Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v17n2p206

Arz Distribution: A Novel One Parameter Model with Bathtub-Shaped Hazard Rate and Application on Covid-19

By Gharaibeh and Al-Ayasrah

15 October 2024

Electronic Journal of Applied Statistical Analysis
Vol. 17, Issue 02, October 2024, 206-224
DOI: 10.1285/i20705948v17n2p206

Arz Distribution: A Novel One Parameter Model with Bathtub-Shaped Hazard Rate and Application on Covid-19

Mohammed M. Gharaibeh*[a] and Rodaina M. Al-Ayasrah[a]

[a]*Department of Mathematics, Al al-Bayt University , Mafraq (25113), Jordan*

15 October 2024

In this paper, a novel one parameter model with bathtub-shaped hazard rate is proposed and called Arz distribution. This distribution is suggested based on the idea of mixture distributions. We investigate several properties of this distribution such as r^{th} moment, moment generating function, skewness, coefficient of variation, kurtosis, index of dispersion, order statistics, Lorenz and Bonferroni curves, Gini index, stochastic ordering, *Rényi* entropy, mean deviations about mean and median. Also, the survival function, hazard function, mean residual life function, reversed hazard function, and odds function are provided with graphical representation. It is found that the hazard function has a bathtub shape even though the distribution has one parameter. The parameter of the distribution is estimated using maximum likelihood method. Application to COVID-19 data set is presented to show the flexibility of the suggested distribution. The application indicates that the proposed distribution is more flexible than some other competitive distributions in fitting such data.

keywords: Mixture distributions, Reliability analysis, Moments, Order statistics, Lorenz and Bonferroni curves, Stochastic ordering, Entropy, Mean deviations, Maximum likelihood estimation.

*Corresponding author: mmgharaibeh@gmail.com.

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

1 Introduction

The idea of mixing distributions is one of the pioneering ideas in the field of statistics and probability. It is used to create a new probability distribution that is more flexible and appropriate than the basic components when applied to some real data sets. This technique has the privilege of increasing the flexibility of the basic distributions without the need of adding more parameters.

The idea of mixing distributions has attracted the interest of many researchers to introduce a new distributions such as: Shanker distribution (Shanker, 2015b), Gharaibeh distribution (Gharaibeh, 2021), Aradhana distribution (Shanker, 2016), Sameera distribution (Alzoubi et al., 2022c), Ishita distribution (Shanker and Shukla, 2017), Akshaya distribution (Shanker, 2017), Akash distribution (Shanker, 2015a), Darna distribution (Al-Omari and Shraa, 2019), Karam distribution (Gharaibeh and Sahtout, 2022), Loai distribution (Alzoubi et al., 2022b), Ola distribution (Al-Ta'ani and Gharaibeh, 2023).

Other researchers used different techniques to generalize and increase the flexibility of the existing distributions by adding more parameters such as: transmuted Aradhana distribution (Gharaibeh, 2020), transmuted Ishita distribution (Gharaibeh and Al-Omari, 2019), transmuted Shanker distribution (Alzoubi et al., 2022a), Topp-Leone Mukherjee-Islam distribution (Al-Omari and Gharaibeh, 2018), Marshall-Olkin extended inverted Kumaraswamy distribution (Usman and Ahsan ul Haq, 2020) and Burr XII modified Weibull distribution (Mdlongwa and Huang, 2017).

In this paper, we adapt the idea of mixing distribution to propose a novel distribution called Arz distribution without adding more parameters. Even though this proposed distribution has one parameter, its hazard function has a bathtub shape. Its superiority in fitting real data is illustrated through an application to Covid-19 data.

The rest of this paper is organized as follows; the probability density function (*pdf*) and cumulative distribution function (*cdf*) of Arz distribution are introduced in Section 2. Reliability analysis functions with graphical representation are given in Section 3. Moments and moment generating function are obtained in Section 4. The distribution of order statistics is investigated in Section 5. Lorenz and Bonferroni curves and Gini index are given in Section 6. Stochastic ordering is explored in Section 7. Rényi entropy and mean deviations are studied in Section 8. The maximum likelihood estimate of the distribution parameter is provided in Section 9. Application to COVID-19 data set is presented in Section 10. The conclusion is summarized in Section 11.

2 Arz Distribution

The distribution $f(x)$ is a mixture of n components distributions $g_1(x), g_2(x), ..., g_n(x)$ if

$$f(x) = \sum_{i=1}^{n} p_i g_i(x), \tag{1}$$

where p_i's are the mixing proportions, such that $\sum_{i=1}^{n} p_i = 1$ and $0 \leq p_i \leq 1$.

Arz distribution is obtained by mixing the following three continuous distributions $Exp(\theta)$, $Gamma(3,\theta)$, and $Gamma(8,\theta)$ with mixing proportions given in Table 1.

Table 1: The pdfs and mixing proportions of basic distributions

Distribution	pdf	Mixing Proportion
$Exp(\theta)$	$g_1(x) = \theta e^{-\theta x} \quad ; x > 0$	$p_1 = \frac{\theta^7}{\theta^7+2\theta^5+5040}$
$Gamma(3,\theta)$	$g_2(x) = \frac{\theta^3 x^2 e^{-\theta x}}{\Gamma(3)}; x > 0$	$p_2 = \frac{2\theta^5}{\theta^7+2\theta^5+5040}$
$Gamma(8,\theta)$	$g_3(x) = \frac{\theta^8 x^7 e^{-\theta x}}{\Gamma(8)}; x > 0$	$p_3 = \frac{5040}{\theta^7+2\theta^5+5040}$

Therefore, the *pdf* of Arz distribution is defined as

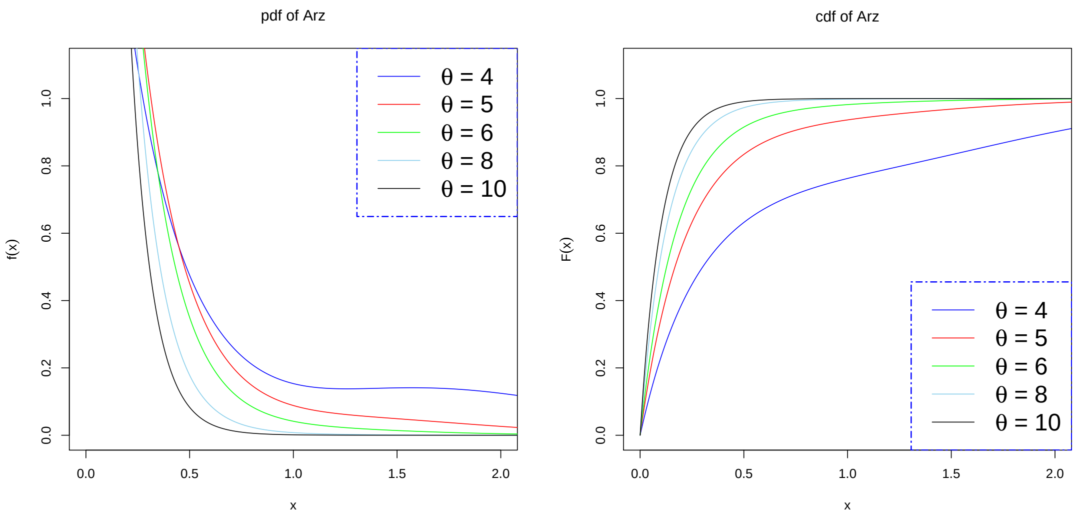
$$f(x) = \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040}\left(x^7 + x^2 + 1\right) e^{-\theta x}; x > 0; \theta > 0. \tag{2}$$

The corresponding *cdf* is given by

$$F(x) = 1 - \left[\frac{\theta^7(x^7+x^2) + \theta^6(7x^6+2x) + 42\theta^5 x^5 + 210\theta^4 x^4 + 840\theta^3 x^3 + 2520\theta^2 x^2 + 5040\theta x}{\theta^7 + 2\theta^5 + 5040} + 1\right] e^{-\theta x}; x; \theta > 0. \tag{3}$$

Figure 1 shows the *pdf* and *cdf* plots of Arz distribution with various values of the distribution parameter. It clear that Arz distribution is right skewed and its *cdf* reaches 1 faster for larger values of θ.

Figure 1: The *pdf* and *cdf* of Arz distribution with different values of θ



3 Reliability Analysis

In this section, we explore some reliability analysis functions for Arz distribution with graphical representation. Using (3), the survival function, $S(x)$, odds function, $O(x)$, and mean residual life function, $MRL(x)$, for Arz distribution, are defined, respectively, as follows

$$S(x) = 1 - F(x) = \left[\frac{\begin{array}{l}\theta^7(x^7+x^2)+\theta^6(7x^6+2x)+42\theta^5x^5 \\ +210\theta^4x^4+840\theta^3x^3+2520\theta^2x^2+5040\theta x\end{array}}{\theta^7+2\theta^5+5040} + 1 \right] e^{-\theta x},$$

$$O(x) = \frac{F(x)}{1-F(x)} = \frac{(\theta^7+2\theta^5+5040)\, e^{\theta x}}{\left[\begin{array}{l}\theta^7(x^7+x^2+1)+\theta^6(7x^6+2x)+\theta^5(42x^5+2)+\theta^4(210x^4) \\ +\theta^3(840x^3)+\theta^2(2520x^2)+\theta(5040x)+5040\end{array}\right]} - 1,$$

$$\begin{aligned} MRL(x) &= E(X-x|X>x) = \frac{1}{1-F(x)}\int_x^\infty (1-F(y))\,dy \\ &= \frac{\left[\begin{array}{l}\theta^7(x^7+x^2+1)+\theta^6(14x^6+4x)+\theta^5(126x^5+6)+\theta^4(840x^4) \\ +\theta^3(4200x^3)+\theta^2(15120x^2)+\theta(35280x)+40320\end{array}\right]}{\left[\begin{array}{l}\theta^8(x^7+x^2+1)+\theta^7(7x^6+2x)+\theta^6(42x^5+2)+\theta^5(210x^4) \\ +\theta^4(840x^3)+\theta^3(2520x^2)+\theta^2(5040x)+\theta 5040\end{array}\right]}. \end{aligned}$$

Clearly, $MRL(0) = E(X)$ which is given in (6).

Based on (2) and (3), the hazard function, $h(x)$, reversed hazard function, $rh(x)$, and cumulative hazard function, $H(x)$, for Arz distribution, are defined, respectively, as follows

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\theta^8\,(x^7+x^2+1)}{\left[\begin{array}{l}\theta^7(x^7+x^2+1)+\theta^6(7x^6+2x)+\theta^5(42x^5+2)+\theta^4(210x^4) \\ +\theta^3(840x^3)+\theta^2(2520x^2)+\theta(5040x)+5040\end{array}\right]},$$

$$rh(x) = \frac{f(x)}{F(x)} = \frac{\theta^8\,(x^7+x^2+1)}{\left[\begin{array}{l}(\theta^7+2\theta^5+5040)(e^{\theta x}-1)-\theta^7(x^7+x^2)-\theta^6(7x^6+2x) \\ -42\theta^5x^5-210\theta^4x^4-840\theta^3x^3-2520\theta^2x^2-5040\theta x\end{array}\right]},$$

$$H(x) = -ln\,(1-F(x)) = \theta x - \ln\left[\frac{\begin{array}{l}\theta^7(x^7+x^2)+\theta^6(7x^6+2x)+\theta^5(42x^5)+\theta^4(210x^4) \\ +\theta^3(840x^3)+\theta^2(2520x^2)+\theta(5040x)\end{array}}{\theta^7+2\theta^5+5040} + 1 \right].$$

Figures 2, 3 and 4 present graphical representation of these reliability functions for different values of the parameter θ. It can be noted that as the value of the parameter

Figure 2: The hazard and odds functions of Arz distribution with different values of θ

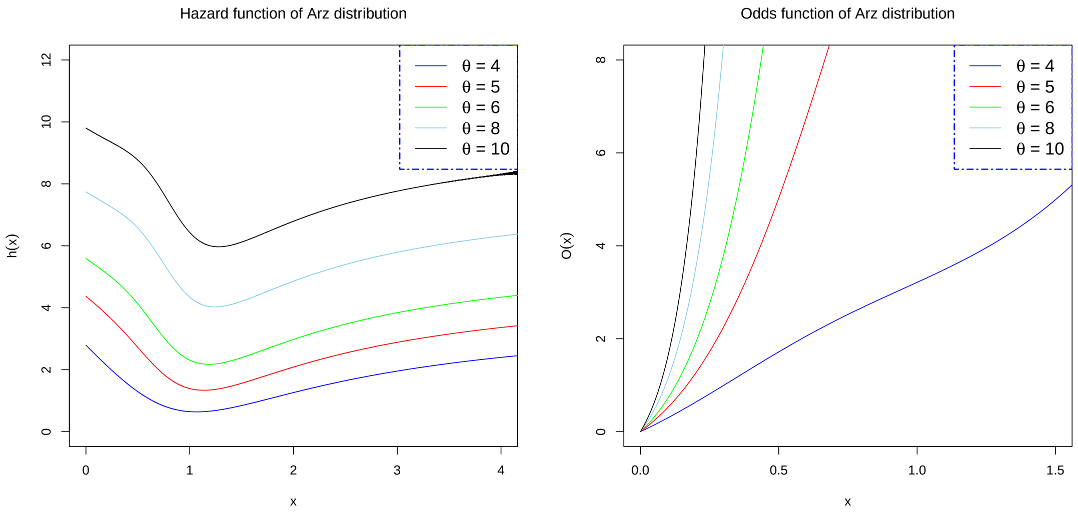
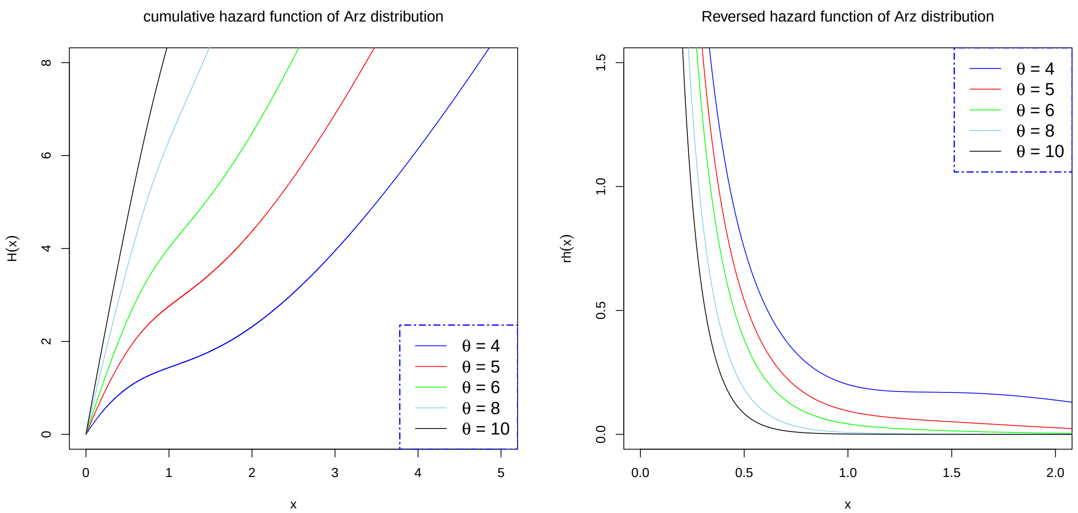


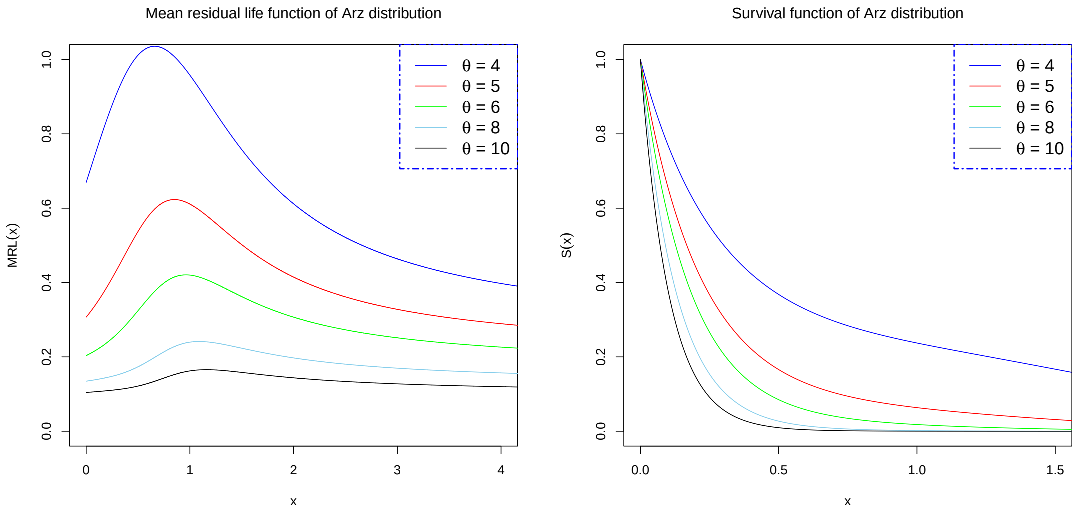
Figure 3: The cumulative and reversed hazard functions of Arz distribution with different values of θ



θ increases, the values of hazard, odd and cumulative hazard functions increase while the values of reversed hazard, mean residual life and survival functions decrease.

The left panel of Figure 2 declares that the hazard function of Arz distribution has a bathtub shape even though the distribution has one parameter. It is worth mentioning

Figure 4: The mean residual life and survival functions of Arz distribution with different values of θ



here that bathtub-shaped is the most important shape every model stumble to have even for distributions with more than one parameter. This gives Arz distribution a privilege over other one parameter distributions.

4 Moments and Associated Measures

In this section, we derive the moment generating function and r^{th} moment of Arz distribution. Also, associated measures including mean, variance, skewness, coefficient of variation, kurtosis and index of dispersion are obtained.

Theorem 1 *Assume that X is a random variable has Arz distribution $(i.e. X \sim Arz(\theta))$, then the moment generating function of X is given by*

$$M_X(t) = \sum_{j=0}^{\infty} \frac{(t/\theta)^j}{j!(\theta^7 + 2\theta^5 + 5040)} \left[\theta^7 j! + \theta^5 (j+2)! + (j+7)!\right] \tag{4}$$

Proof: Using (2.1) and the binomial series, the moment generating function of the Arz distribution can be proved as

$$\begin{aligned}
M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} f(x)dx \\
&= \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \int_0^\infty (x^7 + x^2 + 1)e^{x(t-\theta)}dx \\
&= \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \left[\int_0^\infty e^{x(t-\theta)}dx + \int_0^\infty x^2 e^{x(t-\theta)}dx + \int_0^\infty x^7 e^{x(t-\theta)}dx\right] \\
&= \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \left[\frac{1}{\theta - t} + \frac{2}{(\theta - t)^3} + \frac{5040}{(\theta - t)^8}\right] \\
&= \frac{1}{\theta^7 + 2\theta^5 + 5040} \left[\frac{\theta^7}{\left(1 - \frac{t}{\theta}\right)} + \frac{2\theta^5}{\left(1 - \frac{t}{\theta}\right)^3} + \frac{5040}{\left(1 - \frac{t}{\theta}\right)^8}\right] \\
&= \frac{\theta^7 \sum_{j=0}^\infty \binom{j}{j}(t/\theta)^j + 2\theta^5 \sum_{j=0}^\infty \binom{j+2}{j}(t/\theta)^j + 5040 \sum_{j=0}^\infty \binom{j+7}{j}(t/\theta)^j}{\theta^7 + 2\theta^5 + 5040} \\
&= \sum_{j=0}^\infty \frac{(t/\theta)^j}{j!(\theta^7 + 2\theta^5 + 5040)} \left(\theta^7 j! + \theta^5 (j+2)! + (j+7)!\right)
\end{aligned}$$

Theorem 2 *Let $X \sim Arz(\theta)$, the r^{th} moment of X is given by*

$$E(X^r) = \left[\frac{\theta^7 r! + \theta^5 (r+2)! + (r+7)!}{\theta^{(7+r)} + 2\theta^{(5+r)} + 5040\theta^r}\right]; \qquad r = 1, 2, 3, \dots \tag{5}$$

Proof: Using (2), the r^{th} moment about the origin of Arz distribution can be proved as

$$\begin{aligned}
E(X^r) &= \int_0^\infty x^r f(x)dx \\
&= \int_0^\infty x^r \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \left[x^7 + x^2 + 1\right] e^{-\theta x}dx \\
&= \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \left[\int_0^\infty x^r e^{-\theta x}dx + \int_0^\infty x^{r+2} e^{-\theta x}dx + \int_0^\infty x^{r+7} e^{-\theta x}dx\right] \\
&= \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} \left[\frac{r!}{\theta^{(r+1)}} + \frac{(r+2)!}{\theta^{(r+3)}} + \frac{(r+7)!}{\theta^{(r+8)}}\right] \\
&= \left[\frac{\theta^7 r! + \theta^5 (r+2)! + (r+7)!}{\theta^{(7+r)} + 2\theta^{(5+r)} + 5040\theta^r}\right]
\end{aligned}$$

By substituting $r = 1, 2, 3, 4$ in (2), we get the mean (μ) or expected value of Arz distribution and $E(X^2)$, $E(X^3)$ and $E(X^4)$ as follows

$$\begin{aligned}\mu = E(X) &= \frac{\theta^7 + 6\theta^5 + 40320}{\theta^8 + 2\theta^6 + 5040\theta}, \\ E(X^2) &= \frac{2\theta^7 + 24\theta^5 + 362880}{\theta^9 + 2\theta^7 + 5040\theta^2}, \\ E(X^3) &= \frac{6\theta^7 + 120\theta^5 + 3628800}{\theta^{10} + 2\theta^8 + 5040\theta^3}, \\ E(X^4) &= \frac{24\theta^7 + 720\theta^5 + 39916800}{\theta^{11} + 2\theta^9 + 5040\theta^4}.\end{aligned} \tag{6}$$

Based on these moments, the variance (σ^2), coefficient of variation (CV), index of dispersion (DIS), skewness (SK) and kurtosis (KU) of Arz distribution are, respectively, defined as:

$$\sigma^2 = E(X^2) - \mu^2 = \frac{\theta^{14} + 16\theta^{12} + 12\theta^{10} + 292320\theta^7 + 362880\theta^5 + 203212800}{\left(\theta^8 + 2\theta^6 + 5040\theta\right)^2},$$

$$CV = \frac{\sigma}{\mu} = \frac{\left(\theta^{14} + 16\theta^{12} + 12\theta^{10} + 292320\theta^7 + 362880\theta^5 + 203212800\right)^{\frac{1}{2}}}{\theta^7 + 6\theta^5 + 40320},$$

$$DIS = \frac{\sigma^2}{\mu} = \frac{\theta^{14} + 16\theta^{12} + 12\theta^{10} + 292320\theta^7 + 362880\theta^5 + 203212800}{\theta\left(\theta^{14} + 8\theta^{12} + 12\theta^{10} + 45360\theta^7 + 110880\theta^5 + 203212800\right)},$$

$$\begin{aligned}SK &= \frac{E(X^3) - 3\mu E(X^2) + 2\mu^3}{\sigma^3} \\ &= \frac{\begin{bmatrix} 2\theta^{21} + 60\theta^{19} + 72\theta^{17} + 48\theta^{15} + 25704000\theta^{14} + 61084480\theta^{12} \\ + 45964480\theta^{10} - 41150592000\theta^7 - 6096384000\theta^5 \\ + 2048385024000 \end{bmatrix}}{\left(\theta^{14} + 16\theta^{12} + 12\theta^{10} + 292320\theta^7 + 362880\theta^5 + 203212800\right)^{\frac{3}{2}}},\end{aligned}$$

$$\begin{aligned}KU &= \frac{E(X^4) - 4\mu E(X^3) + 6\mu^2 E(X^2) - 3\mu^4}{\sigma^4} \\ &= \frac{3\begin{bmatrix} 3\theta^{28} + 128\theta^{26} + 408\theta^{24} + 576\theta^{22} + 90921600\theta^{21} + 240\theta^{20} \\ + 397555520\theta^{19} + 628185600\theta^{17} + 322560000\theta^{15} + 150377472000\theta^{14} \\ + 922586112000\theta^{12} + 772208640000\theta^{10} + 1628466094080000\theta^7 \\ + 1741127270400000\theta^5 + 51619302604800000 \end{bmatrix}}{\left(\theta^{14} + 16\theta^{12} + 12\theta^{10} + 292320\theta^7 + 362880\theta^5 + 203212800\right)^2}.\end{aligned}$$

Table 2: The mean, variance, coefficient of variation, skewness, kurtosis and index of dispersion for Arz distribution with different values of the parameter θ.

θ	μ	σ^2	CV	SK	KU	DIS
0.25	32.000	128.001	0.35355	0.7071	3.74999	4.000
0.50	16.000	32.001	0.35356	0.7070	3.74998	2.000
0.75	10.666	14.228	0.35361	0.7062	3.74996	1.334
1.00	7.997	8.016	0.35406	0.7021	3.75027	1.002
1.25	6.390	5.161	0.35552	0.6870	3.7522	0.808
1.50	5.308	3.644	0.35968	0.6446	3.7575	0.687
1.75	4.514	2.785	0.36973	0.5506	3.7588	0.6170
2.00	3.884	2.305	0.39087	0.3926	3.7121	0.593
2.25	3.342	2.063	0.42979	0.2107	3.5237	0.6172
2.50	2.841	1.960	0.49276	0.0988	3.1476	0.690
2.75	2.359	1.892	0.58305	0.1299	2.7118	0.802
3.00	1.900	1.764	0.69910	0.3035	2.4405	0.929
3.25	1.486	1.535	0.83403	0.5778	2.4997	1.033
3.50	1.138	1.235	0.97619	0.9128	2.9701	1.085
3.75	0.868	0.930	1.11087	1.2819	3.8903	1.072
4.00	0.669	0.671	1.22359	1.6665	5.2767	1.002
4.25	0.528	0.473	1.30409	2.0497	7.1142	0.897
4.50	0.428	0.333	1.34899	2.4127	9.3321	0.779
4.75	0.357	0.237	1.36170	2.7354	11.7886	0.663
5.00	0.307	0.172	1.34996	2.9999	14.2765	0.559

These measurements are computed numerically for Arz distribution with different values of the parameter θ and the results are given in Table 2. It can be observed from Table 2 that as the value of θ increases, the values of mean and variance decrease. The skewness values are positive which confirm that Arz distribution is right skewed as shown in Figure 1. The index of dispersion values suggest that Arz distribution is over-dispersed when $\theta < 1.00$ and $3.25 < \theta < 4.00$ and under-dispersed when $1.00 < \theta < 3.25$ and $\theta > 4.00$.

5 Order Statistics

Let $X_{(1)} < X_{(2)} < ... < X_{(n)}$ be the order statistics of a random sample of size n from Arz distribution. Using (2), (3) and binomial series, we get the *pdf* of k^{th} order statistic as

$$\begin{aligned}
f_{(k)}(x) &= n\binom{n-1}{k-1} f(x)[F(x)]^{k-1}[1-F(x)]^{n-k} \\
&= n\binom{n-1}{k-1} f(x)\left[\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j(1-F(x))^j\right][1-F(x)]^{n-k} \\
&= n\binom{n-1}{k-1}\frac{\theta^8}{\theta^7+2\theta^5+5040}\left(x^7+x^2+1\right)e^{-\theta x}\sum_{j=0}^{k-1}(-1)^j e^{-\theta x(n+j-k)} \\
&\times \left[\frac{\begin{array}{l}\theta^7(x^7+x^2)+\theta^6(7x^6+2x)+42\theta^5x^5+210\theta^4x^4+840\theta^3x^3\\ +2520\theta^2x^2+5040\theta x\end{array}}{\theta^7+2\theta^5+5040}+1\right]^{n+j-k} \\
&= n\binom{n-1}{k-1}\theta^8\left(x^7+x^2+1\right)\sum_{j=0}^{k-1}\sum_{l=0}^{n+j-k}(-1)^j e^{-\theta x(n+j-k+1)} \\
&\times \frac{\left[\begin{array}{l}\theta^7(x^7+x^2)+\theta^6(7x^6+2x)+42\theta^5x^5+210\theta^4x^4+840\theta^3x^3\\ +2520\theta^2x^2+5040\theta x\end{array}\right]}{(\theta^7+2\theta^5+5040)^{l+1}}
\end{aligned}$$

6 Lorenz and Bonferroni Curves and Gini Index

The Lorenz curve, $L(q)$, Bonferroni curve, $B(q)$ and Gini index,(G) are useful tools that have applications in economics , reliability , insurance , demography , etc. The general forms are defined as follows

$$L(q) = \frac{1}{\mu}\int_0^p xf(x)dx, \quad B(q) = \frac{1}{q\mu}\int_0^p xf(x)dx, \quad G = 1-2\int_0^1 L(q)dq \quad (7)$$

where $p = F^{-1}(q); q \in (0,1]$ and $\mu = E(X)$.
The Lorenz and Bonferroni curves and Gini index for Arz distribution are defined, respectively, as follows

$$L(q) = 1 - \frac{\begin{bmatrix} \left(p^8 + p^3 + p\right)\theta^8 + \left(8p^7 + 3p^2 + 1\right)\theta^7 + \left(56p^6 + 6p\right)\theta^6 \\ + \left(336p^5 + 6\right)\theta^5 + 1680p^4\theta^4 + 6720p^3\theta^3 + 20160p^2\theta^2 \\ + 40320p\theta + 40320 \end{bmatrix} e^{-p\theta}}{\theta^7 + 6\theta^5 + 40320},$$

$$B(q) = \frac{1}{q} - \frac{\begin{bmatrix} \left(p^8 + p^3 + p\right)\theta^8 + \left(8p^7 + 3p^2 + 1\right)\theta^7 + \left(56p^6 + 6p\right)\theta^6 \\ + \left(336p^5 + 6\right)\theta^5 + 1680p^4\theta^4 + 6720p^3\theta^3 + 20160p^2\theta^2 \\ + 40320p\theta + 40320 \end{bmatrix} e^{-p\theta}}{q\left(\theta^7 + 6\theta^5 + 40320\right)},$$

$$G = \frac{2\begin{bmatrix} \left(p^8 + p^3 + p\right)\theta^8 + \left(8p^7 + 3p^2 + 1\right)\theta^7 + \left(56p^6 + 6p\right)\theta^6 \\ + \left(336p^5 + 6\right)\theta^5 + 1680p^4\theta^4 + 6720p^3\theta^3 + 20160p^2\theta^2 \\ + 40320p\theta + 40320 \end{bmatrix} e^{-p\theta}}{\theta^7 + 6\theta^5 + 40320} - 1,$$

7 Stochastic Ordering

Let U and W be two positive continuous random variables. We say that U is smaller than W in
1- Likelihood ratio order $(U \leq_{LR} W)$ if $\frac{f_U(x)}{f_W(x)}$ decreases in x.
2- Hazard rate order $(U \leq_{HR} W)$ if $h_U(x) \geq h_W(x)$ for all x.
3- Mean residual life order $(U \leq_{MRL} W)$ if $MRL_U(x) \leq MRL_W(x)$ for all x.
4- Stochastic order $(U \leq_{ST} W)$ if $F_U(x) \geq F_W(x)$ for all x.
Shaked and Shanthikumar (1994) showed that

$$\begin{array}{ccc} U \leq_{LR} W \Rightarrow & U \leq_{HR} W & \Rightarrow U \leq_{MRL} W \\ & \Downarrow & \\ & U \leq_{ST} W & \end{array} \tag{8}$$

The following theorem proves that Arz distribution satisfies these four stochastic ordering.

Theorem 3 *Let $U \sim Arz(\theta_1)$ and $W \sim Arz(\theta_2)$. If $\theta_1 > \theta_2$, then $U \leq_{LR} W$ and thus $U \leq_{HR} W, U \leq_{MRL} W$ and $U \leq_{ST} W$.*

Proof: Using (2), we have

$$\frac{f_U(x;\theta_1)}{f_W(x;\theta_2)} = \left[\frac{\theta_1^8\left(\theta_2^7 + 2\theta_2^5 + 5040\right)}{\theta_2^8\left(\theta_1^7 + 2\theta_1^5 + 5040\right)}\right] e^{-x(\theta_1 - \theta_2)}.$$

Therefore,

$$log\frac{f_U(x;\theta_1)}{f_W(x;\theta_2)} = log\left[\frac{\theta_1^8\left(\theta_2^7+2\theta_2^5+5040\right)}{\theta_2^8\left(\theta_1^7+2\theta_1^5+5040\right)}\right]-x(\theta_1-\theta_2),$$

and

$$\frac{\partial}{\partial x}\left[log\frac{f_U(x;\theta_1)}{f_W(x;\theta_2)}\right]=(\theta_2-\theta_1).$$

Thus, for $\theta_1 > \theta_2$, $\frac{\partial}{\partial x}\left[log\frac{f_U(x;\theta_1)}{f_W(x;\theta_2)}\right] < 0$. Therefore, $U \leq_{LR} W$ which implies that $U \leq_{HR} W$, $U \leq_{MRL} W$ and $U \leq_{ST} W$, based on (8).

8 Rényi Entropy and Mean Deviations

In this section, we study the Rényi entropy and mean deviations of Arz distribution.

8.1 Rényi entropy

Rényi entropy can be used as a measure of uncertainty which is defined by (Renyi, 1961) as

$$\Re(\kappa) = \frac{1}{1-\kappa}log\int_0^{\infty}\left(f(x)\right)^{\kappa}dx\,;\kappa>0,\kappa\neq 1. \tag{9}$$

Theorem 4 *The Rényi entropy of Arz distribution random variable X is given by*

$$\Re(\kappa)=\frac{1}{1-\kappa}log\left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040}\right)^{\kappa}\sum_{i=1}^{\kappa}\sum_{j=1}^{i}\binom{\kappa}{i}\binom{i}{j}\frac{(2i+5j)!}{(\kappa\theta)^{2i+5j+1}}\right]$$

Proof: Using (2), (9) and the binomial series, we have

$$\begin{aligned}
\Re(\kappa) &= \frac{1}{1-\kappa} log \int_0^\infty \left[\frac{\theta^8}{\theta^7+2\theta^5+5040} \left(1+x^2+x^7\right) e^{-\theta x} \right]^\kappa dx \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \int_0^\infty \left(1+x^2+x^7\right)^\kappa e^{-\theta\kappa x} dx \right] \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \int_0^\infty \left(1+x^2(1+x^5)\right)^\kappa e^{-\theta\kappa x} dx \right] \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \int_0^\infty \sum_{i=1}^{\kappa} \binom{\kappa}{i} (x^2(1+x^5))^i e^{-\theta\kappa x} dx \right] \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \int_0^\infty \sum_{i=1}^{\kappa} \binom{\kappa}{i} x^{2i} \sum_{j=1}^{i} \binom{i}{j} x^{5j} e^{-\theta\kappa x} dx \right] \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \sum_{i=1}^{\kappa} \sum_{j=1}^{i} \binom{\kappa}{i} \binom{i}{j} \int_0^\infty \int_0^\infty x^{2i+5j} e^{-\theta\kappa x} dx \right] \\
&= \frac{1}{1-\kappa} log \left[\left(\frac{\theta^8}{\theta^7+2\theta^5+5040} \right)^\kappa \sum_{i=1}^{\kappa} \sum_{j=1}^{i} \binom{\kappa}{i} \binom{i}{j} \frac{(2i+5j)!}{(\kappa\theta)^{2i+5j+1}} \right]
\end{aligned}$$

8.2 Mean Deviation about Mean and Median

Mean deviations about the median and mean are defined, respectively, as

$$Dev(\tilde{M}) = \int_0^\infty |x-\tilde{M}| f(x) dx = \mu - 2\int_0^{\tilde{M}} x f(x) dx \tag{10}$$

$$Dev(\mu) = \int_0^\infty |x-\mu| f(x) dx = 2\mu F(x) - 2\int_0^{\mu} x f(x) dx \tag{11}$$

where $\tilde{M}$ is population median.

Theorem 5 *The mean deviation about the median of Arz distribution is given by*

$$Dev(\tilde{M}) = \frac{2e^{-\tilde{M}\theta}}{\theta(\theta^7+2\theta^5+5040)} \begin{bmatrix} \left(\tilde{M}^8+\tilde{M}^3+\tilde{M}\right)\theta^8 + \left(8\tilde{M}^7+3\tilde{M}^2+1\right)\theta^7 \\ + \left(56\tilde{M}^6+6\tilde{M}\right)\theta^6 + \left(336\tilde{M}^5+6\right)\theta^5 \\ + 1680\tilde{M}^4\theta^4 + 6720\tilde{M}^3\theta^3 + 20160\tilde{M}^2\theta^2 \\ + 40320\tilde{M}\theta + 40320 \end{bmatrix} - \mu$$

Proof: By plugging (2) and (6) in (10), we have

$$
\begin{aligned}
Dev(\tilde{M}) &= \mu - 2\int_0^{\tilde{M}} x f(x)dx \\
&= \frac{\theta^7+6\theta^5+40320}{\theta^8+2\theta^6+5040\theta} - \frac{2\theta^8}{\theta^7+2\theta^5+5040}\int_0^{\tilde{M}} \left(x^8+x^3+x\right)e^{-\theta x}dx \\
&= \frac{\theta^7+6\theta^5+40320}{\theta^8+2\theta^6+5040\theta} - \frac{2(\theta^7+6\theta^5+40320)}{\theta^8+2\theta^6+5040\theta} \\
&+ \frac{2\begin{bmatrix} \left(\tilde{M}^8+\tilde{M}^3+\tilde{M}\right)\theta^8 + \left(8\tilde{M}^7+3\tilde{M}^2+1\right)\theta^7 \\ +\left(56\tilde{M}^6+6\tilde{M}\right)\theta^6 + \left(336\tilde{M}^5+6\right)\theta^5 + 1680\tilde{M}^4\theta^4 \\ +6720\tilde{M}^3\theta^3 + 20160\tilde{M}^2\theta^2 + 40320\tilde{M}\theta + 40320 \end{bmatrix} e^{-\tilde{M}\theta}}{\theta^8+2\theta^6+5040\theta}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2\begin{bmatrix} \left(\tilde{M}^8+\tilde{M}^3+\tilde{M}\right)\theta^8 + \left(8\tilde{M}^7+3\tilde{M}^2+1\right)\theta^7 \\ +\left(56\tilde{M}^6+6\tilde{M}\right)\theta^6 + \left(336\tilde{M}^5+6\right)\theta^5 + 1680\tilde{M}^4\theta^4 \\ +6720\tilde{M}^3\theta^3 + 20160\tilde{M}^2\theta^2 + 40320\tilde{M}\theta + 40320 \end{bmatrix} e^{-\tilde{M}\theta} - (\theta^7+6\theta^5+40320)}{\theta^8+2\theta^6+5040\theta} \\
&= \frac{2e^{-\tilde{M}\theta}}{\theta(\theta^7+2\theta^5+5040)}\begin{bmatrix} \left(\tilde{M}^8+\tilde{M}^3+\tilde{M}\right)\theta^8 + \left(8\tilde{M}^7+3\tilde{M}^2+1\right)\theta^7 \\ +\left(56\tilde{M}^6+6\tilde{M}\right)\theta^6 + \left(336\tilde{M}^5+6\right)\theta^5 + 1680\tilde{M}^4\theta^4 \\ +6720\tilde{M}^3\theta^3 + 20160\tilde{M}^2\theta^2 + 40320\tilde{M}\theta + 40320 \end{bmatrix} - \mu
\end{aligned}
$$

Theorem 6 *The mean deviation about the mean of Arz distribution is given by*

$$
Dev(\mu) = \frac{2e^{-\theta\mu}}{\theta^8+2\theta^6+5040\theta}\begin{bmatrix} (\mu^8+\mu^3+\mu)\theta^8 + (7\mu^7+2\mu^2)\theta^7 + (49\mu^6+4\mu)\theta^6 \\ +(294\mu^5+4)\theta^5 + 1470\mu^4\theta^4 + 5880\mu^3\theta^3 \\ +17640\mu^2\theta^2 + 35280\mu\theta + 35280 \end{bmatrix}
$$

Proof: By plugging (2) and (6) in (11), we have

$$\begin{aligned}
Dev(\mu) &= 2\mu F(\mu) - 2\int_0^{\mu} x f(x) dx \\
&= 2\mu - 2\mu \left[\frac{\theta^7(\mu^7+\mu^2)+\theta^6(7\mu^6+2\mu)+42\theta^5\mu^5 + 210\theta^4\mu^4+840\theta^3\mu^3+2520\theta^2\mu^2+5040\theta\mu}{\theta^7+2\theta^5+5040} + 1 \right] e^{-\theta\mu} \\
&- \frac{2\theta^8}{\theta^7+2\theta^5+5040}\int_0^{\mu} \left(x^8+x^3+x\right) e^{-\theta x} dx \\
&= \frac{2e^{-\theta\mu}}{\theta^8+2\theta^6+5040\theta} \left[\begin{array}{l} \left(\mu^8+\mu^3+\mu\right)\theta^8 + \left(8\mu^7+3\mu^2+1\right)\theta^7 + \left(56\mu^6+6\mu\right)\theta^6 \\ + \left(336\mu^5+6\right)\theta^5 + 1680\mu^4\theta^4 + 6720\mu^3\theta^3 + 20160\mu^2\theta^2 \\ + 40320\mu\theta + 40320 \end{array} \right] \\
&- \left(\theta^7+2\theta^5+5040\right) \left[\frac{\theta^7(\mu^7+\mu^2)+\theta^6(7\mu^6+2\mu)+42\theta^5\mu^5+210\theta^4\mu^4 + 840\theta^3\mu^3+2520\theta^2\mu^2+5040\theta\mu}{\theta^7+2\theta^5+5040} + 1 \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{2e^{-\theta\mu}}{\theta^8+2\theta^6+5040\theta} \left[\begin{array}{l} \left(\mu^8+\mu^3+\mu\right)\theta^8 + \left(8\mu^7+3\mu^2+1\right)\theta^7 + \left(56\mu^6+6\mu\right)\theta^6 \\ + \left(336\mu^5+6\right)\theta^5 + 1680\mu^4\theta^4 + 6720\mu^3\theta^3 + 20160\mu^2\theta^2 \\ + 40320\mu\theta + 40320 \end{array} \right] \\
&- \left[\begin{array}{l} \theta^7(\mu^7+\mu^2+1)+\theta^6(7\mu^6+2\mu)+\theta^5(42\mu^5+2)+210\theta^4\mu^4+840\theta^3\mu^3 \\ + 2520\theta^2\mu^2+5040\theta\mu+5040 \end{array} \right] \\
&= \frac{2e^{-\theta\mu}}{\theta^8+2\theta^6+5040\theta} \left[\begin{array}{l} \left(\mu^8+\mu^3+\mu\right)\theta^8 + \left(7\mu^7+2\mu^2\right)\theta^7 + \left(49\mu^6+4\mu\right)\theta^6 \\ + \left(294\mu^5+4\right)\theta^5 + 1470\mu^4\theta^4 + 58800\mu^3\theta^3 \\ + 17640\mu^2\theta^2 + 35280\mu\theta + 35280 \end{array} \right]
\end{aligned}$$

9 Maximum Likelihood Estimation

Assume that $X_1, X_2, ..., X_n$ is a random sample from Arz distribution. Based on the pdf in (2), the likelihood function of Arz distribution can be defined as

$$\begin{aligned}
L(\theta|x_1, x_2, ..., x_n) = \prod_{i=1}^{n} f(x_i|\theta) &= \prod_{i=1}^{n} \frac{\theta^8}{\theta^7+2\theta^5+5040}\left(1+x_i^2+x_i^7\right) e^{-\theta x_i} \\
&= \left(\frac{\theta^8}{\theta^7+2\theta^5+5040}\right)^n \prod_{i=1}^{n}\left(1+x_i^2+x_i^7\right) e^{-\theta\sum_{i=1}^{n} x_i}
\end{aligned}$$

The log likelihood function is

$$\begin{aligned} L^* &= \ln L(\theta|x_1, x_2, ..., x_n) \\ &= n \ln\left(\frac{\theta^8}{\theta^7 + 2\theta^5 + 5040}\right) + \sum_{i=1}^{n} \ln\left(x_i^7 + x_i^2 + 1\right) - \theta \sum_{i=1}^{n} x_i \\ &= 8n\ln(\theta) - n\ln(\theta^7 + 2\theta^5 + 5040) + \sum_{i=1}^{n} \ln(x_i^7 + x_i^2 + 1) - n\theta\bar{x} \end{aligned}$$

where, $\bar{x} = \sum_{i=1}^{n} x_i/n$. Therefore,

$$\frac{\partial L^*}{\partial \theta} = \frac{8n}{\theta} - \frac{n(7\theta^6 + 10\theta^4)}{\theta^7 + 2\theta^5 + 5040} - n\bar{x}$$

The maximum likelihood estimate of θ is the solution of $\frac{\partial L^*}{\partial \theta} = 0$ which can be obtained by solving the following polynomial equation

$$\bar{x}\theta^8 - \theta^7 + 2\bar{x}\theta^6 - 6\theta^5 + 5040\bar{x}\theta - 40320 = 0$$

10 Application to COVID-19 Data

In this section, we present an analysis of covid- 19 data set, which illustrate the goodness of fit of the proposed Arz distribution, in comparison with some existing distributions. The data set contains 29 observations represent the number of deaths in Jordan due to COVID-19 during the period from (1/2/2021) to (1/3/2021). These observations are 10, 8, 10, 10, 8, 7, 10, 6, 10, 16, 10, 12, 11,11, 18, 18, 12, 9, 16, 15, 11, 16, 19, 22, 16, 23, 25, 26. This data is available at the World Health Organization (WHO) website, (https://covid19.who.int/). The goodness of fit for Arz distribution is compared with the following one parameter distributions:

- Lindley distribution: $f(x) = \frac{\theta^2}{\theta+1}(x+1)e^{-x\theta}; x > 0; \theta > 0.$
- Exponential distribution: $f(x) = \theta e^{-x\theta}; x > 0; \theta > 0.$
- Shanker distribution (Shanker, 2015b): $f(x) = \frac{\theta^2}{\theta^2+1}(\theta + x)e^{-x\theta}; x > 0; \theta > 0.$
- Ishita distribution (Shanker and Shukla, 2017): $f(x) = \frac{\theta^3}{\theta^3+2}(x^2 + \theta)e^{-x\theta}; x > 0; \theta > 0.$
- Aradhana distribution (Shanker, 2016): $f(x) = \frac{\theta^3}{\theta^2+2\theta+2}(1 + x)^2 e^{-x\theta}; x > 0; \theta > 0.$
- Akash distribution (Shanker, 2015a): $f(x) = \frac{\theta^3}{\theta^2+2}(1 + x^2)e^{-x\theta}; x > 0; \theta > 0.$

Table 3 shows the values of $-2log$ likelihood $(-2logL)$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogorov Smirnov (KS) statistic and its $p-value$ for all fitted distributions. It can be seen that the Arz distribution has the lowest values of the statistic $(-2LogL, AIC, BIC, KS)$ and the highest $p-value$. Thus, we can conclude that Arz distribution is more appropriate for modeling this data set, comparing with the other considered models.

Table 3: $-2logL, AIC, BIC, KS$ statistic and its $p-value$ for all fitted distributions.

Distribution	$-2logL$	***AIC***	***BIC***	***KS***	***p-value***
Lindley	188.2067	190.2067	191.5389	0.2714	0.0323
Exponential	202.7782	204.7782	206.1104	0.37016	0.0009
Shanker	185.8036	187.8036	189.1358	0.2542	0.0537
Ishita	177.5740	179.574	180.9062	0.1939	0.2434
Aradhana	180.1030	182.103	183.4352	0.2088	0.174
Akash	178.2318	180.2318	181.564	0.19688	0.2278
Arz	**169.8472**	**171.8472**	**173.1795**	**0.18734**	**0.2795**

The MLEs, standard errors, and confidence intervals (CI) for the parameters of fitted distributions are obtained and given in Table 4.

Table 4: The MLEs of the parameters of Arz and other fitted distributions and their confidence intervals.

Distribution	***MLE***	***Standard Error***	***CI lower limit***	***CI upper limit***
Lindley	0.1367	0.0183	0.1008	0.1725
Exponential	0.0728	0.0138	0.0458	0.0998
Shanker	0.1434	0.0190	0.1062	.1806
Ishita	0.2172	0.0236	0.1709	0.2635
Aradhana	0.2037	0.0222	0.1602	0.2472
Akash	0.2150	0.0233	0.1693	0.2607
Arz	**0.5818**	**0.0389**	**0.5056**	**0.6580**

11 Conclusion

This paper introduced a new probability distribution with one parameter. It is called Arz distribution which is a mixture of $Exp(\theta)$, $Gamma(3, \theta)$ and $Gamma(8, \theta)$. We studied many properties of the proposed distribution such as the r^th moment, moment generating function, variance, mean, skewness, coefficient of variation, kurtosis, index of dispersion, survival function, reversed hazard function, hazard function, odds function, cumulative hazard function, mean residual life function, order statistics, Lorenz and Bonferroni curves, Gini index, stochastic ordering, Rényi entropy, mean deviations, and maximum likelihood estimate of the distribution parameter. We found that Arz distribution is skewed to the right and the values of cdf, odds function, hazard function, and cumulative hazard function increase as the parameter value increases, while the values of mean, variance, reversed hazard function, mean residual life function, and survival function decrease as the parameter value increases. It was observed with one parameter, the hazard rate function of Arz distribution took the shape of a bathtub. The Arz distribution is applied to real data set related to COVID-19. It is found that Arz distribution is more appropriate for modeling this data than the other compared distributions.

References

Al-Omari, A. and Gharaibeh, M. (2018). Topp-Leone Mukherjee-Islam distribution: properties and applications. *Pakistan Journal of Statistics*, 34(6): 479–494.

Al-Omari, A. and Shraa, D. (2019). Darna distribution: Properties and application. *Electronic Journal of Applied Statistical Analysis*, 12(2): 520–541.

Al-Ta'ani, O. and Gharaibeh, M. M. (2023). Ola distribution: A new one parameter model with applications to engineering and covid-19 data. *Applied Mathematics & Information Sciences*, 17(2): 243–252.

Alzoubi, L., Gharaibeh, M., and Alzghool, R. (2022a). Transmuted Shanker distribution: properties and applications. 48:195–212.

Alzoubi, L. M. A., Gharaibeh, M. M., and Alkhazaalh, A. M. (2022b). Loai distribution : Properties , parameters estimation and application to covid - 19 real data. *Mathematical Statistician and Engineering Applications*, 71: 1231–1255.

Alzoubi, L. M. A., Gharaibeh, M. M., Alkhazaalh, A. M., and Benrabia, M. (2022c). Estimation of Sameera distribution parameters with applications to real data. *Applied Mathematics & Information Sciences*, 16: 1057–1071.

Gharaibeh, M. M. (2020). Transmuted Aradhana distribution: Properties and application. *Jordan Journal of Mathematics and Statistics (JJMS)*, 13(2): 287–304.

Gharaibeh, M. M. (2021). Gharaibeh distribution and its applications. *Journal of Statistics Applications & Probability*, 10(2): 441–452.

Gharaibeh, M. M. and Al-Omari, A. I. (2019). Transmuted Ishita distribution and its applications. *Journal of Statistics Applications & Probability*, 8(2): 67–81.

Gharaibeh, M. M. and Sahtout, M. O. (2022). Karam distribution: A new lifetime distribution with real data application. *Journal of Applied Probability & Statistics*, 17(1): 1–16.

Mdlongwa, P., O.-B. A. A. and Huang, S. (2017). The Burr XII modified weibull distribution: model, properties and applications. *Electronic Journal of Applied Statistical Analysis*, 10(1): 118–145.

Renyi, A. (1961). On measures of entropy and information. *In Proceedings of the*

Fourth Berkeley Symposium on Mathematical Statistics and Probability, 1: 547–561.

Shaked, M. and Shanthikumar, J. (1994). *Stochastic Orders and Their Applications.* Academic Press, New York.

Shanker, R. (2015a). Akash distribution and its applications. *International Journal of Probability and Statistics*, 4(3): 65–75.

Shanker, R. (2015b). Shanker distribution and its applications. *International Journal of Statistics and Applications*, 5(6): 338–348.

Shanker, R. (2016). Aradhana distribution and its applications. *International Journal of Statistics and Applications*, 6(1): 23–34.

Shanker, R. (2017). Akshaya distribution and its application. *American Journal of Mathematics and Statistics*, 7(2): 51–59.

Shanker, R. and Shukla, K. (2017). Ishita distribution and its applications. *Biometrics & Biostatistics International Journal*, 5(2): 1 – 9.

Usman, R. M. and Ahsan ul Haq, M. (2020). The Marshall-Olkin extended inverted Kumaraswamy distribution: Theory and applications. *Journal of King Saud University - Science*, 32(1):356–365.