_{j=0}^{\rho}\binom{\rho}{j}(\alpha\Gamma(\alpha-1))^j\beta^{(2\rho-j-1)}\Gamma(\alpha(\rho-j)-\rho+2j+1)}{\rho^{\alpha(\rho-j)-\rho+2j+1}((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho}
\end{aligned}$$

Thus, Rényi and Tsallis entropies are defined as

$$R_\rho = \frac{1}{1-\rho} log \left[\frac{\sum_{j=0}^{\rho} \binom{\rho}{j} (\alpha\Gamma(\alpha-1))^j \beta^{(2\rho-j-1)} \Gamma(\alpha(\rho-j)-\rho+2j+1)}{\rho^{\alpha(\rho-j)-\rho+2j+1}((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho} \right]$$

$$T_\rho = \frac{1}{\rho-1} \left[1 - \frac{\sum_{j=0}^{\rho} \binom{\rho}{j} (\alpha\Gamma(\alpha-1))^j \beta^{(2\rho-j-1)} \Gamma(\alpha(\rho-j)-\rho+2j+1)}{\rho^{\alpha(\rho-j)-\rho+2j+1}((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^\rho} \right]$$

Table 2: Numerical results for Shannon, Rényi and Tsallis entropies for LBBD using different values of α and β with δ=5.

α	β	R^ρ	S^ρ	T^ρ	α	β	R^ρ	S^ρ	T^ρ
1.5	**0.5**	**0.397397**	**3.394330**	**0.250000**	**1.7**	**0.5**	**0.316355**	**3.615781**	**0.250000**
1.5	**0.6**	**0.505011**	**2.943372**	**0.249998**	**1.7**	**0.6**	**0.405589**	**3.160058**	**0.249999**
1.5	**0.7**	**0.610034**	**2.564902**	**0.249991**	**1.7**	**0.7**	**0.494290**	**2.776689**	**0.249996**
1.5	**0.8**	**0.709727**	**2.245314**	**0.249969**	**1.7**	**0.8**	**0.580215**	**2.446280**	**0.249986**
1.5	**0.9**	**0.801924**	**1.960771**	**0.249902**	**1.7**	**0.9**	**0.661527**	**2.172919**	**0.249958**
1.5	**1.0**	**0.884893**	**1.708012**	**0.249730**	**1.7**	**1.0**	**0.736706**	**1.915220**	**0.249882**
1.5	**1.1**	**0.957240**	**1.480913**	**0.249331**	**1.7**	**1.1**	**0.804477**	**1.683218**	**0.249702**
1.5	**1.2**	**1.017838**	**1.274950**	**0.248476**	**1.7**	**1.2**	**0.863770**	**1.472400**	**0.249308**
1.5	**1.3**	**1.065776**	**1.086687**	**0.246763**	**1.7**	**1.3**	**0.913671**	**1.279334**	**0.248502**
1.5	**1.4**	**1.100316**	**0.913457**	**0.243527**	**1.7**	**1.4**	**0.953402**	**1.101355**	**0.246947**
1.5	**1.5**	**1.120864**	**0.753144**	**0.237709**	**1.7**	**1.5**	**0.982293**	**0.936350**	**0.244093**
1.5	**1.6**	**1.126939**	**0.604046**	**0.227685**	**1.7**	**1.6**	**0.999768**	**0.782616**	**0.239076**
1.5	**1.7**	**1.118160**	**0.464769**	**0.211046**	**1.7**	**1.7**	**1.005328**	**0.638762**	**0.230578**
1.5	**1.8**	**1.094222**	**0.334162**	**0.184319**	**1.7**	**1.8**	**0.998537**	**0.503634**	**0.216654**
1.5	**1.9**	**1.054891**	**0.211261**	**0.142615**	**1.7**	**1.9**	**0.979019**	**0.376268**	**0.194500**
1.5	**2.0**	**0.999986**	**0.095252**	**0.079207**	**1.7**	**2.0**	**0.946442**	**0.255850**	**0.160157**
1.6	**0.5**	**0.354189**	**3.503154**	**0.250000**	**1.8**	**0.5**	**0.283177**	**3.730095**	**0.250000**
1.6	**0.6**	**0.452087**	**3.049472**	**0.249999**	**1.8**	**0.6**	**0.364576**	**3.273274**	**0.249999**
1.6	**0.7**	**0.548482**	**2.668336**	**0.249994**	**1.8**	**0.7**	**0.446270**	**2.888366**	**0.249998**
1.6	**0.8**	**0.640891**	**2.351131**	**0.249979**	**1.8**	**0.8**	**0.526249**	**2.556112**	**0.249991**
1.6	**0.9**	**0.727316**	**2.064087**	**0.249935**	**1.8**	**0.9**	**0.602838**	**2.286075**	**0.249973**
1.6	**1.0**	**0.806134**	**1.808886**	**0.249820**	**1.8**	**1.0**	**0.674623**	**2.026066**	**0.249924**
1.6	**1.1**	**0.876018**	**1.579398**	**0.249549**	**1.8**	**1.1**	**0.740391**	**1.791655**	**0.249807**
1.6	**1.2**	**0.935871**	**1.371097**	**0.248962**	**1.8**	**1.2**	**0.799096**	**1.578356**	**0.249547**
1.6	**1.3**	**0.984785**	**1.180545**	**0.247776**	**1.8**	**1.3**	**0.849828**	**1.382763**	**0.249010**
1.6	**1.4**	**1.022006**	**1.005068**	**0.245513**	**1.8**	**1.4**	**0.891788**	**1.202226**	**0.247961**
1.6	**1.5**	**1.046907**	**0.842549**	**0.241404**	**1.8**	**1.5**	**0.924270**	**1.034644**	**0.246014**
1.6	**1.6**	**1.058962**	**0.691282**	**0.234258**	**1.8**	**1.6**	**0.946648**	**0.878325**	**0.242550**
1.6	**1.7**	**1.057733**	**0.549873**	**0.222285**	**1.8**	**1.7**	**0.958364**	**0.731884**	**0.236618**
1.6	**1.8**	**1.042858**	**0.417168**	**0.202876**	**1.8**	**1.8**	**0.958915**	**0.594176**	**0.226786**
1.6	**1.9**	**1.014031**	**0.292200**	**0.172315**	**1.8**	**1.9**	**0.947852**	**0.464240**	**0.210963**
1.6	**2.0**	**0.971002**	**0.174156**	**0.125434**	**1.8**	**2.0**	**0.924766**	**0.341266**	**0.186159**

Table 2 shows the numerical results for Shannon, Rényi and Tsallis entropies for LBBD using different values of α of 1.5, 1.6, 1.7 and 1.8 and values of β of 0.5-2.0 (step=0.1). Based on this table, it is possible to see that the Rény, Tsallis and Shannon entropy values are increasing as the values of α increase and they are decreasing as the values of β are increasing. The Shannon entropy values are increasing as the values of α increase. They are decreasing as the values of β increase. The Tsallis entropy values are increasing as the values of α increase. These values are decreasing as the values of β increase.

10 Parameters Estimation

Let $X_1, X_2, ..., X_n$ be a random sample from LBBD. By using (3.1), the likelihood function of LBBD can be found as

$$\begin{aligned} L &= \prod_{i=1}^{n}\left[\frac{(\alpha\Gamma(\alpha-1)\beta^2 x_i + \beta^{\alpha+1}x_i^{\alpha-1})}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}e^{-\beta x_i}\right] \\ &= \frac{e^{-\beta\sum_{i=1}^{n}x_i}}{[(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)]^n}\left[\prod_{i=1}^{n}\left[\alpha\Gamma(\alpha-1)\beta^2 x_i + \beta^{\alpha+1}x_i^{\alpha-1}\right]\right] \end{aligned}$$

Thus, the log-likelihood function is

$$\begin{aligned} \ell &= ln(L) = ln\left\{\frac{e^{-\beta\sum_{i=1}^{n}x_i}}{[(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)]^n}\left[\prod_{i=1}^{n}\left[\alpha\Gamma(\alpha-1)\beta^2 x_i + \beta^{\alpha+1}x_i^{\alpha-1}\right]\right]\right\} \\ &= \sum_{i=1}^{n}\left[ln(\alpha\Gamma(\alpha-1)\beta^2 x_i + \beta^{\alpha+1}x_i^{\alpha-1})\right] - nln[(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)] - \beta\sum_{i=1}^{n}x_i \end{aligned}$$

The derivatives with respect to the parameters α and β, respectively, are

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} &= \sum_{i=1}^{n}\left[\frac{\beta^{\alpha}x_i^{\alpha-1}(ln(\beta)+ln(x_i)) + \beta x_i(\alpha\Gamma'(\alpha-1)+\Gamma(\alpha-1))}{\beta^{\alpha}x_i^{\alpha-1}+\alpha\beta\Gamma(\alpha-1)x_i}\right] \\ &\quad -n\left(\frac{(\alpha-\beta+\alpha\beta)\Gamma'(\alpha-1)+(\beta+1)\Gamma(\alpha-1)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right) \\ \frac{\partial\ell}{\partial\beta} &= \sum_{i=1}^{n}\left[\frac{\beta^{\alpha}(\alpha+1)x_i^{\alpha-1}+2\alpha\beta\Gamma(\alpha-1)x_i}{\beta^{\alpha+1}x_i^{\alpha-1}+\alpha\beta^2\Gamma(\alpha-1)x_i}\right] - n\left(\frac{\alpha-1}{\alpha-\beta+\alpha\beta}\right) - \sum_{i=1}^{n}x_i \end{aligned}$$

The MLEs of α and β are the solutions of the nonlinear system of Equations $\frac{\partial\ell}{\partial\alpha}=0$ and $\frac{\partial\ell}{\partial\beta}=0$, which can be solved numerically as there is no exact solution. The Ordinary least square (OLS) and weighted least square estimator (WLS) are two more techniques for estimating the model parameters. They are defined as

$$\begin{aligned} R_{OLS} &= \sum_{i=1}^{n}\left[F_l(x_{(i)}) - \frac{i}{n+1}\right]^2 \\ &= \sum_{i=1}^{n}\left[\frac{\left(\alpha\Gamma(\alpha-1)(1-(\beta x_{(i)}+1)e^{-\beta x_{(i)}}) + \beta\gamma(\alpha,\beta x_{(i)})\right)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} - \frac{i}{n+1}\right]^2 \end{aligned}$$

Thus, the OLS estimators of α and θ are the solutions of the following equations

$$\frac{\partial R_{OLS}}{\partial\alpha} = 0, \quad \frac{\partial R_{OLS}}{\partial\theta} = 0$$

The WLS of LBLD is defined as

$$W_{WLS} = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n+1-i)} \left[\frac{\left(\begin{array}{c} (\alpha\Gamma(\alpha-1)(1-(\beta x_{(i)}+1)e^{-\beta x_{(i)}}) \\ +\beta\gamma(\alpha,\beta x_{(i)}) \end{array} \right)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} - \frac{i}{n+1} \right]^2$$

Again, the WLS estimators of α and θ are the solutions of the following equations

$$\frac{\partial W_{WLS}}{\partial \alpha} = 0, \quad \frac{\partial W_{WLS}}{\partial \theta} = 0$$

The MPS estimators, $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$, of α and β can be obtained by maximizing the geometric mean of the spacings, that is,

$$\Psi_i(\alpha,\beta) = F_l(x_{(i)}|\alpha,\beta) - F_l(x_{(i-1)}|\alpha,\beta),\ i=1,...,n,$$

$$FM(\alpha,\beta|x) = \left(\prod_{i=1}^{n+1} \Psi_i(\alpha,\beta) \right)^{\frac{1}{n+1}}$$

Now, the natural logarithm of (10.1) gives

$$NL(\alpha,\beta|x) = \frac{1}{n+1} \sum_{i=1}^{n+1} \left[-ln((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)) + ln \left(\begin{array}{c} \alpha\Gamma(\alpha-1)(1-(\beta x_{(i)}+1)e^{\beta x_{(i)}}) + \beta\gamma(\alpha,\beta x_{(i)}) \\ -\beta\gamma(\alpha,\beta x_{(i-1)}) - \alpha\Gamma(\alpha-1)(1-(\beta x_{(i-1)}+1))e^{-\beta x_{(i-1)}} \end{array} \right) \right]$$

$\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$ can be obtained by solving the following nonlinear system of equations with respect to the parameters α and β.

$$\frac{\partial NL(\alpha,\beta|x)}{\partial \alpha} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{\Delta_1(x_{(k)}|\alpha,\beta) - \Delta_1(x_{(k-1)}|\alpha,\beta)}{\Psi_i(\alpha,\beta)} = 0$$

$$\frac{\partial NL(\alpha,\beta|x)}{\partial \beta} = \frac{1}{n+1} \sum_{i=1}^{n+1} \frac{\Delta_2(x_{(k)}|\alpha,\beta) - \Delta_2(x_{(k-1)}|\alpha,\beta)}{\Psi_i(\alpha,\beta)} = 0,$$

where

$$\Delta_1(x_{(.)}|\alpha,\beta) = \frac{\partial G(x_{(.)}|\alpha,\beta)}{\partial \alpha} \tag{10.1}$$

$$\Delta_2(x_{(.)}|\alpha,\beta) = \frac{\partial G(x_{(.)}|\alpha,\beta)}{\partial \beta} \tag{10.2}$$

The Cramer-Von Mises (CVM) estimators of α and β can be obtained by minimizing CVM^2.

$$CVM^2 = \int_{-\infty}^{\infty} [F_n(x) - F_l^*(x)]^2 \, dF_l^*(x)$$

$$= \sum_{i=0}^{n} \left[F_l(x_{(i)},\alpha,\beta) - \frac{2i-1}{2n} \right]^2 + \frac{1}{12n}$$

$$= \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{\left(\alpha\Gamma(\alpha-1)(1-(\beta x_{(i)}+1)e^{-\beta_{(i)}}) + \beta\gamma(\alpha,\beta x_{(i)}) \right)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} - \frac{2i-1}{2n} \right]^2$$

These estimators are the solutions of the following system of nonlinear equations

$$\begin{aligned}\sum_{i=0}^{n}\left[2F_l(x_{(i)},\alpha,\beta)-\frac{2i-1}{n}\right]\Lambda_1(x_{(i)}|\alpha,\beta) &= 0\\ \sum_{i=0}^{n}\left[2F_l(x_{(i)},\alpha,\beta)-\frac{2i-1}{n}\right]\Lambda_2(x_{(i)}|\alpha,\beta) &= 0,\end{aligned}$$

where

$$\Lambda_1(x_{(.)}|\alpha,\beta) = \frac{\partial F_l(x_{(.)}|\alpha,\beta)}{\partial\alpha}, \quad \Lambda_2(x_{(.)}|\alpha,\beta)=\frac{\partial F_l(x_{(.)}|\alpha,\beta)}{\partial\beta}$$

The Anderson-Darling (AD) estimators of α and β can be obtained by minimizing:

$$\begin{aligned}AD(\alpha,\beta) &= -n-\frac{1}{n}\sum_{i=0}^{n}(2i-1)\left\{ln[F_l(x_{(k)};\alpha,\beta\eta)]+ln\bar{F}_l(x_{(n+1-i)};\alpha,\beta)\right\}\\ &= -n-\sum_{i=0}^{n}\frac{(2i-1)}{n}\left[\begin{array}{c} ln\frac{\alpha\Gamma(\alpha-1)(1-(\beta x_i+1)e^{-\beta x_i})+\beta\gamma(\alpha,\beta x_i)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\\ +ln\left[\frac{(\alpha\Gamma(\alpha-1)(1-(\beta x_{n+1-i}+1)e^{-\beta x_{n+1-i}})+\beta\gamma(\alpha,\beta x_{n+1-i}))}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]\end{array}\right]\end{aligned}$$

These estimators are the solutions of the following system of nonlinear equations

$$\frac{\partial AD(\alpha,\beta)}{\partial\alpha}=0, \quad \frac{\partial AD(\alpha,\beta)}{\partial\beta}=0$$

11 Stress-Strength Reliability

Assume that X and Y are two independent random variables that are observed from the LBBD. The life of a component with a random strength Y and a random stress X is explained by stress-strength reliability.

Theorem 11.1 *Let the random variables X and Y be independently selected from the LBBD. The stress-strength reliability is given by*

$$p(Y<X) = \sum_{i=0}^{n}\frac{(-1)^i}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^2}\left(\begin{array}{c}\frac{\alpha^2(\Gamma(\alpha-1))^2\Gamma(i+4)}{i!(i+2)}\\ +\frac{\alpha\Gamma(\alpha-1)\beta\Gamma(\alpha+i+2)}{i!(i+2)}\\ +\frac{\alpha\Gamma(\alpha-1)\beta\Gamma(\alpha+i+2)}{i!(\alpha+i)}\\ +\frac{\beta^2\Gamma(2\alpha+i)}{i!(\alpha+i)}\end{array}\right)$$

Proof 11.1

$$
\begin{aligned}
p(Y<X) &= \int_0^\infty \int_0^x \frac{(\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1})}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \\
&\quad \times \frac{(\alpha\Gamma(\alpha-1)\beta^2 y + \beta^{\alpha+1}y^{\alpha-1})}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} e^{-\beta x} e^{-\beta y} dy dx \\
&= \left[\frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]^2 \int_0^\infty \int_0^x (\ \alpha\beta^2\Gamma(\alpha-1)x + \beta^{\alpha+1}x^{\alpha-1}\ ) \\
&\quad \times (\ \alpha\beta^2\Gamma(\alpha-1)y + \beta^{\alpha+1}y^{\alpha-1}\ ) e^{-\beta x} e^{-\beta y} dy dx \\
&= \left[\frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]^2 \int_0^\infty \int_0^x \left[\alpha\beta^2\Gamma(\alpha-1)x + \beta^{\alpha+1}x^{\alpha-1}\right] e^{-\beta x} \\
&\quad \times \sum_{i=0}^{n} \left(\frac{(-1)^i \alpha\beta^{i+2}\Gamma(\alpha-1)y^{i+1} + (-1)^i \beta^{\alpha+i+1} y^{\alpha+i-1}}{i!} \right) dy dx \\
&= \left[\frac{1}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]^2 \sum_{i=0}^{n} \int_0^\infty \left[\alpha\beta^2\Gamma(\alpha-1)x + \beta^{\alpha+1}x^{\alpha-1}\right] \\
&\quad \times \left(\frac{(-1)^i \alpha\beta^{i+2}\Gamma(\alpha-1)x^{i+2}}{i!(i+2)} + \frac{(-1)^i \beta^{\alpha+i+1} x^{\alpha+i}}{i!(\alpha+i)} \right) e^{-\beta x} dx \\
&= \sum_{i=0}^{n} \left[\frac{(-1)^i}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right]^2 \int_0^\infty \left(\begin{array}{c} \frac{\alpha^2(\Gamma(\alpha-1))^2\beta^{i+4}x_i^{i+3}}{i!(i+2)} \\ +\frac{\alpha\Gamma(\alpha-1)\beta^{\alpha+i+3}x^{\alpha+i+1}}{i!(i+2)} \\ +\frac{\alpha\Gamma(\alpha-1)\beta^{\alpha+i+3}x^{\alpha+i+1}}{i!(\alpha+i)} \\ +\frac{\beta^{2\alpha+i+2}x^{2\alpha+i-1}}{i!(\alpha+i)} \end{array} \right) e^{-\beta x} dx \\
&= \sum_{i=0}^{n} \frac{1}{((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1))^2} \left(\begin{array}{c} \frac{(-1)^i\alpha^2(\Gamma(\alpha-1))^2\Gamma(i+4)}{i!(i+2)} + \frac{(-1)^i\alpha\Gamma(\alpha-1)\beta\Gamma(\alpha+i+2)}{i!(i+2)} \\ +\frac{(-1)^i\alpha\Gamma(\alpha-1)\beta\Gamma(\alpha+i+2)}{i!(\alpha+i)} + \frac{(-1)^i\beta^2\Gamma(2\alpha+i)}{i!(\alpha+i)} \end{array} \right)
\end{aligned}
$$

12 Harmonic Mean and Mode

This section introduces the harmonic mean and the mode of the LBBD in the following theorems.

Theorem 12.1 *The harmonic mean of the LBBD is given by*

$$H = \frac{\alpha\beta+\beta^2}{(\alpha-\beta+\alpha\beta)} \tag{12.1}$$

Proof 12.1

$$
\begin{aligned}
H &= \int_0^\infty \frac{1}{x} f_l(x) dx = \int_0^\infty \frac{1}{x} \left(\frac{(\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \right) dx \\
&= \frac{\beta}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \int_0^\infty \left[\alpha\beta\Gamma(\alpha-1)e^{-\beta x} + \beta^\alpha x^{\alpha-2} e^{-\beta x}\right] dx \\
&= \frac{\beta}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)} \left[\alpha\Gamma(\alpha-1) + \beta\Gamma(\alpha-1)\right] = \frac{\alpha\beta+\beta^2}{(\alpha-\beta+\alpha\beta)}
\end{aligned}
$$

Theorem 12.2 *The mode of the LBBD is the solution of the Equation*

$$\alpha\beta\Gamma(\alpha-1)(\beta x-1) + \beta^\alpha x^{\alpha-2}(\beta x-\alpha+1) = 0 \tag{12.2}$$

Proof 12.2

$$
\begin{aligned}
f_l(x;\alpha,\beta) &= \frac{(\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1})}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}e^{-\beta x} \\
ln f_l(x;\alpha,\beta) &= ln\left(\frac{(\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1})}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}e^{-\beta x}\right) \\
&= ln(\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1}) \\
&\quad -ln((\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)) - \beta x \\
\frac{\partial ln f_l(x;\alpha,\beta)}{\partial x} &= \frac{\alpha\Gamma(\alpha-1)\beta^2 + (\alpha-1)\beta^{\alpha+1}x^{\alpha-2}}{\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1}} - \beta \overset{set}{=} 0 \\
\beta &= \frac{\alpha\Gamma(\alpha-1)\beta^2 + (\alpha-1)\beta^{\alpha+1}x^{\alpha-2}}{\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1}} \\
\alpha\Gamma(\alpha-1)\beta^2 x + \beta^{\alpha+1}x^{\alpha-1} &= \alpha\Gamma(\alpha-1)\beta + (\alpha-1)\beta^{\alpha}x^{\alpha-2} \\
0 &= \alpha\beta\Gamma(\alpha-1)(\beta x - 1) + \beta^{\alpha}x^{\alpha-2}(\beta x - \alpha + 1)
\end{aligned}
$$

Figure 4 shows the plot of Equation (12.2). It shows that there is one solution to this equation. But it can not be found explicitly.

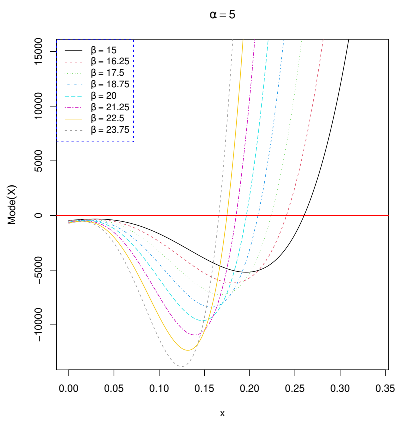

Figure 4: Mode Equation plot

13 Mean Absolute Deviations about Mean and Median

Theorem 13.1 *Let $X \sim f_l(x;\alpha,\beta)$, the mean deviations about the mean (MD_A) and median (MD_D) are*

$$
MD_A = \frac{\begin{pmatrix} 2(\alpha((\alpha-1)\beta+2)(\beta\gamma(\alpha,\beta\mu)+\alpha\Gamma(\alpha-1)(1-e^{-\beta\mu}(\beta\mu+1))) \\ -((\alpha-1)\beta+\alpha)(\beta\gamma(\alpha+1,\beta\mu)+\alpha\Gamma(\alpha-1)(e^{-\beta\mu}(-\beta\mu(\beta\mu+2)-2)+2))) \end{pmatrix}}{\beta((\alpha-1)\beta+\alpha)^2\Gamma(\alpha-1)}
$$

$$
MD_D = \frac{\begin{pmatrix} (-2\beta\gamma(\alpha+1,\beta M)+\alpha((\alpha-1)\beta+2)\Gamma(\alpha-1) \\ +\alpha\Gamma(\alpha-1)e^{-\beta M}(2\beta M(\beta M+2)-4e^{\beta M}+4)) \end{pmatrix}}{(\beta((\alpha-1)\beta+\alpha)\Gamma(\alpha-1))}
$$

Proof 13.1

$$
\begin{aligned}
MD_A &= 2\mu F(\mu) - 2\int_0^{\mu} x f(x)dx \\
&= 2\left(\frac{2\alpha+\beta\alpha(\alpha-1)}{\beta(\alpha-\beta+\alpha\beta)}\right)\left(\frac{\alpha\Gamma(\alpha-1)(1-(\beta\mu+1)e^{-\beta\mu})+\beta\gamma(\alpha,\beta\mu)}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right) \\
&\quad -2\int_0^{\mu} x\left(\frac{(\alpha\Gamma(\alpha-1)\beta^2 x+\beta^{\alpha+1}x^{\alpha-1})e^{-\beta x}}{(\alpha-\beta+\alpha\beta)\Gamma(\alpha-1)}\right)dx \\
&= \frac{\left(\begin{array}{c} 2(\alpha((\alpha-1)\beta+2)(\beta\gamma(\alpha,\beta\mu)+\alpha\Gamma(\alpha-1)(1-e^{-\beta\mu}(\beta\mu+1))) \\ -((\alpha-1)\beta+\alpha)(\beta\gamma(\alpha+1,\beta\mu)+\alpha\Gamma(\alpha-1)(e^{-\beta\mu}(-\beta\mu(\beta\mu+2)-2)+2))) \end{array}\right)}{\beta((\alpha-1)\beta+\alpha)^2\Gamma(\alpha-1)} \\
MD_D &= \frac{\left(\begin{array}{c} (-2\beta\gamma(\alpha+1,\beta M)+\alpha((\alpha-1)\beta+2)\Gamma(\alpha-1) \\ +\alpha\Gamma(\alpha-1)e^{-\beta M}(2\beta M(\beta M+2)-4e^{\beta M}+4)) \end{array}\right)}{(\beta((\alpha-1)\beta+\alpha)\Gamma(\alpha-1))}
\end{aligned}
$$

14 Simulation Study

In this section, a simulation study is performed to evaluate the efficiency and accuracy of the methods applied for estimating the parameters of the LBBD distribution. We have used the R software R Core Team (2021) to do this study. $N = 1500$ samples are generated for this purpose, each of sizes 100, 200, 300, 400 and 500 for the values of $(\alpha,\ \beta) = (2,\ 3)$. For each sample, the estimates of the parameter space $\phi = (\alpha, \theta)$, mean square error (MSE) and the bias are obtained. Then, we calculate the average bias (AB) and the average of mean squares errors (AMSE) as follows: $AB(\hat{\phi}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)$, $AMSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\phi}-\phi)^2$. The results of this simulation are summarized in Table 3.

As it can be seen from Table 3, the estimates are approximately unbiased and consistent. It shows that the OLS method is preferred to be uses for AMSE. The AD method is better to estimate α for large sample sizes, where the OLS is better for small sample sizes. For estimating β the CVM is preferred, in general.

15 Real Data Applications

This section compares the proposed distribution's goodness of fit to a few other existing distributions in order to demonstrate the flexibility of the proposed distribution utilizing three real-life time data sets. The first data set represents the number of million revolutions before failure for each of the 23 ball bearings in the life tests (Kalbfleisch and Lawless (1991)). The second data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli (Bjerkedal (1960)). The third data set represents he number of cycles until failure of the yarn (Picciotto (1970)). The following distributions are used for this comparison:

- Loai distribution (Alzoubi et al. (2022b)):

$$
f(x) = \frac{\theta^2}{\alpha+1}\left[\frac{1}{2}\alpha\theta x^2 + \frac{1}{\theta+1}(1+x)\right]e^{-\theta x};\ x>0, \theta>0, \alpha>0
$$

- Benrabia distribution (Benrabia and Alzoubi (2022)): (See (2.1))

Table 3: Average bias and mean square error for estimated LBBD parameters

Method	n	$\hat{\alpha}$	$\hat{\beta}$	$AB(\hat{\alpha})$	$AMSE(\hat{\alpha})$	$AB(\hat{\beta})$	$AMSE(\hat{\beta})$
MLE		**2.55695**	**3.01929**	**0.55695**	**1.76421**	**0.01929**	**0.09229**
OLS		**2.11667**	**3.04759**	**0.11667**	**0.06603**	**0.04759**	**0.03735**
WLS		**2.34496**	**2.99241**	**0.34496**	**2.33594**	**-0.00759**	**0.09359**
MPS	**100**	**3.17630**	**3.10356**	**1.17630**	**1.01587**	**0.10356**	**0.09948**
AD		**2.38417**	**2.99048**	**0.38417**	**3.21017**	**-0.00952**	**0.08365**
CVM		**4.53127**	**2.99991**	**2.53127**	**6.1728**	**-0.00009**	**0.10970**
MLE		**2.17985**	**3.00584**	**0.17985**	**0.76408**	**0.00584**	**0.03779**
OLS		**2.12727**	**3.03726**	**0.12727**	**0.04669**	**0.03726**	**0.01860**
WLS		**2.13993**	**2.98850**	**0.13993**	**0.76580**	**-0.01150**	**0.04122**
MPS	**200**	**2.36907**	**3.06118**	**0.36907**	**1.02837**	**0.06118**	**0.04375**
AD		**2.15447**	**3.00078**	**0.15447**	**0.70978**	**0.00078**	**0.04122**
CVM		**2.22862**	**3.00864**	**0.22862**	**1.02919**	**0.00864**	**0.04946**
MLE		**2.09260**	**2.99793**	**0.09260**	**0.35633**	**-0.00207**	**0.02674**
OLS		**2.13118**	**3.03763**	**0.13118**	**0.04159**	**0.03763**	**0.01308**
WLS		**2.07531**	**2.99080**	**0.07531**	**0.37224**	**-0.00920**	**0.02997**
MPS	**300**	**2.19949**	**3.03728**	**0.19949**	**0.46666**	**0.03728**	**0.02875**
AD		**2.07559**	**2.99265**	**0.07559**	**0.40528**	**-0.00735**	**0.03004**
CVM		**2.11640**	**3.00204**	**0.11640**	**0.47169**	**0.00204**	**0.03264**
MLE		**2.09415**	**3.00858**	**0.09415**	**0.29036**	**0.00858**	**0.02045**
OLS		**2.12877**	**3.04085**	**0.12877**	**0.03716**	**0.04085**	**0.01021**
WLS		**2.07925**	**3.00207**	**0.07925**	**0.29721**	**0.00207**	**0.02177**
MPS	**400**	**2.18072**	**3.02941**	**0.18072**	**0.30833**	**0.02941**	**0.01878**
AD		**2.05881**	**2.99598**	**0.05881**	**0.26554**	**-0.00402**	**0.01986**
CVM		**2.07835**	**2.99973**	**0.07835**	**0.30427**	**-0.00027**	**0.02406**
MLE		**2.06812**	**3.00754**	**0.06812**	**0.23742**	**0.00754**	**0.01600**
OLS		**2.12170**	**3.04268**	**0.12170**	**0.03396**	**0.04268**	**0.00905**
WLS		**2.05706**	**3.00361**	**0.05706**	**0.23476**	**0.00361**	**0.01704**
MPS	**500**	**2.12846**	**3.02123**	**0.12846**	**0.24650**	**0.02123**	**0.01615**
AD		**2.04539**	**2.99768**	**0.04539**	**0.21260**	**-0.00232**	**0.01706**
CVM		**2.06695**	**2.99941**	**0.06695**	**0.25305**	**-0.00059**	**0.01967**

- Lindley distribution (Ghitany et al. (2008)):
  $f(x) = \frac{\alpha^2(1+x)e^{-\alpha x}}{1+\alpha}$; $x > 0, \alpha > 0$
- Gamma distribution (?): $f(x) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$; $x > 0, \alpha, \beta > 0$
- Sameera distribution (Alzoubi et al. (2022a)):

$$f(x) = \left(\frac{\alpha^2\beta^2}{1+\alpha^2\beta} + \frac{x^{\alpha-1}\beta^\alpha}{(1+\alpha^2\beta)\Gamma(\alpha)}\right)e^{-\beta x};\ x > 0, \alpha, \beta > 0$$

- Exponential distribution (Kingman (1982)): $f(x) = \theta e^{-\theta x}$; $x > 0, \theta > 0$.

Table 4: The number of million revolutions before failure for each of the 23 ball bearings in the life tests

17.88	28.920	33.000	41.520	42.120	45.600	48.800	51.840
51.960	54.120	55.560	67.800	68.440	68.640	68.880	84.120
93.120	98.640	105.120	105.840	127.920	128.040	173.40	

Table 5: The survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli

12	15	22	24	24	32	32	33	34	38	38	43
44	48	52	53	54	54	55	56	57	58	58	59
60	60	60	60	61	62	63	65	65	67	68	70
70	72	73	75	76	76	81	83	84	85	87	91
95	96	98	99	109	110	121	127	129	131	143	146
146	175	175	211	233	258	258	263	297	341	341	376

Table 6: The data are the number of cycles until failure of the yarn

86	146	251	653	98	249	400	292	131	169	175	176	76
264	15	364	195	262	88	264	157	220	42	321	180	198
38	20	61	121	282	224	149	180	325	250	196	90	229
166	38	337	65	151	341	40	40	135	597	246	211	180
93	315	353	571	124	279	81	186	497	182	423	185	229
400	338	290	398	71	246	185	188	568	55	55	61	244
20	284	393	396	203	829	239	236	286	194	277	143	198
264	105	203	124	137	135	350	193	188				

Table 7: Summary for data used for LBBD

Min.	**1st Qu**	**Median**	**Mean**	**3rd Qu**	**Max.**
17.880	**47.200**	**66.800**	**72.230**	**95.880**	**173.400**
12.00	**54.75**	**70.00**	**99.82**	**112.75**	**376.00**
15.0	**129.2**	**195.5**	**222.0**	**282.5**	**829.0**

Tables 8 - 10 clarify that the suggested distribution has the lowest values of $-ln(L)$, AIC, CAIC, BIC, HQIC, and KS with the highest p-value. Therefore the suggested distribution is preferred over the competence distributions. The 95% CIs of α and β are calculated in these tables.

Table 8: Application I of LBBD

Distr.	-ln(L)	AIC	CAIC	BIC	HQIC	KS	pv	Par	Est	SE	95% CI Lower	95% CI Upper
LBBD	113.590	231.181	231.781	233.452	231.752	0.182	0.388	α	3.852	5.327	-6.590	14.300
								β	0.028	0.004	0.019	0.036
Benrabia	127.940	260.094	260.726	262.487	249.484	0.297	0.027	α	0.014	0.003	0.008	0.019
								β	2.795	5.667	-8.313	13.903
Lindley	114.736	232.489	232.681	233.634	232.078	0.193	0.314	α	0.027	0.004	0.019	0.035
Gamma	119.060	241.959	242.534	244.138	245.359	0.239	0.121	α	0.171	0.051	0.071	0.270
								β	12.075	3.511	5.194	18.955
Exponential	129.037	260.198	260.401	261.405	250.051	0.307	0.020	θ	0.014	0.004	0.006	0.022
Loai	121.697	257.395	247.632	253.213	247.536	0.285	0.038	α	1.142	0.508	0.147	2.137
								β	0.025	0.007	0.011	0.038

Table 9: Application II of LBBD

Distr.	-ln(L)	AIC	CAIC	BIC	HQIC	KS	p-Value	Par	Est.	SE	95% CI Lower	95% CI Upper
LBBD	156.440	312.880	627.759	630.364	628.813	0.109	0.355	α	7.357	10.700	-13.698	28.412
								β	0.022	0.001	0.021	0.023
Berabia	201.714	403.487	405.427	407.704	406.334	0.241	0.006	α	1.993	0.001	1.991	1.995
								β	0.010	2.726	-5.334	5.354
Lindley	197.260	394.580	395.520	396.658	395.973	0.129	0.159	α	0.020	0.002	0.017	0.023
Gamma	197.147	394.353	396.293	398.570	397.200	0.205	0.026	α	1.481	0.004	1.474	1.488
								β	0.101	0.338	-0.561	0.763
Sameera	201.403	402.866	404.806	407.083	405.712	0.136	0.140	α	1.924	0.795	0.366	3.482
								β	0.018	0.004	0.010	0.026
Loai	197.548	395.155	397.095	399.372	398.001	0.174	0.067	α	1.391	0.001	1.390	1.392
								β	0.026	0.117	-0.204	0.256

Table 10: Application III of LBBD

Distr.	-ln(L)	AIC	CAIC	BIC	HQIC	KS	pv	Par	Est.	SE	95% CI Lower	95% CI Upper
LBBD	314.539	631.359	631.483	640.569	633.117	0.102	0.245	α	5.837	10.700	-15.220	26.892
								β	0.009	0.001	0.008	0.010
Berabia	322.127	646.255	646.378	655.465	648.013	0.197	0.001	α	0.004	0.0004	0.004	0.005
								β	2.681	4.347	-5.839	11.201
Lindley	314.636	630.671	630.795	639.881	632.429	0.110	0.182	α	0.009	0.001	0.008	0.010
Gamma	314.667	631.235	631.358	640.445	632.993	0.111	0.167	α	2.189	0.282	1.636	2.742
								β	0.010	0.001	0.007	0.013
Sameera	317.025	636.051	636.174	645.261	637.809	0.122	0.103	α	4.367	0.447	3.491	5.244
								β	0.018	0.002	0.015	0.021
Loai	314.489	630.557	630.681	639.767	632.315	0.089	0.400	α	0.012	0.001	0.010	0.014
								β	1.601	1.491	-1.321	4.523

16 Conclusions

This paper proposed the length biased of Benrabia distribution. The basic properties of this distribution are derived, such as the moments and their related measures, the harmonic mean, and the mode. In addition to the reliability analysis functions, the pdfs of the minimum, maximum and the k^{th} order statistics are derived as well as the quantile function. The mean absolute deviations about mean and median are, also derived. The MLE, OLS, WLS, MPS, CVM and AD methods of estimating parameters are discussed, these methods are tested through a simulation study shows that the estimators are unbiased and consistent. To prove the goodness of fit for these distributions, three real data applications were illustrated compared to other distributions showing that the suggested distribution fits the real data better than the competence distributions.

References

Al-Omari, A. and Alanzi, A. (2021). Inverse Length Biased Maxwell Distribution: Statistical Inference with an Application . *Computer Systems Science and Engineering*, 39(1):147–164.

Al-Omari, A., Alhyasat, K., Ibrahim, K., and Abu-Bakar, M. (2019). Power length biased Suja distribution: properties and application . *Electronic Journal of Applied Statistical Analysis*, 12(2):429–452.

Aleem, M., Sufyan, M., and Khan, N. (2013). A Class of Modified Weighted Weibull Distribution and its properties. *American Review of Mathematics and Statistics*, 1:29–37.

Alqallaf, F., Ghitany, M., and Agostinelli, C. (2015). Weighted exponential distribution: Different methods of estimations. *Applied Math. Inform*, 9:1167–1173.

Alsmairan, I. and Al-Omari, A. (2020). Weighted Suja distribution with application to ball bearings data. *Life Cycle Reliability and Safety Engineering*, 9(2):195–211.

Alzoubi, L., Gharaibeh, M., Alkhazaalh, A., and Benrabia, M. (2022a). Estimation of Sameera Distribution Parameters with Applications to Real Data. *Applied Mathematics Information Sciences*, 16:1057–1071.

Alzoubi, L., Gharaibeh, M., Alkhazaalh, A., and Berabia, M. (2022b). Loai distribution: Properties, parameters estimation and application to covid-19 real data. *Mathematical Statistician and Engineering Applications*, 7(4):1231–1255.

Asgharzadeh, A., Bakouch, H., Nadarajah, S., and Sharafia, F. (2016). A new weighted Lindley Distribution with application . *Brazilian Journal of Probability and Statistics*, 30(1):1–27.

Badmus, N., Bamiduro, T., and Ogunobi, S. (2014). Lehmann Type-II Weighted Weibull distribution . *International Journal Physical Science*, 9:71–78.

Bashir, S. and Rasul, M. (2015). Some Properties of the Weighted Lindley distribution. *International Journal of Economics and Business Review*, 3:11–17.

Benrabia, M. and Alzoubi, L. (2022). Benrabia Distribution: Properties and Applications. *Electronic Journal of Applied Statistical Analysis*, 15(2):300–317.

Bjerkedal, T. (1960). Acquisition of Resistance in Guinea Pigs Infected with Different Doses of Virulent Tubercle Bacilli. *American Journal of Epidemiology*, 72(1):130–148.

Cox, D. (1968). Some Sampling Problems in Technology. In: Johnson, N.L. and Smith, H., Eds., New Developments in Survey Sampling.

Das, K. and Roy, T. (2011). On Some Length-Biased Weighted Weibull Distribution. *Advances in Applied Science Research*, 2:465–475.

Fatima, K. and Ahmad, S. P. (2017). Weighted Inverse Rayleigh distribution. *International Journal of Statistics and Systems,*, 1(12):119–137.

Fisher, R. (1934). The Effect of Methods of Ascertainment upon the Estimation of Frequencies. *The Annals of Human Genetics*, 6:13–25.

Ghitany, M., Atieh, B., and Nadarajah, S. (2008). Lindley Distribution and its Applications. *Mathematics and Computers in Simulation*, 78:493–506.

Jan, U., Fatima, K., and Ahmad, S. P. (2017). On weighted Ailamujia distribution and its applications to lifetime data. *Journal of Statistics Applications and Probability*, 6:619–633.

Kalbfleisch, J. and Lawless, J. (1991). Regression models for right truncated data with applications to aids incubation times and reporting lags. *Statistica Sinica*, 1(1):19–32.

Kersey, J. (2010). Weighted inverse weibull and beta-inverse weibull distribution. Master thesi, University of Georgia Southern.

Kingman, J. (1982). The coalescent. *Stochastic Processes and their Applications*, 13(3):235–248.

Patil, G. and Ord, J. (1976). On size-biased sampling and related form-invariant weighted distributions. *Sankhyā: The Indian Journal of Statistics*, 38(1):48–61.

Picciotto, R. (1970). *Tensile Fatigue Characteristics of a Sized Polyester/viscose Yarn and Their Effect on Weaving Performance*. North Carolina State University at Raleigh.

R Core Team (2021). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Rao, C. (1965). On Discrete Distributions Arising out of Methods of Ascertainment. *Sankhya: The Indian Journal of Statistics, Series A (1961-2002)*, 27(2/4):311–324.

Rashwan, N. (2013). The Double Weighted Rayleigh Distribution Properties and Estimation. *International Journal of Scientific and Engineering Research*, 4:1084–1089.

Roman, R. (2010). Theoretical properties and estimation in weighted weibull and related distributions. Electronic theses and dissertations, Georgia Southern University.

Saghir, A., Hamedani, G., Tazeem, S., and Khadim, A. (2017a). Weighted Distribution: A breif review, perespective and characterization. *International Journal of Statistics and Probability*, 6(3):109–131.

Saghir, A., Khadim, A., and Lin, Z. (2017b). The Maxwell-length-biased distribution: Properties and Estimation. *Journal of Statistical Theory and Practice,*, 11(1):26–40.

Sharma, V., Dey, S., Singh, S., and Manzoor, U. (2018). On length and area biased maxwell distributions . *Communications in Statistics - Simulation and Computation*, 47(5):1506–1528.

Shen, Y., Ning, J., and Qin, J. (2009). Analyzing length-biased data with semiparametric transformation and accelerated failure time models. *Journal of the American Statistical Association*, 104(487):1192–1202.

Shi, X., Oluyede, B., and Pararai, M. (2012). Theoretical Properties of Weighted Generalized Rayleigh and Related Distributions . *Theoretical Mathematics and Applications*, 2(2):45–62.

Warren, W. (1975). *Statistical Distributions in Forestry and Forest Products Research*, chapter 5, pages 369–384. D. Reidel Publishing Company.

Ye, Y., Oluyede, B., and Pararai, M. (2012). Weighted Generalized Beta Distribution of the Second Kind and Related Distributions. *Journal of Statistical and Econometric Methods*, 1(1):13–31.

Zelen, M. (1974). Problems in cell kinetics and the early detection of disease. *Reliability and Biometry, SIAM*, 56(3):701–706.