Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v18n2p402

Inference and Application of Modified Lehmann Type-II Fréchet Distribution using Progressively Censored Data

By Prakash et al.

15 October 2025

Electronic Journal of Applied Statistical Analysis
Vol. 18, Issue 2, October 2025, 402-430
DOI: 10.1285/i20705948v18n2p402

Inference and Application of Modified Lehmann Type-II Fréchet Distribution using Progressively Censored Data

Aman Prakash[a], John Abonongo[*b], and Raj Kamal Maurya[a]

[a]*Department of Mathematics, Sardar Vallabhbhai National Institute of Technology, Surat, India*
[b]*Department of Statistics and Actuarial Science, C. K. Tedam University of Technology and Applied Sciences, Navrongo, Ghana*

15 October 2025

In this article, we introduce a new three-parameter Fréchet distribution via the modified Lehmann Type-II class of distributions and investigate its properties, inferential methods, and real-world application. Fundamental distributional properties such as the quantile function, moments, moment generating function, entropy, and order statistics have been discussed. Inferential results have been obtained within the classical and Bayesian frameworks, utilizing a progressive censoring scheme. In the classical estimation framework, maximum likelihood estimation and maximum product spacing estimation are considered to obtain the estimates using the Newton-Raphson method. In addition, approximate confidence intervals have been derived using maximum likelihood estimation estimates. Meanwhile, we have used informative and non-informative prior via likelihood and product spacing functions to find the Bayes estimates in the Bayesian framework. Since the posterior distributions cannot be expressed in closed form, we employ a combination of Gibbs sampling and the Metropolis-Hastings algorithm to obtain the Bayes estimates. Furthermore, credible intervals are constructed using the Bayes estimates under informative and non-informative Prior. A comprehensive simulation study is carried out to assess the performance of the proposed estimation techniques. To demonstrate the practical utility of the proposed model, a real-life dataset is analyzed, showcasing the effectiveness of the proposed methodologies.

*Corresponding authors: jabonongo@cktutas.edu.gh

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

keywords: Fréchet distribution, Progressive censoring scheme, Maximum likelihood estimation, Maximum product spacing estimation, Bayesian estimation.

1 Introduction

Developing new probability distributions allows researchers to explore real-life problems to adequately fit asymmetric and complex random events. This has led to the development of several models in the literature. The most straightforward and handy models introduced in research are the Lehmann Type-I (L-I) and Lehmann Type-II (L-II) by Lehmann (1953). The L-I model is often discussed in favour of power function (PF) distribution. The PF is defined as the simple exponentiation of any baseline model with cumulative distribution function (CDF) defined as

$$F(x;\theta,\boldsymbol{\Delta}) = [G(x,\boldsymbol{\Delta})]^{\theta},\ x>0,$$

where $\theta>0$ is a shape parameter, $\boldsymbol{\Delta}$ is a vector of the parameter(s), and $G(x;\boldsymbol{\Delta})$ is the baseline distribution. Gupta et al. (1998) practised the L-I class on exponential distribution. The simplicity and usefulness of the PF distribution have attracted researchers to explore its further applications, extensions, and generalizations in different areas of research. Cordeiro and De Castro (2011) proposed a dual transformation of the L-II with CDF given by

$$F(x;\theta,\boldsymbol{\Delta}) = 1-[1-G(x;\boldsymbol{\Delta})]^{\theta},\ x>0,$$

where θ, $\boldsymbol{\Delta}$ and $G(x;\boldsymbol{\Delta})$ are as defined earlier. The merit of the L-II distribution is its closed-form features, which enable one to derive and study its properties easily. Both models (L-I and L-II) have been extensively explored in the literature. Among them is Dallas (1976), who established a relationship between PF and Pareto distributions. Meniconi and Barry (1996) found the PF as the best-fitted model for electronic components data independence of record values-based characterization was discussed by Chang (2007). Ahsanullah et al. (2013) also discussed the characterization of the PF distribution based on lower record values, and Naveed-Shahzad et al. (2015) derived the moments by using different methods.

Moreover, L-II G family-based development has gained much attention recently. Arshad et al. (2021) introduced a bathtub-shaped failure rate model and explored its application using engineering data. Tomazella et al. (2022) discussed several mathematical properties of L-II Fréchet distribution and explored its application to aircraft maintenance data, and Awodutire et al. (2020) studied some statistical measures of the generalized half-logistic and explored its application to sports data. Also, Badmus et al. (2014) proposed the weighted Weibull via L-II and applied it to textile engineering data, Ogunde et al. (2020) extended the Gumbel Type-II via exponentiated L-II G class and applied it to biological data, and Lehmann Type-II Lomax distribution has been proposed and studied by Isa et al. (2023). Recently, a new extension of the L-II class has been proposed by Balogun et al. (2021), known as the modified Lehmann Type-II (ML-II) G class of distributions. The CDF is given by

$$F(x;\alpha,\gamma,\boldsymbol{\Delta}) = 1 - \left[\frac{1-G(x;\boldsymbol{\Delta})}{1-\gamma G(x;\boldsymbol{\Delta})}\right]^{\alpha}, \; x \in \mathbb{R}, \tag{1}$$

where $G(x;\boldsymbol{\Delta}) \in (0,1)$ is the CDF of any arbitrary baseline model, $\boldsymbol{\Delta}$ is a vector of parameter(s), dependent on $(m \times 1)$ with $-\infty < \gamma < 1$, and $\alpha > 0$ are the scale and shape parameters, respectively. They proposed the modified Lehmann Type-II-exponential (ML-II-Exp) distribution as a special case. Elshahhat et al. (2022) applied the Type-II Lehmann Fréchet via progressive Type-II censoring to survival data. Nevertheless, this work is motivated by the flexibility of the Fréchet distribution in the literature. The Fréchet distribution is one of the commonly used distributions in extreme value theory. It can model different failure rates compared to other known classical distributions. Some of its applications can be found in fields like physics, engineering, and biology, among others. Extensions of the Fréchet distribution can be found in Barreto-Souza et al. (2011), Chakraborty et al. (2019), Salah et al. (2020), Al-Babtain et al. (2020). Other extensions have been the three-parameter Kumaraswamy-Fréchet distribution (KFD) by Shahbaz et al. (2012) and Yousof et al. (2018). Tomazella et al. (2022) showed that the KFD has a simple structure, but its parameters are non-identifiable. Therefore, they proposed the Type-II Lehmann Fréchet distribution (LDF-TII). Most of the above studies have been done on the properties and applications of the proposed distribution. Also, the parameters have been estimated in the above work without using a censoring scheme. So, in this study, the main goal is to propose an extension of the one parameter Fréchet via the ML-II class of distribution and perform inference and applications under the progressive Type-II censoring scheme. To the best of our knowledge, this distribution has not been introduced in the literature and neither has it been considered under progressive Type-II censoring. The purpose of a life testing experiment in reliability is to assess a product's durability by subjecting it to operational conditions until it fails. In life testing experiments, the goal is to obtain data on the time to failure to estimate the reliability of a product and predict its performance over time. Performing a life-testing experiment is mainly dependent on time and cost. Assume a life test is conducted to monitor items until they fail. These items could be any life-testing experiments that will be observed until their failure. However, we recognize that the lifetime of items may not always be precisely recorded. Due to time and cost constraints, it is not feasible to continue the experiment until all items have failed. As a result, censored data is helpful for life-testing experiments. Censored data consists of observations that are only partially observed. A Type-I censoring scheme and a Type-II censoring scheme are the two most popular censoring schemes. In the Type-I censoring scheme, failures occur randomly, but the experimental duration is pre-fixed. In contrast, the Type-II censoring scheme observes a fixed number of failures with varying experimental durations. The main disadvantage of these schemes is that it is useless if the experimenter wishes to remove units at any other point than the endpoint during the experiment. Some testing units should be eliminated early to utilize the surviving units for other experiments (see, Cohen (1963)). Because of these limitations, it is impossible to prevent survival units from losing contact with experimental units or breaking unexpectedly at places other than the endpoint. To address these issues, researchers have proposed progressive censoring schemes (PCS), which are more flexible and allow experimenters to remove live units during the experiment. A more detailed discussion of these censoring schemes can be found in Balakrishnan and Aggarwala (2000). In this study, we focus on a progressive Type-II censoring scheme. It is described as follows: suppose a test starts with n identical testing units, and $m(m < n)$ is a prefixed integer. When the first failure $x_{1:m:n}$ is observed, R_1 units are randomly removed from the remaining $n-1$ surviving units. When $x_{2:m:n}$ of the second failure, R_2 units are randomly removed from the remaining $n - R_1 - 2$ surviving units. It is continued in this fashion and is terminated when m^{th} failure occurs at time $x_{m:m:n}$, and $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are removed from the test. $R = (R_1, R_2, \cdots, R_m)$ is pre-fixed scheme. While a number of studies have been done by many researchers using PCS, one may refer to Maurya et al. (2019), Mahto et al. (2020) and Prakash et al. (2024) for more details. The rest of the paper is organized as follows: In section 2, we proposed a three-parameter Modified Lehmann Type-II Fréchet distribution by extending the one parameter Fréchet distribution. The statistical properties such as quantile function, moments and moment generating function, entropy and order statistics of the proposed distribution have been discussed in Section 3. Further, we have used classical and Bayesian methods for inference purposes. In section 4, we have discussed the maximum likelihood estimation (MLE) and maximum product spacing estimation (MPSE) for parameter estimation, and by using MLE estimates, we constructed the approximate confidence intervals for the model parameters. Whereas in section 5, Bayesian estimates have been obtained by using informative and non-informative prior (INIP) via the likelihood function as well as the product spacing function. Furthermore, to provide the application of the proposed methodologies in real life, we have taken real data on the mortality rate of Japanese children under five years of age and discussed in section 7. From the simulation results, it has been observed that the proposed methods provide satisfactory results for the real data. Lastly, the concluding remarks of the study have been pointed out in section 8.

2 Modified Lehmann Type-II Fréchet Distribution

In this section, we extend the one parameter Fréchet distribution by the idea of Balogun et al. (2021). Considering the one parameter Fréchet distribution with CDF and probability density function (PDF) defined as $G(x;\beta) = e^{-x^{-\beta}}$ and $g(x;\beta) = \beta x^{-(\beta+1)} e^{-x^{-\beta}}$ for $\beta > 0$ respectively as the baseline distribution in equation (1), we obtain the proposed distribution known as the Modified Lehmann Type-II Fréchet (MLIIFr) distribution. The CDF is given by

$$F(x;\alpha,\beta,\gamma) = 1 - \left[\frac{1 - e^{-x^{-\beta}}}{1 - \gamma e^{-x^{-\beta}}}\right]^{\alpha}, \; x > 0, \; \alpha > 0, \; \beta > 0, \; -\infty < \gamma < 1, \tag{2}$$

where γ is a scale parameter, and α and β are shape parameters. The PDF is given by

$$f(x;\alpha,\beta,\gamma) = \alpha\beta(1-\gamma)x^{-(\beta+1)}e^{-x^{-\beta}}\left(1-e^{-x^{-\beta}}\right)^{\alpha-1}\left(1-\gamma e^{-x^{-\beta}}\right)^{-(1+\alpha)}, \; x>0. \quad (3)$$

The corresponding hazard rate function is given by

$$h(t;\alpha,\beta,\gamma) = \alpha\beta(1-\gamma)t^{-(\beta+1)}e^{-t^{-\beta}}\left(1-e^{-t^{-\beta}}\right)^{-1}\left(1-\gamma e^{-t^{-\beta}}\right)^{-1}, \; t>0. \quad (4)$$

The plot presented in Figure (1) indicates that the MLIIFr distribution exhibits heavy tails for all parameter combinations. It is a highly flexible and adaptable model suitable for a wide range of real-world data, especially in scenarios involving skewed distributions and heavy tails. Its ability to model different tail behaviors and peak shapes makes it valuable for extreme value modelling and reliability analysis. Also, from the hazard rate plot of the MLIIFr distribution it is evident of s adaptability to different real-world situations. It can model scenarios with increasing, decreasing, or non-monotonic hazard rates, making it suitable for various reliability, survival analysis, and risk management applications. This versatility is crucial for accurately capturing the behavior of systems over time and predicting their reliability and failure patterns effectively. It is appropriate

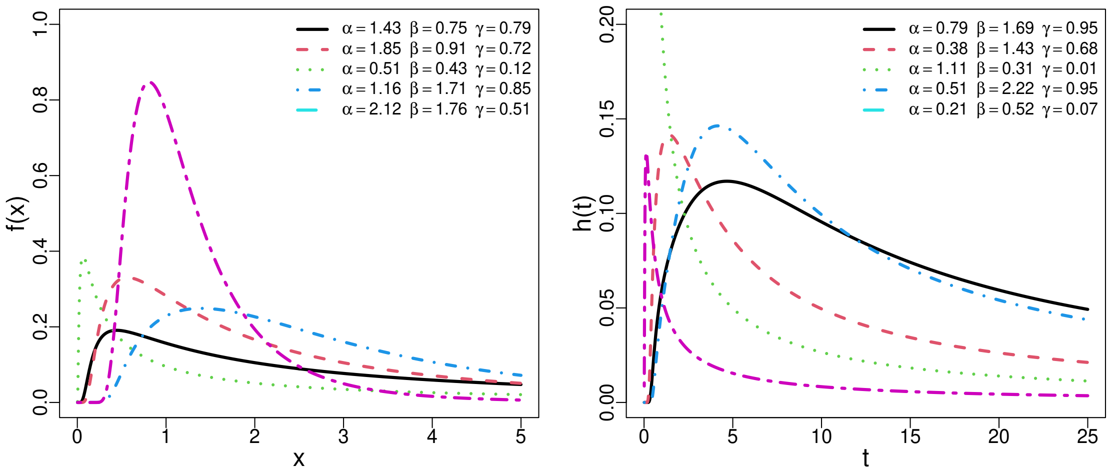

Figure 1: Plots of the PDF $f(x)$ and hazard rate $h(t)$ functions of the MLIIFr distribution

to obtain the linear representation of the PDF of the proposed model before going ahead with deriving the statistical properties. Therefore, making use of the following generalized binomial expansion

$$(1-z)^k = \sum_{i=0}^{\infty}(-1)^i\binom{k}{i}z^i, \; |z|<1 \; \text{ and } \; (1-z)^{-k} = \sum_{j=0}^{\infty}\binom{k+j-1}{j}z^j, \; |z|<1. \quad (5)$$

We have

$$\left(1-e^{-x^{-\beta}}\right)^{\alpha-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\left(e^{-x^{-\beta}}\right)^i \quad (6)$$

and

$$\left(1-\gamma e^{-x^{-\beta}}\right)^{-(1+\alpha)}=\sum_{j=0}^{\infty}\binom{j-\alpha-2}{j}\left(\gamma e^{-x^{-\beta}}\right)^{j}. \tag{7}$$

Substituting equation (6) and (7) into equation (3) and after some simplifications, we obtain the linear representation of the MLIIFr distribution as

$$f(x;\alpha,\beta,\gamma)=\alpha\beta(1-\gamma)\sum_{i,j=0}^{\infty}\Psi_{ij}x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{i+j+1}, \tag{8}$$

where $\Psi_{ij}=(-1)^i\gamma^j\binom{\alpha-1}{i}\binom{j-\alpha-2}{j}$.

3 Statistical Properties

In this section, statistical properties of the MLIIFr distribution, including the quantile function, moments, moment generating function, Renyi entropy, and order statistics, have been discussed.

3.1 Quantile function

The quantile function is important in describing the random variable of a distribution. It helps in generating random samples which are useful in simulations. It can also compute shape measures such as skewness and kurtosis. The quantile function of the MLIIFr distribution for $u\in(0,1)$ is given by

$$X_u=\left[-\ln\left(\frac{(1-u)^{1/\alpha}-1}{\gamma(1-u)^{1/\alpha}-1}\right)\right]^{-1/\beta}.$$

3.2 Moments

The moments of distribution are important in estimating measures of variation like the variance, standard deviation, coefficient of variation, mean deviation, median deviation, kurtosis, skewness and so on.
The k^{th} non-central moment by definition is given as

$$\mu_k^{'}=\int_0^{\infty}x^kf(x)dx. \tag{9}$$

Substituting equation (8) into equation (9), we have

$$\mu_k^{'}=\alpha\beta(1-\gamma)\sum_{i.,j=0}^{\infty}\Psi_{ij}\int_0^{\infty}x^kx^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{i+j+1}.$$

Using of the identity $\Gamma(s) = \int_0^\infty y^{s-1}e^{-y}dy$, the k^{th} non-central moment of the MLIIFr distribution can be written as

$$\mu_k^{'} = \alpha(1-\gamma)\sum_{i,j=0}^{\infty}\Psi_{ij}(i+j+1)^{-(1-k/\beta)}\Gamma\left(1-k/\beta\right),\ \beta > k.$$

The k^{th} incomplete moment, by definition, is given as

$$M_k(x) = \int_0^y x^k f(x)dx. \tag{10}$$

Substituting equation (8) into equation (10), we have

$$M_k(x) = \alpha\beta(1-\gamma)\sum_{i,j=0}^{\infty}\Psi_{ij}\int_0^y x^k x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{i+j+1}. \tag{11}$$

After some algebra and making use of the incomplete gamma function

$\Gamma(a,y) = \int_0^y x^{a-1}e^{-x}dx$, we obtain the k^{th} incomplete moment of the MLIIFr as

$$M_k(x) = \alpha(1-\gamma)\sum_{i,j=0}^{\infty}\Psi_{ij}(i+j+1)^{-(1-k/\beta)}\Gamma\left(1-k/\beta,(i+j+1)x^{-\beta}\right),\ \beta > k.$$

3.3 Moment generating function

The moment generating function (MGF) helps in determining the moments of a random variable. By definition, the moments of a random variable X are given as $M_x(z) = \mathbb{E}(e^{zx})$ if only it exists. Applying series expansion, $M_x(z) = \sum_{k=0}^{\infty}\frac{z^k}{k!}\mu_k^{'}$. Therefore,

$$M_x(z) = \alpha(1-\gamma)\sum_{i,j,k=0}^{\infty}\Psi_{ij}\left(\frac{z^k(i+j+1)^{-(1-k/\beta)}}{k!}\right)\Gamma\left(1-k/\beta\right),\ \beta > k.$$

3.4 Entropy

It is an essential measure of uncertainty associated with a random variable. The Shannon entropy was proposed by Shannon (1948) and defined by $I_S(X) = \mathbb{E}[-log(f(x))]$ which is the most widely used in the application. Nevertheless, Rényi entropy, which was proposed by Rényi (1961), has gained more attention in recent times. This is because the Rényi entropy is capable of generalizing several other entropy measures, including Shannon entropy (see, Csiszár and Körner (2011)). The Rényi entropy of random variable X with pdf is defined by

$$I_R(X) = \frac{1}{1-\delta}\log\left\{\int_0^\infty f^\delta(x)\right\}dx,\ \delta > 0 \text{ and } \delta \neq 0. \tag{12}$$

From equation (12), we can write

$$f^{\delta}(x)=\alpha^{\delta}\beta^{\delta}(1-\gamma)^{\delta}x^{-\delta(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\delta}\left(1-e^{-x^{-\beta}}\right)^{\delta(\alpha-1)}\left(1-\gamma e^{-x^{-\beta}}\right)^{-\delta(1+\alpha)}.$$

Using binomial expansion and after some algebra, we have

$$f^{\delta}(x)=\alpha^{\delta}\beta^{\delta}(1-\gamma)^{\delta}\sum_{a,b=0}^{\infty}\varPhi_{ab}x^{-\delta(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\delta+a+b},$$

where $\varPhi_{ab}=(-1)^{a}\gamma^{b}\binom{\delta(\alpha-1)}{a}\binom{\delta(1+\alpha)+b-1}{b}$. Therefore,

$$\int_{0}^{\infty}f^{\delta}(x)dx=\alpha^{\delta}\beta^{\delta}(1-\gamma)^{\delta}\sum_{a,b=0}^{\infty}\varPhi_{ab}\int_{0}^{\infty}x^{-\delta(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\delta+a+b}dx.$$

After some algebra manipulations and making use of the gamma function, $\Gamma(s)=\int_{0}^{\infty}y^{s-1}e^{-y}dy$, we obtain

$$I_{R}(X)=\frac{1}{1-\delta}\log\left\{\frac{\alpha^{\delta}\beta^{\delta-1}(1-\gamma)^{\delta}}{(\delta+a+b)^{\frac{\delta(\beta+1)-1}{\beta}}}\sum_{a,b=0}^{\infty}\varPhi_{ab}\Gamma\left(\frac{\delta(\beta+1)-1}{\beta}\right)\right\}.$$

3.5 Order Statistics

In life tests of components and reliability studies, order statistics are very useful. Let $x_1,x_2,\ldots,x_n$ be a random sample of size n that follows the MLIIFr distribution and $x_{1:n}<x_{2:n}<\ldots<x_{n:n}$ be the corresponding order statistics. Then, the PDF of the i^{th} order statistics is defined as

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)\sum_{w=0}^{n-i}(-1)^{w}\binom{n-i}{w}(F(x))^{w+i-1}. \tag{13}$$

Substituting equations (2) and (3) in equation (13), we obtain

$$\begin{aligned}f_{i:n}(x)=&\frac{\alpha\beta(1-\gamma)n!}{(i-1)!(n-i)!}x^{-(\beta+1)}e^{-x^{-\beta}}\left(1-e^{-x^{-\beta}}\right)^{\alpha-1}\left(1-\gamma e^{-x^{-\beta}}\right)^{-(1+\alpha)}\\&\times\sum_{w=0}^{n-i}(-1)^{w}\binom{n-i}{w}\left[1-\left(\frac{1-e^{-x^{-\beta}}}{1-\gamma e^{-x^{-\beta}}}\right)^{\alpha}\right]^{w+i-1}.\end{aligned}$$

The PDF of the n^{th} order statistics is defined as

$$f_{n:n}=n[F(x)]^{n-1}f(x). \tag{14}$$

Substituting equations (2) and (3) in equation (14), will give

$$f_{n:n}=n\alpha\beta(1-\gamma)\left[1-\left(\frac{1-e^{-x^{-\beta}}}{1-\gamma e^{-x^{-\beta}}}\right)^{\alpha}\right]^{n-1}x^{-(\beta+1)}e^{-x^{-\beta}}\left(1-e^{-x^{-\beta}}\right)^{\alpha-1}\left(1-\gamma e^{-x^{-\beta}}\right)^{-(1+\alpha)}.$$

The PDF of the first-order statistics is defined as

$$f_{1:n} = n[1 - F(x)]^{n-1} f(x). \tag{15}$$

Substituting equations (2) and (3) in equation (15) will give

$$f_{n:n} = n\alpha\beta(1-\gamma)\left[\left(\frac{1-e^{-x^{-\beta}}}{1-\gamma e^{-x^{-\beta}}}\right)^{\alpha}\right]^{n-1} x^{-(\beta+1)} e^{-x^{-\beta}} \left(1-e^{-x^{-\beta}}\right)^{\alpha-1} \left(1-\gamma e^{-x^{-\beta}}\right)^{-(1+\alpha)}.$$

4 Classical Estimation

In this section, the MLE method and MPSE method have been discussed as a classical estimation. Also, the approximate confidence interval (ACI) has been constructed using MLE estimates.

4.1 Maximum Likelihood Estimation

The MLEs of the model parameters of the MLIIFr distribution are derived from a progressively censored sample in this section. The Type-II PCS $x_{1:m:n}, \ldots, x_{m:m:n}$ is constructed from the MLIIFr distribution. For simplicity, we have denoted data as $(x_1, \ldots, x_m)$, where $x_i = x_{i:m:n},\ i = 1, \ldots, m$ throghout the paper. Based on the observed data, the likelihood function (LF) can be written as

$$L(x; \alpha, \beta, \gamma) \propto \prod_{i=1}^{m} f(x_i; \alpha, \beta, \gamma) \left(1 - F(x_i; \alpha, \beta, \gamma)\right)^{R_i}. \tag{16}$$

Substituting equation (2) and equation (3) into equation (16), we have

$$L(x; \alpha, \beta, \gamma) \propto \prod_{i=1}^{m} \alpha\beta(1-\gamma) x_i^{-\beta-1} e^{-x_i^{-\beta}} \left(1-e^{-x_i^{-\beta}}\right)^{\alpha-1} \left(1-\gamma e^{-x_i^{-\beta}}\right)^{-\alpha-1} \left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}. \tag{17}$$

The log-likelihood function by taking proportional constant A can be written as

$$\log L = \log A + \sum_{i=1}^{m} \log\alpha + \log\beta + \log(1-\gamma) - (\beta+1)\log x_i - x_i^{-\beta} + (\alpha-1)\log\left(1-e^{-x_i^{-\beta}}\right)$$
$$- (\alpha+1)\log\left(1-\gamma e^{-x_i^{-\beta}}\right) + \alpha R_i \log\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right).$$

For finding the MLE estimates of the model parameter, we need to compute the first partial derivatives of $\log L$ with respect to α, β and γ and equate with zero. So, we obtain the normal equations, which are given as

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{m} \frac{1}{\alpha} + \log(1 - e^{-x_i^{-\beta}}) - \log(1 - \gamma e^{-x_i^{-\beta}}) + R_i \log\left(\frac{1 - e^{-x_i^{-\beta}}}{1 - \gamma e^{-x_i^{-\beta}}}\right). \tag{18}$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{m} \frac{1}{\beta} - \log x_i + \frac{(1-\alpha)\log x_i}{x_i^{-\beta}(e^{-x_i^{-\beta}} - 1)} - \frac{\gamma \log x_i(\alpha+1)}{x_i^{-\beta}(\gamma - e^{-x_i^{-\beta}})} + \frac{(1-\gamma)\alpha R_i \log x_i e^{-x_i^{-\beta}}}{x_i^{-\beta}(e^{-x_i^{-\beta}} - 1)(\gamma - e^{-x_i^{-\beta}})}. \tag{19}$$

$$\frac{\partial \log L}{\partial \gamma} = \sum_{i=1}^{m} \frac{1}{\gamma - 1} + \frac{\alpha R_i + \alpha + 1}{(e^{-x_i^{-\beta}} - \gamma)}. \tag{20}$$

MLEs of the parameters α, β, and γ can be calculated by solving the above equations (18)-(20). As the above expressions are complex, finding explicit solutions for the parameters is difficult, so a numerical approximation technique is used to obtain the required solution. We have used the Newton-Raphson iteration method to find model parameter estimates.

4.1.1 Approximate Confidence Interval

Due to the complex equations (18)-(20), the asymptotic confidence intervals (ACIs) of the model parameters are challenging to determine. As a result, $100(1-\psi)\%$ approximate confidence intervals of α, β, and γ have been obtained using a variance-covariance matrix for the corresponding MLEs by inverting the observed Fisher information matrix. The inverse of the observed Fisher information matrix is given as

$$\hat{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\gamma}) = \begin{pmatrix} -\frac{\partial^2 \log L}{\partial \alpha^2} & -\frac{\partial^2 \log L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \log L}{\partial \alpha \partial \gamma} \\ -\frac{\partial^2 \log L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L}{\partial \beta^2} & -\frac{\partial^2 \log L}{\partial \beta \partial \gamma} \\ -\frac{\partial^2 \log L}{\partial \gamma \partial \alpha} & -\frac{\partial^2 \log L}{\partial \gamma \partial \beta} & -\frac{\partial^2 \log L}{\partial \gamma^2} \end{pmatrix}^{-1}_{(\hat{\alpha}, \hat{\beta}, \hat{\gamma})} = \begin{pmatrix} \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\beta}) & \mathrm{cov}(\hat{\alpha}, \hat{\gamma}) \\ \mathrm{cov}(\hat{\beta}, \hat{\alpha}) & \mathrm{var}(\hat{\beta}) & \mathrm{cov}(\hat{\beta}, \hat{\gamma}) \\ \mathrm{cov}(\hat{\gamma}, \hat{\alpha}) & \mathrm{cov}(\hat{\gamma}, \hat{\beta}) & \mathrm{var}(\hat{\gamma}) \end{pmatrix},$$

where the elements of the matrix are given as

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \sum_{i=1}^{m} \frac{-1}{\alpha^2}; \quad \frac{\partial^2 \log L}{\partial \gamma^2} = \sum_{i=1}^{m} -\frac{1}{(\gamma-1)^2} + \frac{(\alpha+\alpha R_i+1)}{\left(\gamma - e^{-x_i^{-\beta}}\right)^2},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \beta} = \frac{\partial^2 \log L}{\partial \beta \partial \alpha} = \sum_{i=1}^{m} \frac{\log x_i}{x_i^{-\beta}(1-e^{-x_i^{-\beta}})} + \frac{\gamma \log x_i}{x_i^{-\beta}(e^{-x_i^{-\beta}}-\gamma)} + \frac{(1-\gamma)R_i \log x_i e^{-x_i^{-\beta}}}{x_i^{-\beta}(e^{-x_i^{-\beta}}-1)(\gamma - e^{-x_i^{-\beta}})},$$

$$\frac{\partial^2 \log L}{\partial \alpha \partial \gamma} = \frac{\partial^2 \log L}{\partial \gamma \partial \alpha} = \sum_{i=1}^{m} \frac{1}{(e^{-x_i^{-\beta}}-1)} + \frac{R_i}{(e^{-x_i^{-\beta}}-\gamma)},$$

$$\begin{aligned}\frac{\partial^2 \log L}{\partial \beta^2} = \sum_{i=1}^{m} -\frac{1}{\beta^2} &- \frac{(\alpha-1)(\log x_i)^2 \left(e^{-x_i^{-\beta}}(x_i^{-\beta}-1) - x_i^{-\beta}\right)}{x_i^{-2\beta}\left(e^{-x_i^{-\beta}}-1\right)^2} + \frac{\gamma(\alpha+1)(\log x_i)^2\left(\gamma x_i^{-\beta} + (1-x_i^{-\beta})e^{-x_i^{-\beta}}\right)}{x_i^{-2\beta}\left(\gamma - e^{-x_i^{-\beta}}\right)^2} \\ &+ \frac{\alpha(1-\gamma)R_i(\log x_i)^2 e^{-x_i^{-\beta}} \left(x_i^{-\beta}(e^{-x_i^{-\beta}}-1)-1\right)\gamma - e^{-x_i^{-\beta}}\left((x_i^{-\beta}-1)e^{-x_i^{-\beta}} - x_i^{-\beta}\right)}{x_i^{-2\beta}\left(\gamma - e^{-x_i^{-\beta}}\right)^2\left(e^{-x_i^{-\beta}}-1\right)^2} - \frac{(\log x_i)^2}{x_i^{-\beta}},\end{aligned}$$

$$\frac{\partial^2 \log L}{\partial \beta \partial \gamma} = \frac{\partial^2 \log L}{\partial \gamma \partial \beta} = \sum_{i=1}^{m} \frac{(\alpha+\alpha R_i+1)(\log x_i)e^{-x_i^{-\beta}}}{x_i^{-\beta}\left(\gamma - e^{-x_i^{-\beta}}\right)^2}.$$

Therefore, the ACIs of α, β and γ are $\left[\hat{\alpha} \pm Z_{\frac{\psi}{2}}\sqrt{\text{var}(\hat{\alpha})}\right]$, $\left[\hat{\beta} \pm Z_{\frac{\psi}{2}}\sqrt{\text{var}(\hat{\beta})}\right]$ and $\left[\hat{\gamma} \pm Z_{\frac{\psi}{2}}\sqrt{\text{var}(\hat{\gamma})}\right]$, respectively. Where $\text{var}(\hat{\alpha})$, $\text{var}(\hat{\beta})$ and $\text{var}(\hat{\gamma})$ are the elements of the leading diagonal in $\hat{I}^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and $Z_{\frac{\psi}{2}}$ is the $\left(\frac{\psi}{2}\right) th$ quantile of the standard normal distribution. Despite the capabilities of MLE and its wide use in parameter estimation, MPSE provides a better alternative in specific contexts. In statistical analysis, MPSE is beneficial because it is robust in handling misspecification, outliers, and spacing distribution. Since MLE and MPSE can handle different scenarios, selecting one over the other should be determined by the specific features of the data and the model. The following subsection discusses the MPSE.

4.2 Maximum Product Spacing Estimation

In this subsection, we have used the MPSE to estimate the parameters of the statistical model. This method maximizes the geometric mean of the spacings in the data, i.e., the difference between the CDF values at neighbouring points. There is a similarity between MPSE and MLE. The MLE selects the parameter values that maximize the likelihood function, just as MPSE does, by maximizing the product of the CDF gaps at adjacent ordered points. Based on the probability integral transform, MPSE assumes that independent samples derived from a random variable are uniformly distributed over its CDF. A specific quantitative measure of uniformity has been used to select parameter values for MPSE to ensure that the observed data is consistent. Ng et al. (2012) introduced MPSE under Type-II PCS. According to Anatolyev and Kosenok (2005), MPSE is more efficient in small samples with skewed or heavy-tailed distributions than MLE. Based on the Type-II progressive censoring scheme, product spacing (PS) is defined as follows:

$$S = \prod_{i=1}^{m+1} \left[F(x_i;\alpha,\beta,\gamma) - F(x_{i-1};\alpha,\beta,\gamma)\right] \prod_{i=1}^{m} \left[1 - F(x_i;\alpha,\beta,\gamma)\right]^{R_i},$$

$$S = D_1 \cdot D_{m+1} \prod_{i=2}^{m} \left[F(x_i;\alpha,\beta,\gamma) - F(x_{i-1};\alpha,\beta,\gamma)\right] \prod_{i=1}^{m} \left[1 - F(x_i;\alpha,\beta,\gamma)\right]^{R_i}, \tag{21}$$

where

$$D_i = \begin{cases} D_1 = F(x_1), \\ D_i = F(x_i) - F(x_{i-1}); \quad i = 2,\ldots,m. \\ D_{m+1} = 1 - F(x_m). \end{cases}$$

The equation (21) can be written as

$$S = D_1 \cdot D_{m+1} \prod_{i=2}^{m} (D_i) \prod_{i=1}^{m} (1 - F(x_i;\mu,\sigma))^{R_i}.$$

D_i represents the spacing, as introduced by Cheng and Amin (1983). By using equation (2), the PS function can be written as

$$S = \left(1 - \frac{1 - e^{-x_1^{-\beta}}}{1 - \gamma e^{-x_1^{-\beta}}}\right)^{\alpha} \left(\frac{1 - e^{-x_m^{-\beta}}}{1 - \gamma e^{-x_m^{-\beta}}}\right)^{\alpha} \prod_{i=2}^{m} \left[\left(\frac{1 - e^{-x_{i-1}^{-\beta}}}{1 - \gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha} - \left(\frac{1 - e^{-x_i^{-\beta}}}{1 - \gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right] \prod_{i=1}^{m} \left(\frac{1 - e^{-x_i^{-\beta}}}{1 - \gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}. \tag{22}$$

$$\begin{aligned} \log S = {} & \alpha \log\left(1 - \frac{1 - e^{-x_1^{-\beta}}}{1 - \gamma e^{-x_1^{-\beta}}}\right) + \alpha \log\left(\frac{1 - e^{-x_m^{-\beta}}}{1 - \gamma e^{-x_m^{-\beta}}}\right) + \sum_{i=2}^{m} \alpha \log\left[\left(\frac{1 - e^{-x_{i-1}^{-\beta}}}{1 - \gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha} - \left(\frac{1 - e^{-x_i^{-\beta}}}{1 - \gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right] \\ & + \sum_{i=1}^{m} \alpha R_i \log\left(\frac{1 - e^{-x_i^{-\beta}}}{1 - \gamma e^{-x_i^{-\beta}}}\right). \end{aligned}$$

By maximizing the PS function for α, β and γ, we can find the estimates of model parameters as follows:

$$\frac{\partial \log S}{\partial \alpha} = \log\left(1 - \frac{1-e^{-x_1^{-\beta}}}{1-\gamma e^{-x_1^{-\beta}}}\right) + \log\left(\frac{1-e^{-x_m^{-\beta}}}{1-\gamma e^{-x_m^{-\beta}}}\right) + \sum_{i=1}^{m} R_i \log\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)$$
$$+ \sum_{i=2}^{m} \frac{\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha} \log\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right) - \left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha} \log\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)}{\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha} - \left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha}} = 0.$$

$$\frac{\partial \log S}{\partial \beta} = -\alpha\left(\frac{\log(x_1) e^{-x_1^{-\beta}}}{x_1^{\beta}(\gamma - e^{-x_1^{-\beta}})} + \frac{\log(x_m) e^{-x_m^{-\beta}}(\gamma-1)}{x_m^{\beta}(e^{-x_m^{-\beta}}-1)(\gamma - e^{-x_m^{-\beta}})}\right) - \sum_{i=1}^{m} \alpha R_i \left(\frac{\log(x_i) e^{-x_i^{-\beta}}(\gamma-1)}{x_i^{\beta}(e^{-x_i^{-\beta}}-1)(\gamma - e^{-x_i^{-\beta}})}\right)$$
$$- \sum_{i=2}^{m} \frac{\alpha(\gamma - e^{-x_i^{-\beta}})\left(\frac{e^{-x_i^{-\beta}}-1}{e^{-x_i^{-\beta}}-\gamma}\right)^{\alpha}\left(\frac{\log(x_i)e^{-x_i^{-\beta}}}{x_i^{\beta}(\gamma - e^{-x_i^{-\beta}})} + \frac{\log(x_i)(e^{-x_i^{-\beta}}-1)e^{-x_i^{-\beta}}}{x_i^{\beta}(\gamma - e^{-x_i^{-\beta}})^2}\right)}{(e^{-x_i^{-\beta}}-1)}$$
$$+ \sum_{i=2}^{m} \frac{\alpha(\gamma - e^{-x_{i-1}^{-\beta}})\left(\frac{e^{-x_{i-1}^{-\beta}}-1}{e^{-x_{i-1}^{-\beta}}-\gamma}\right)^{\alpha}\left(\frac{\log(x_{i-1})e^{-x_{i-1}^{-\beta}}}{x_{i-1}^{\beta}(\gamma - e^{-x_{i-1}^{-\beta}})} + \frac{\log(x_{i-1})(e^{-x_{i-1}^{-\beta}}-1)e^{-x_{i-1}^{-\beta}}}{x_{i-1}^{\beta}(\gamma - e^{-x_{i-1}^{-\beta}})^2}\right)}{(e^{-x_{i-1}^{-\beta}}-1)} = 0.$$

$$\frac{\partial \log S}{\partial \gamma} = -\alpha\left(\frac{e^{-x_1^{-\beta}}-1}{(\gamma-1)(\gamma - e^{-x_1^{-\beta}})}\right) - \frac{\alpha}{\gamma - e^{-x_m^{-\beta}}} - \alpha\sum_{i=1}^{m}\frac{R_i}{\gamma - e^{-x_i^{-\beta}}}$$
$$+ \alpha\sum_{i=2}^{m} \frac{\left(\gamma - e^{-x_i^{-\beta}}\right)\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha} + \left(e^{-x_i^{-\beta}} - \gamma\right)\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}}{\left(\gamma - e^{-x_{i-1}^{-\beta}}\right)\left(\gamma - e^{-x_i^{-\beta}}\right)\left[\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha} - \left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}\right]} = 0.$$

The above equations are in complex form, and solving these equations analytically is challenging. Thus, Newton-Raphson iterative numerical methods have been used to solve these equations. We discuss the Bayesian framework for estimating model parameters in the following section.

5 Bayesian Estimation

This section explores the Bayesian estimation of the model parameters under progressively censored data. We focus on deriving Bayesian estimates using the LF and PS with associated credible intervals. Bayesian inference is highly dependent on the choice of prior distributions. However, no specific guidelines exist for choosing optimal priors for unknown parameters. We have used INIP for estimation purposes. The squared error loss function (SELF) has been considered in this section, but one can also use alternative loss functions for estimation purposes. In SELF, overestimations and underestimations are treated equally, and errors are handled quadratically, so larger errors have a much greater impact than smaller ones. As a result, SELF is very sensitive to deviations from the actual parameter value. SELF can be defined as $\mathcal{L}(\zeta, \hat{\zeta}) = (\zeta - \hat{\zeta})^2$, where $\hat{\zeta}$ denote the estimated value of parameter $\zeta = (\alpha, \beta, \gamma)$.

5.1 Informative and non-informative Prior

In this subsection, we use piecewise independent gamma priors for the model parameters α and β, while non-informative prior has been used for γ. It is important to note that Jeffrey's priors are challenging due to the complex nature of the Fisher information matrix. Therefore, we have used independent gamma priors. A gamma prior is flexible since it can accommodate informative and non-informative priors by selecting the appropriate hyper-parameters. Furthermore, depending on the parameter values, it offers flexibility in their shape. Additionally, independent gamma priors are simple, which can reduce computational complexity. As a result, we can adopt it as appropriate prior to the model parameter. Thus, the prior of α, β, and γ are given as

$$\pi_1(\alpha) \propto \alpha^{a_1-1}e^{-a_2\alpha},\ \alpha, a_1, a_2 > 0;$$
$$\pi_2(\beta) \propto \beta^{b_1-1}e^{-b_2\beta},\ \beta, b_1, b_2 > 0; \quad \pi_3(\gamma) \propto 1,\ -\infty < \gamma < 1.$$

Where (a_1, a_2) and (b_1, b_2) are the hyper-parameters of α and β respectively. The joint prior for ζ can be written as

$$\pi(\zeta) \propto \alpha^{a_1-1}\beta^{b_1-1}e^{-a_2\alpha-b_2\beta}.$$

5.1.1 Using likelihood function

Following the Bayes theorem, based on the likelihood function (17), the joint posterior density of α, β and γ can be written as

$$\pi_1^*(\zeta|x) \propto \alpha^{a_1+m-1}\beta^{b_1+m-1}(1-\gamma)^m e^{-a_2\alpha-b_2\beta}\prod_{i=1}^{m} x_i^{-\beta-1}e^{-x_i^{-\beta}}\left(1-e^{-x_i^{-\beta}}\right)^{\alpha-1+\alpha R_i}\left(1-\gamma e^{-x_i^{-\beta}}\right)^{-\alpha-1-\alpha R_i}.$$

The conditional posterior density (CPD) of α, β and γ can be written as

$$\pi_1^*(\alpha|\beta,\gamma,x) \propto \alpha^{a_1+m-1}e^{-\alpha\left(a_2-\sum_{i=1}^{m}(1+R_i)\log(1-e^{-x_i^{-\beta}})+(1+R_i)\log(1-\gamma e^{-x_i^{-\beta}})\right)}, \tag{23}$$

$$\pi_1^*(\beta|\alpha,\gamma,x) \propto \beta^{b_1+m-1}e^{b_2\beta}\prod_{i=1}^{m} x_i^{-\beta-1}e^{-x_i^{-\beta}}\left(1-e^{-x_i^{-\beta}}\right)^{\alpha-1+\alpha R_i}\left(1-\gamma e^{-x_i^{-\beta}}\right)^{-\alpha-1-\alpha R_i}, \tag{24}$$

$$\pi_1^*(\gamma|\alpha,\beta,x) \propto (1-\gamma)^m\prod_{i=1}^{m}\left(1-\gamma e^{-x_i^{-\beta}}\right)^{-\alpha-1-\alpha R_i}. \tag{25}$$

From the above equations (23)-(25), we can see that the CPD of α can be written as a gamma density function. Whereas the CPDs of β and γ can not be expressed in any well-known distribution. So, it is difficult to generate the samples of β and γ directly. However, the CPD plots of β and γ are presented in Figure (2). It shows that these distributions follow a nearly symmetrical pattern. Therefore, a Markov chain Monte Carlo (MCMC) method, Metropolis-Hastings (MH) algorithm, has been used to generate MCMC samples. So, we can use here Gibbs sampling with the MH algorithm to find the Bayes estimates. The algorithm procedure is given in subsection (5.1.3).

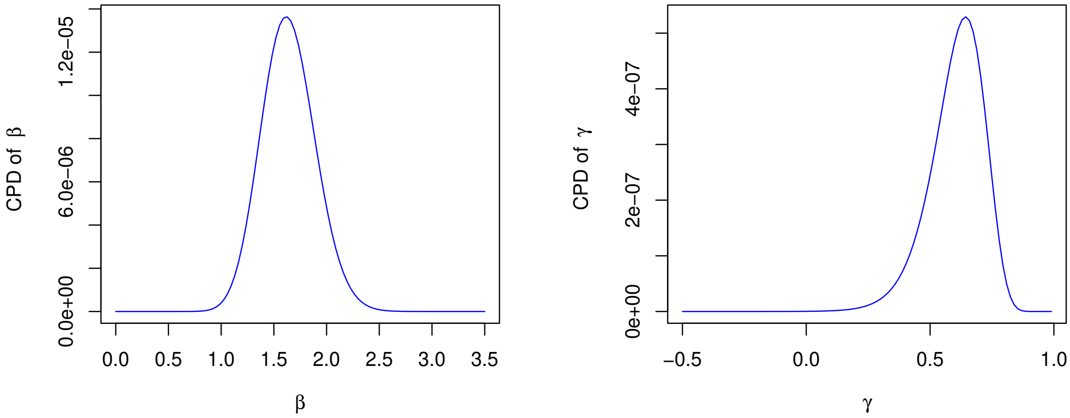

Figure 2: CPD plot of β and γ based on LF

5.1.2 Using product spacing function

As we have obtained the Bayes estimates using LF, in a similar way, the joint posterior density of α, β and γ based on a PS function (22) can be written as

$$\pi_2^*(\zeta|x) \propto \alpha^{a_1-1}\beta^{b_1-1}e^{-a_2\alpha-b_2\beta}\left(1-\frac{1-e^{-x_1^{-\beta}}}{1-\gamma e^{-x_1^{-\beta}}}\right)^{\alpha}\left(\frac{1-e^{-x_m^{-\beta}}}{1-\gamma e^{-x_m^{-\beta}}}\right)^{\alpha}$$
$$\prod_{i=2}^{m}\left[\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}-\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right]\prod_{i=1}^{m}\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}.$$

The CPD of α, β and γ can be written as

$$\pi_2^*(\alpha|\beta,\gamma,x) \propto \alpha^{a_1-1}e^{-a_2\alpha}\left(1-\frac{1-e^{-x_1^{-\beta}}}{1-\gamma e^{-x_1^{-\beta}}}\right)^{\alpha}\left(\frac{1-e^{-x_m^{-\beta}}}{1-\gamma e^{-x_m^{-\beta}}}\right)^{\alpha}$$
$$\prod_{i=2}^{m}\left[\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}-\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right]\prod_{i=1}^{m}\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}, \quad (26)$$

$$\pi_2^*(\beta|\alpha,\gamma,x) \propto \beta^{b_1-1}e^{-b_2\beta}\left(1-\frac{1-e^{-x_1^{-\beta}}}{1-\gamma e^{-x_1^{-\beta}}}\right)^{\alpha}\left(\frac{1-e^{-x_m^{-\beta}}}{1-\gamma e^{-x_m^{-\beta}}}\right)^{\alpha}$$
$$\prod_{i=2}^{m}\left[\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}-\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right]\prod_{i=1}^{m}\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}, \quad (27)$$

$$\pi_2^*(\gamma|\alpha,\beta,x) \propto \left(1-\frac{1-e^{-x_1^{-\beta}}}{1-\gamma e^{-x_1^{-\beta}}}\right)^{\alpha}\left(\frac{1-e^{-x_m^{-\beta}}}{1-\gamma e^{-x_m^{-\beta}}}\right)^{\alpha}$$
$$\prod_{i=2}^{m}\left[\left(\frac{1-e^{-x_{i-1}^{-\beta}}}{1-\gamma e^{-x_{i-1}^{-\beta}}}\right)^{\alpha}-\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha}\right]\prod_{i=1}^{m}\left(\frac{1-e^{-x_i^{-\beta}}}{1-\gamma e^{-x_i^{-\beta}}}\right)^{\alpha R_i}. \quad (28)$$

The CPDs for α, β and γ, as given in equations (26)-(28) can not be expressed in any standard statistical distribution. However, from the CPDs plot presented in Figure (3), we can use the normal distribution to generate posterior samples. We use the MH algorithm to find the Bayes estimates. A hybrid MCMC algorithm is given in the next subsection to obtain the Bayes (point or interval) estimates for model parameters.

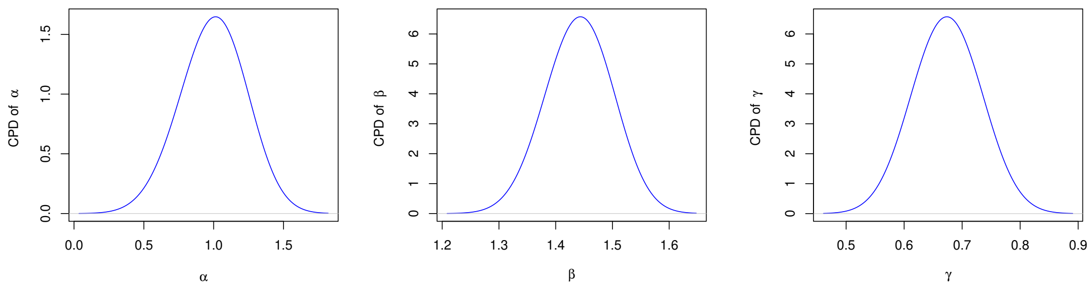

Figure 3: CPD plot of α, β and γ based on PS

5.1.3 Hybrid MCMC Algorithm

In this subsection, we introduced the MCMC algorithm that integrates Gibbs sampling and the MH algorithm. The Gibbs sampling method is employed for updating the parameter α, while the MH algorithm is utilized to update the parameters β and γ.

The process for generating MCMC samples based on the LF of α, β and γ involves the following steps:

1. Set the initial guesses $(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}) = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$.

2. Set $p = 1$.

3. Obtain α^p from $\text{Gamma}\Big(m + a_1, \Big(a_2 - \sum_{i=1}^{m}(1 + R_i)\log(1 - e^{-x_i^{-\beta}}) + (1 + R_i)\log(1 - \gamma e^{-x_i^{\beta}})\Big)\Big)$.

4. Obtain β^* and γ^* from $N(\hat{\beta}, \text{var}(\hat{\beta}))$ and $N(\hat{\gamma}, \text{var}(\hat{\gamma}))$, respectively, then apply the MH algorithm as follows:
   a) Calculate
   $$R_\beta = \frac{\pi_1^*\left(\beta^*|\alpha^p, \gamma^{p-1}, x\right)}{\pi_1^*\left(\beta^{p-1}|\alpha^p, \gamma^{p-1}, x\right)} \quad \text{and} \quad R_\gamma = \frac{\pi_1^*\left(\gamma^*|\alpha^p, \beta^{p-1}, x\right)}{\pi_1^*\left(\gamma^{p-1}|\alpha^p, \beta^{p-1}, x\right)}.$$
   b) Obtain $W_\beta = \min\{1, R_\beta\}$ and $W_\gamma = \min\{1, R_\gamma\}$.
   c) Obtain u_1 and u_2 from the uniform $U(0, 1)$ distribution.
   d) If $u_1 \leq W_\beta$, set $\beta^p = \beta^*$, else set $\beta^p = \beta^{p-1}$. Similarly, if $u_2 \leq W_\gamma$, set $\gamma^p = \gamma^*$, else set $\gamma^p = \gamma^{p-1}$.

5. Set $p = p + 1$.

6. Repeat steps (3-5) G times in order to obtain enough observations.

To obtain the desired estimates, we first discard the initial G_0 samples to eliminate any potential bias from the initial values or burn-in period. This process ensures that the remaining samples are more representative of the target distribution. The estimates for the parameters α, β, and γ under the SELF are then calculated as follows: $\hat{\alpha} = \frac{1}{G-G_0}\sum_{p=G_0+1}^{G}\alpha^p$, $\hat{\beta} = \frac{1}{G-G_0}\sum_{p=G_0+1}^{G}\beta^p$ and $\hat{\gamma} = \frac{1}{G-G_0}\sum_{p=G_0+1}^{G}\gamma^p$.

To develop the Bayesian MCMC estimates using the PS function for parameters such as α, β, and γ. We can easily follow the same steps of the MH algorithm as discussed in subsection (5.1.3).

5.2 Credible Interval

The following steps have been followed to obtain the Bayesian credible intervals (CIs) for model parameters (α, β, γ):

1. Arrange α^p, β^p and γ^p in increasing order for $p = 1, \ldots, G - G_0$.

2. Determine the level of significance ψ.

3. We can get the $(1 - \psi)\%$ CIs for α, β and γ as $\left(\alpha^{\left[\frac{\psi(N-N_0)}{2}\right]}, \alpha^{\left[(G-G_0)(1-\frac{\psi}{2})\right]}\right)$, $\left(\beta^{\left[\frac{\psi(G-G_0)}{2}\right]}, \beta^{\left[(G-G_0)(1-\frac{\psi}{2})\right]}\right)$ and $\left(\gamma^{\left[\frac{\psi(G-G_0)}{2}\right]}, \gamma^{\left[(G-G_0)(1-\frac{\psi}{2})\right]}\right)$.

Using the above steps, we can obtain the CIs of model parameters in both LF and PS approaches.

6 Simulation Study

A Monte Carlo simulation is used in this section to compare the results of the above estimation method for MLIIFr under progressively censored data. We have generated a progressively Type-II censored sample, $(x_1, \ldots, x_m)$ by using a pre-fixed censoring scheme $(R_1, \ldots, R_m)$. For sample generation, we have used the algorithm proposed by Balakrishnan and Sandhu (1995). The following censoring schemes have been used in simulations:

$$\begin{aligned}
\text{Scheme I: } R &= \left(n-m, 0^{*m-1}\right), \\
\text{Scheme II: } R &= \left(0^{*m-1}, n-m\right), \\
\text{Scheme III: } R &= \left(\frac{n-m}{2}, 0^{*m-2}, \frac{n-m}{2}\right).
\end{aligned}$$

This study calculates point and interval estimations based on the above censoring schemes and different combinations of (n, m). The estimates of model parameters have been calculated for classical and Bayesian frameworks. ACIs and CIs have been obtained using MLE and Bayes estimates under INIP via LF and PS. In addition, interval estimation is calculated for a significance level of $\psi = 0.05$. The actual value of the model parameter is taken $(\alpha, \beta, \gamma) = (1.05, 1.45, 0.65)$. The value of hyper parameters $(a_1, a_2) = (2.1, 2)$ and $(b_1, b_2) = (4, 2.75)$ has been taken for informative prior in Bayesian estimates. We select hyper-parameters in such a way that the prior mean corresponds to the expected value of each parameter. A more detailed explanation can be found in Kundu (2008). The simulation is performed 1000 times for each combination of (n, m) and scheme. The simulations have been conducted using R software. Table 1 provides the average estimates and mean squared errors (MSEs) using MPSE. Table 2 provides the point and interval estimates along with ACIs and coverage probabilities (CPs) using MLE. Parameter estimates and associated MSEs of model parameters in the Bayesian framework, along with CIs and CPs, are noted in Table 3 and 4 via LF and PS, respectively. MSEs have been used to compare point estimates, while interval estimates are compared through interval lengths. The difference between upper and lower bounds determines the interval length. In the classical framework, the MSEs using MPSE are lower than MLE, as seen in Table 1 and 2. In the Bayesian framework, Table 3 and 4 show that Bayes estimates via PS have smaller MSEs than via LF in most cases. Further, the MSE value is minimal compared to all methods when we use Bayes estimates via PS in most cases. In addition, we can also observe that by decreasing the value of m for a fixed n, MSEs value increases in most cases as shown in Table 1-4 and MSEs decreases by increasing the value of n. In interval estimation, we can see from Table 2-4 that the interval lengths of CIs are shorter than ACIs, and CPs are at a nominal level in both methods. CPs of confidence intervals are often nearly identical to their actual values of 0.95.

Table 1: Average estimates and MSEs for various combinations of (n, m) by using MPSE

n	m	R	$\hat{\alpha}$	MSE(α)	$\hat{\beta}$	MSE(β)	$\hat{\gamma}$	MSE(γ)
35	30	I	1.0878	0.2412	1.6311	0.1852	0.7232	0.0537
	25		1.0557	0.2514	1.6711	0.1873	0.7228	0.0601
	20		0.9941	0.2538	1.6859	0.2035	0.7394	0.0694
35	30	II	0.9196	0.2193	1.7091	0.1361	0.6753	0.0562
	25		0.9120	0.2282	1.6232	0.1436	0.7202	0.0655
	20		0.9371	0.2487	1.6195	0.1484	0.7683	0.0685
35	30	III	0.9559	0.2316	1.6994	0.1667	0.6613	0.0547
	25		0.8962	0.2497	1.6682	0.1768	0.6854	0.0626
	20		0.8892	0.2638	1.6684	0.2035	0.7141	0.0694
45	40	I	1.0617	0.1715	1.6563	0.1202	0.6746	0.0508
	35		1.1105	0.1834	1.6505	0.1543	0.7079	0.0531
	30		1.0305	0.2316	1.6965	0.1694	0.7043	0.0564
45	40	II	0.9733	0.1734	1.6661	0.1233	0.6492	0.0496
	35		0.9327	0.1996	1.6538	0.1273	0.6726	0.0516
	30		0.9544	0.2097	1.6099	0.1334	0.7298	0.0563
45	40	III	0.9912	0.2009	1.6865	0.1184	0.6544	0.0481
	35		0.9431	0.2048	1.6839	0.1231	0.6473	0.0582
	30		0.9384	0.2155	1.6534	0.1329	0.6774	0.0598
55	50	I	1.1108	0.1351	1.6401	0.1169	0.6784	0.0382
	45		1.1278	0.1498	1.6404	0.1251	0.6914	0.0401
	40		1.1301	0.1504	1.6437	0.1276	0.6955	0.0475
55	50	II	0.9902	0.1479	1.6713	0.0958	0.6401	0.0449
	45		0.9783	0.1654	1.6373	0.1013	0.6502	0.0569
	40		0.9404	0.1896	1.6204	0.1203	0.6617	0.0561
55	50	III	1.0242	0.1537	1.6741	0.0811	0.6389	0.0327
	45		0.9759	0.1775	1.6646	0.0944	0.6372	0.0486
	40		0.9788	0.1855	1.6401	0.1073	0.6646	0.0518

7 Application

In this section, we demonstrate the usefulness and flexibility of the MLIIFr distribution using mortality data of Japanese children under five years. Also, the performance of the MLIIFr distribution is compared with other distributions. The performance of the distributions about providing better parametric fit to the dataset is compared using the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramér-von Misses (W^*), Anderson-Darling (A^*) and Kolmogorov-Smirnov (KS) statistics. The distribution with the least value of these measures provides a reasonable fit to the dataset. The fit for the MLIIFr distribution is compared with other distributions, including the two-parameters Weibull, Fréchet, Weibull exponential (WE), Power-Lomax, Dagum and Modified Lehmann Type-II Exponential (MLII-Ex) distributions. The CDFs of the competing models are presented in Table 5. The data consists of the mortality rate of children in Japan under five years of age from 1976 to 2022. The data has been represented in Table 6 and taken from **https://data.worldbank.org/indicator/SH.DYN.MORT** (accessed on 29 Aug 2024). Table 7 shows the goodness of fit and information criteria

Table 2: Average estimates, MSEs, ACIs and CPs for various combinations of (n, m) by using MLE

n	m	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	ACI(α)	ACI(β)	ACI(γ)
35	30	I	1.0863	1.5441	0.6236	(0.4441,1.7284)	(0.8003,2.2877)	(0.0503,0.9691)
			0.2824	0.1991	0.1164	84.5	90.8	86.6
	25		1.0631	1.5282	0.6061	(0.4290,1.6970)	(0.7846,2.2718)	(0.0430,0.9551)
			0.2915	0.2051	0.1212	83.8	90.2	83.4
	20		1.0961	1.5189	0.6551	(0.4650,1.7270)	(0.7754,2.2624)	(0.0549,1.2550)
			0.3112	0.2089	0.1303	81.4	93.3	81.9
35	30	II	1.0653	1.4991	0.6988	(0.4829,1.6476)	(0.79949,2.1985)	(0.0764,0.9821)
			0.2331	0.1472	0.0941	80.7	86.7	84.1
	25		1.0699	1.5194	0.7573	(0.5091,1.6307)	(0.8112,2.2276)	(0.1413,0.9723)
			0.2438	0.1576	0.1034	78.7	94.1	82.2
	20		0.9823	1.5312	0.7945	(0.4009,1.5637)	(0.8063,2.2558)	(0.1504,0.9385)
			0.2616	0.1744	0.1112	94.1	94.8	86.2
35	30	III	1.0536	1.5273	0.6299	(0.4186,1.6887)	(0.7837,2.2710)	(0.0253,0.96346)
			0.2481	0.1916	0.1354	88.3	89.5	85.8
	25		1.1338	1.4989	0.6928	(0.5649,1.7027)	(0.7936,2.2042)	(0.0815,0.9041)
			0.2661	0.1941	0.1483	86.4	84.4	78.3
	20		0.9791	1.5825	0.6756	(0.3700,1.5882)	(0.7859,2.3792)	(0.0747,0.8959)
			0.2884	0.2221	0.1709	78.9	89.6	87.8
45	40	I	1.1426	1.5367	0.5686	(0.4401,1.8452)	(0.8794,2.1940)	(0.0174,0.9548)
			0.1821	0.1344	0.0887	83.5	93.1	88.2
	35		1.1311	1.5087	0.6337	(0.4984,1.7636)	(0.8517,2.1658)	(0.1206,0.9468)
			0.1921	0.1604	0.1021	83.7	87.2	77.4
	30		1.1214	1.5329	0.5896	(0.5240,1.7179)	(0.8776,2.1882)	(0.0284,0.8508)
			0.2523	0.1843	0.1121	87.2	86.3	79.2
45	40	II	1.1621	1.4681	0.6929	(0.5269,1.7973)	(0.8547,2.0816)	(0.2348,0.9510)
			0.1976	0.1417	0.0885	83.4	89.9	83.4
	35		1.0387	1.4996	0.6813	(0.4478,1.6295)	(0.8760,2.1232)	(0.1212,0.92414)
			0.2142	0.1592	0.0931	83.1	92.3	87.2
	30		1.1021	1.4371	0.6803	(0.4907,1.7094)	(0.8315,2.0424)	(0.0335,0.9743)
			0.2567	0.1736	0.1047	83.9	88.2	81.2
45	40	III	1.0588	1.6152	0.5935	(0.3437,1.7739)	(0.8996,2.3308)	(0.0380,0.9650)
			0.2071	0.1474	0.1114	85.2	95.1	91.3
	35		1.0659	1.5876	0.6182	(0.4349,1.6969)	(0.8898,2.2855)	(0.0161,0.8204)
			0.2128	0.1547	0.1227	82.7	87.6	82.8
	30		1.1262	1.5336	0.6995	(0.4913,1.7611)	(0.8549,2.2124)	(0.1374,0.9616)
			0.2215	0.1625	0.1433	88.1	93.5	84.8
55	50	I	1.1318	1.5099	0.6147	(0.4511,1.8125)	(0.9300,2.0899)	(0.1519,0.9865)
			0.1621	0.1136	0.0689	88.2	89.4	86.4
	45		1.0621	1.5647	0.5675	(0.4260,1.6981)	(0.9682,2.1612)	(0.0649,0.9700)
			0.1346	0.1316	0.0841	80.9	92.9	83.7
	40		1.1243	1.4738	0.6404	(0.4774,1.7625)	(0.9080,2.0395)	(0.1934,0.9874)
			0.1158	0.1318	0.0933	85.3	86.4	78.8
55	50	II	1.1268	1.5138	0.6274	(0.4552,1.7984)	(0.9317,2.0958)	(0.1427,0.9008)
			0.1774	0.1006	0.0743	82.1	91.2	84.7
	45		1.0444	1.5286	0.6341	(0.4192,1.6697)	(0.9427,2.1146)	(0.0903,0.9778)
			0.1856	0.1163	0.1019	82.3	87.4	86.5
	40		1.1449	1.4419	0.6618	(0.5287,1.7611)	(0.8910,1.9928)	(0.0491,0.9644)
			0.1968	0.1214	0.1088	81.8	92.1	81.5
55	50	III	1.0667	1.5941	0.6041	(0.4146,1.7189)	(0.9772,2.2107)	(0.1161,0.9091)
			0.1974	0.0929	0.0773	79.9	89.2	89.6
	45		1.0787	1.5168	0.6341	(0.3973,1.7602)	(0.9075,2.1261)	(0.1248,0.8643)
			0.1993	0.1087	0.0971	85.2	91.5	91.2
	40		1.0782	1.5404	0.6201	(0.4917,1.6647)	(0.9470,2.1339)	(0.0349,0.9154)
			0.2064	0.1099	0.1014	86.1	89.2	90.2

Table 3: Average estimates, MSEs, CIs and CPs for various combinations of (n, m) by Bayes estimation using LF

n	m	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	CI(α)	CI(β)	CI(γ)
35	30	I	1.0335	1.5841	0.5984	(0.6996,1.4337)	(1.5334,1.6396)	(0.5478,0.6538)
			0.1043	0.0039	0.0044	98.25	96.16	97.72
	25		1.0443	1.3822	0.6947	(0.7931,1.6089)	(1.4368,1.6269)	(0.4460,0.6383)
			0.1233	0.0046	0.0049	94.57	97.35	95.63
	20		1.1268	1.4285	0.5996	(0.7013,1.6538)	(1.3670,1.5867)	(0.5363,0.6536)
			0.1448	0.0056	0.0054	91.67	98.28	95.26
35	30	II	1.0937	1.3864	0.6285	(0.7445,1.5100)	(1.3193,1.5473)	(0.4945,0.6802)
			0.0947	0.0036	0.0039	93.34	97.92	94.71
	25		1.1016	1.5316	0.6271	(0.7090,1.6057)	(1.4670,1.5957)	(0.5677,0.6898)
			0.1032	0.0045	0.0043	96.52	94.87	98.17
	20		1.1903	1.4425	0.6124	(0.7225,1.6031)	(1.4393,1.5412)	(0.5590,0.7799)
			0.1197	0.0053	0.0052	95.52	97.38	94.98
35	30	III	1.1313	1.4417	0.5805	(0.7677,1.5677)	(1.1922,1.5919)	(0.4860,0.6914)
			0.1036	0.0044	0.0043	97.25	97.71	98.89
	25		1.0598	1.4528	0.5995	(0.6903,1.5167)	(1.2983,1.5885)	(0.4394,0.6558)
			0.1097	0.0061	0.0051	95.72	98.96	99.23
	20		1.1621	1.3969	0.6957	(0.6128,1.5123)	(1.3442,1.5535)	(0.6402,0.7499)
			0.1134	0.0072	0.0061	93.62	96.82	97.37
45	40	I	1.0688	1.6145	0.6384	(0.7605,1.4362)	(1.6549,1.7673)	(0.4785,0.6978)
			0.0643	0.0031	0.0033	95.82	98.51	98.92
	35		1.0511	1.4782	0.6507	(0.6782,1.4213)	(1.4198,1.5385)	(0.5978,0.7018)
			0.0795	0.0037	0.0036	98.36	96.26	98.12
	30		1.1122	1.5901	0.6136	(0.7482,1.5504)	(1.4343,1.7429)	(0.5529,0.6678)
			0.0811	0.0041	0.0041	98.06	96.82	99.24
45	40	II	1.0838	1.3897	0.6041	(0.8031,1.6872)	(1.3332,1.5415)	(0.5475,0.7507)
			0.0753	0.0029	0.0029	95.81	97.96	95.82
	35		1.0285	1.4579	0.6832	(0.7181,1.3987)	(1.3987,1.6117)	(0.4396,0.7445)
			0.0787	0.0034	0.0032	98.82	96.72	98.26
	30		1.1036	1.4003	0.7111	(0.7567,1.8553)	(1.3434,1.6603)	(0.5398,0.7763)
			0.0827	0.0039	0.0039	96.27	98.17	95.92
45	40	III	1.0897	1.5527	0.6301	(0.7212,1.3455)	(1.3942 ,1.7063)	(0.5021,0.7008)
			0.0682	0.0035	0.0035	98.95	95.29	94.98
	35		1.1104	1.5682	0.6613	(0.7381,1.5978)	(1.4288,1.7429)	(0.5142,0.7209)
			0.0727	0.0043	0.0039	96.37	98.71	96.26
	30		0.9605	1.5256	0.6638	(0.6504,1.3307)	(1.4719,1.5808)	(0.5180,0.7131)
			0.0758	0.0052	0.0045	98.45	97.68	96.88
55	50	I	1.0136	1.5347	0.7178	(0.7348,1.3542)	(1.4787,1.5844)	(0.5685,0.7847)
			0.0234	0.0017	0.0028	96.47	97.85	97.73
	45		1.0372	1.4638	0.6894	(0.7540,1.3722)	(1.4120,1.6150)	(0.4319,0.7364)
			0.0341	0.0024	0.0031	95.67	96.33	97.88
	40		1.1205	1.4094	0.6985	(0.7735,1.5378)	(1.3500,1.5695)	(0.4944,0.7521)
			0.0503	0.0032	0.0037	95.82	97.35	96.68
55	50	II	1.0186	1.5314	0.6136	(0.8179,1.5502)	(1.3502,1.5762)	(0.4656,0.7540)
			0.0212	0.0015	0.0025	95.36	98.07	96.17
	45		1.0922	1.5271	0.6554	(0.7839,1.4633)	(1.3666,1.5838)	(0.4981,0.7116)
			0.0254	0.0023	0.0029	94.82	98.36	96.59
	40		1.1086	1.5789	0.6236	(0.7829,1.2316)	(1.3125,1.6526)	(0.4683,0.7755)
			0.0324	0.0029	0.0033	97.38	95.97	94.92
55	50	III	1.0489	1.5213	0.6639	(0.7769,1.3641)	(1.3225,1.6223)	(0.5171,0.7107)
			0.0227	0.0021	0.0028	98.88	95.91	95.63
	45		1.0796	1.4684	0.6811	(0.896,1.3754)	(1.4219,1.6864)	(0.4276,0.7378)
			0.0322	0.0027	0.0036	96.96	98.37	96.84
	40		1.0598	1.5638	0.6177	(0.7564,1.4197)	(1.4138,1.6201)	(0.3570,0.6760)
			0.0408	0.0034	0.0039	97.93	97.85	95.55

Table 4: Average estimates, MSEs, CIs and CPs for various combinations of (n, m) by Bayes estimation using PS

n	m	R	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	CI(α)	CI(β)	CI(γ)
35	30	I	1.0512	1.4075	0.5854	(0.5171,1.2482)	(1.4321,1.6812)	(0.4906,0.6532)
			0.0823	0.0025	0.0033	98.27	95.18	95.92
	25		0.9053	1.4177	0.5739	(0.4499,1.1535)	(1.4217,1.6366)	(0.5228,0.6601)
			0.0924	0.0031	0.0036	95.88	97.38	98.79
	20		0.8772	1.4953	0.5649	(0.5175,1.1285)	(1.3451,1.6454)	(0.4994,0.6561)
			0.1061	0.0038	0.0041	99.03	95.27	96.83
35	30	II	1.0731	1.5373	0.6018	(0.6837,1.1898)	(1.3763,1.5916)	(0.4439,0.6601)
			0.0749	0.0021	0.0031	96.96	98.63	97.95
	25		1.1606	1.4261	0.5816	(0.5321,1.2180)	(1.4056,1.5211)	(0.5134,0.6537)
			0.0834	0.0027	0.0036	97.92	97.65	95.85
	20		0.8729	1.4943	0.6166	(0.4188,1.1923)	(1.4267,1.5655)	(0.5848,0.6503)
			0.0947	0.0032	0.0039	95.16	99.37	96.77
35	30	III	1.0725	1.4409	0.5829	(0.4027,1.1357)	(1.3746,1.5157)	(0.5280,0.6643)
			0.0946	0.0027	0.0037	97.96	95.88	98.91
	25		0.9706	1.4832	0.6992	(0.5354,1.0930)	(1.4656,1.5878)	(0.6397,0.7541)
			0.1021	0.0031	0.0041	95.29	98.55	97.63
	20		1.1024	1.5262	0.6867	(0.5161,1.3727)	(1.4562,1.5664)	(0.6172,0.7505)
			0.1093	0.0041	0.0048	98.47	99.26	96.91
45	40	I	1.0556	1.4121	0.6717	(0.6027,1.0794)	(1.3524,1.5613)	(0.6421,0.7475)
			0.0539	0.0016	0.0023	95.72	98.36	95.99
	35		0.8994	1.5066	0.6794	(0.6366,1.1969)	(1.4503,1.6602)	(0.6116,0.7252)
			0.0645	0.0019	0.0027	97.74	95.85	99.63
	30		1.1851	1.5216	0.6259	(0.6270,1.1419)	(1.4550,1.5920)	(0.5727,0.6860)
			0.0679	0.0023	0.0031	95.79	98.16	97.19
45	40	II	1.0883	1.5283	0.6303	(0.6450,1.1543)	(1.4707,1.5775)	(0.5783,0.6733)
			0.0582	0.0019	0.0021	95.09	97.61	98.87
	35		1.0353	1.4171	0.6331	(0.5839,1.0928)	(1.3648,1.4706)	(0.5752,0.6891)
			0.0623	0.0021	0.0036	98.94	95.82	97.18
	30		1.0516	1.5806	0.6089	(0.5909,1.1047)	(1.5220,1.6384)	(0.5512,0.6748)
			0.0732	0.0026	0.0041	99.01	98.52	95.99
45	40	III	1.0812	1.3849	0.7135	(0.5863,1.1255)	(1.3416,1.4280)	(0.6123,0.7674)
			0.0613	0.0019	0.0028	95.26	98.88	97.71
	35		0.9812	1.4133	0.6973	(0.6202,1.1353)	(1.40613,1.5016)	(0.5201,0.7157)
			0.0749	0.0021	0.0031	97.91	99.16	95.86
	30		1.1336	1.5082	0.6731	(0.6581,1.1962)	(1.4468,1.5720)	(0.6202,0.7254)
			0.0812	0.0025	0.0038	96.58	97.91	95.82
55	50	I	1.0878	1.4397	0.6632	(0.6311,1.1406)	(1.3676,1.4700)	(0.5558,0.6715)
			0.0133	0.0011	0.0018	98.92	97.31	95.83
	45		1.1272	1.4662	0.6553	(0.6080,1.1823)	(1.4065,1.5359)	(0.5975,0.7085)
			0.0173	0.0012	0.0024	94.99	98.97	98.12
	40		1.2336	1.4655	0.6593	(0.6304,1.1964)	(1.4140,1.5186)	(0.6018,0.7068)
			0.0274	0.0015	0.0026	97.51	96.16	96.93
55	50	II	0.9616	1.5055	0.6962	(0.7109,1.2155)	(1.4606,1.5521)	(0.6305,0.7537)
			0.0254	0.0009	0.0014	95.05	98.62	97.35
	45		1.1023	1.4182	0.6814	(0.7550,1.1645)	(1.3648,1.4725)	(0.6275,0.7364)
			0.0312	0.0012	0.0018	99.86	97.15	95.72
	40		1.1065	1.5062	0.6644	(0.7255,1.1601)	(1.4446,1.5662)	(0.5982,0.7002)
			0.0419	0.0013	0.0023	95.93	97.45	98.23
55	50	III	1.0521	1.4722	0.6612	(0.6523,1.1427)	(1.4119,1.5281)	(0.5673,0.6948)
			0.0265	0.0013	0.0017	98.52	95.65	97.41
	45		1.1015	1.4963	0.6915	(0.6148,1.1573)	(1.4483,1.5430)	(0.6211,0.7230)
			0.0289	0.0013	0.0019	95.14	98.52	98.05
	40		0.9546	1.4746	0.6379	(0.5909,1.1145)	(1.4154,1.5274)	(0.6196,0.7160)
			0.0345	0.0016	0.0026	96.64	98.07	97.82

of the fitted distributions. It can be seen that the MLIIFr distribution provides a reasonable fit to the dataset among the other distributions fitted since it has the least AIC, BIC, K-S, A^*, W^* and $-2\log L$ values. Also, the p value is the maximum in the case of MLIIFr distribution than others. Figure 4 shows the QQ, empirical, and histogram plots for the Data. Using QQ plots, data can be assessed by comparing sample quantiles (blue dots) with theoretical quantiles (red line). An alignment of the points along the red line indicates a good fit. Yellow shading indicates the 95% bootstrap interval, which indicates the range of expected variation. Deviations within expected variability lie in this area, while those beyond are outside the theoretical distribution. As we can see from the QQ plot, most points are within the bootstrap interval. Furthermore, the empirical CDF plot shows that the data fits the MLIIFr distribution well. As seen in the histogram, there is a peak at the lower values followed by a long tail at the higher values. Additionally, the PDF plot shows that MLIIFr distribution is useful in modelling data with heavy tails and asymmetry.

To study under Type-II PCS, we assumed $m = 31$. We have noted the reading for all censoring schemes R discussed in section 6. Both classical and Bayesian methods have been used to calculate the estimates for data, and results are noted in Table 8. We can see that CI has a shorter interval length than ACI. The real data estimates are consistent with the simulation study. Further, we generated 15000 MCMC samples of (α, β, γ) using LF for trace and histogram plots based on all three schemes. Each trace and histogram plot presented in Figures (5)-(7), the symmetric Bayes estimate and its bounds of 95% CI are expressed by solid red horizontal lines. In addition, the Bayes estimates are plotted with dashed blue vertical lines in histogram plots. As a result, we can conclude that the proposed estimation methodologies based on data provide a good demonstration of the modified Lehmann Type-II Fréchet distribution lifetime model.

Table 5: Competing distributions

Model	CDF
1. Weibull	$F(x) = 1 - e^{-\beta x^{\alpha}}, \quad x \geq 0, \alpha, \beta > 0.$
2. Fréchet	$F(x) = e^{-\alpha x^{-\beta}}, \quad x \geq 0, \alpha, \beta > 0.$
3. Dagum	$F(x) = (1 + \lambda x^{-\alpha})^{-\beta}, \quad x \geq 0, \alpha, \beta, \lambda > 0.$
4. WE	$F(x) = 1 - e^{-\left(\frac{\alpha\beta x}{\gamma}\right)^{k}}, \quad x \geq 0, \alpha, \beta, \gamma, k > 0.$
5. MLII-Ex	$F(x) = 1 - \left(\frac{e^{-\theta x}}{1-\alpha+\alpha e^{-\theta x}}\right)^{\beta}, \quad x \geq 0, -\infty < \alpha < 1, \theta, \beta > 0.$
6. Power-Lomax	$F(x) = 1 - \lambda^{\alpha}\left(x^{\beta} + \lambda\right)^{-\alpha}, \quad x \geq 0, \alpha, \beta, \lambda > 0.$

Table 6: Japan's infant mortality rate from 1976 to 2022

12.5	11.8	11.1	10.5	9.9	9.3	8.8	8.3	7.9	7.5	7.1	6.8	6.6	6.5	6.3	6.2
6.1	6.0	5.9	5.7	5.5	5.2	5.0	4.7	4.5	4.3	4.1	4.0	3.9	3.7	3.6	3.5
3.4	3.3	3.2	3.2	3.0	2.9	2.8	2.8	2.7	2.6	2.5	2.5	2.4	2.3	2.3	

Table 7: Information criteria and goodness of fit of Data

Model	$-2\log L$	AIC	BIC	W^*	A^*	KS (p value)
MLIIFr	216.6489	222.6458	228.1962	0.1070	0.6987	0.0852(0.8838)
Dagum	232.9447	228.9447	264.4952	0.1270	0.8120	0.1099(0.6230)
Fréchet	251.5899	255.5899	259.2902	0.1082	0.8279	0.2663(0.0025)
Power-Lomax	217.6711	223.6711	229.2215	0.1097	0.7848	0.1066(0.6597)
WE	219.6698	227.6698	235.0704	0.1378	0.9619	0.1141(0.5734)
Weibull	219.6697	223.6697	229.3700	0.1378	0.9613	0.1145(0.5692)
MLII-Ex	252.9192	258.9198	264.4698	1.8926	10.1589	0.3554(0.0002)

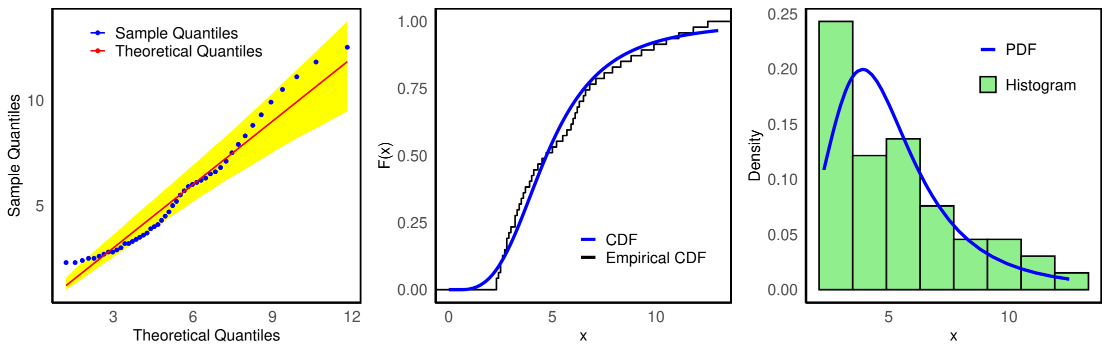

Figure 4: QQ plot (left), empirical CDF plot (centre) and histogram plot (right) for Data

Table 8: Parameter estimates based on PCS for Data using the proposed methods

Scheme	Parameter	MLE	MPSE	INIP(LF)	INIP(PS)	CI(LF)	CI(PS)	ACI
I	α	0.9641	1.1603	0.9304	0.9936	(0.6176,1.4191)	(0.7797,1.1692)	(0.0000,2.8617)
	β	3.5462	3.4273	3.5231	3.2216	(3.0917,3.9954)	(3.0273,3.4446)	(0.9289,6.1632)
	γ	0.9968	0.9971	0.9959	0.9984	(0.9921,0.9985)	(0.9977,0.9991)	(0.9912,0.9999)
II	α	0.1408	0.1406	0.1392	0.1252	(0.1128,0.1734)	(0.1023,0.1520)	(0.0721,0.2094)
	β	5.2861	5.3054	5.2681	5.2712	(4.7727,5.7815)	(5.2385,5.2996)	(4.2133,6.3587)
	γ	0.9967	0.9971	0.9957	0.9975	(0.9912,0.9985)	(0.9966,0.9985)	(0.9917,0.9999)
III	α	0.2254	0.2287	0.2227	0.2601	(0.1658,0.3076)	(0.2306,0.2981)	(0.0579,0.3928)
	β	4.9853	4.9839	5.0027	4.9553	(4.3076,5.7748)	(4.9408,4.9818)	(2.7445,7.2262)
	γ	0.9971	0.9973	0.9963	0.9981	(0.9917,0.9989)	(0.9973,0.9986)	(0.9908,0.9999)

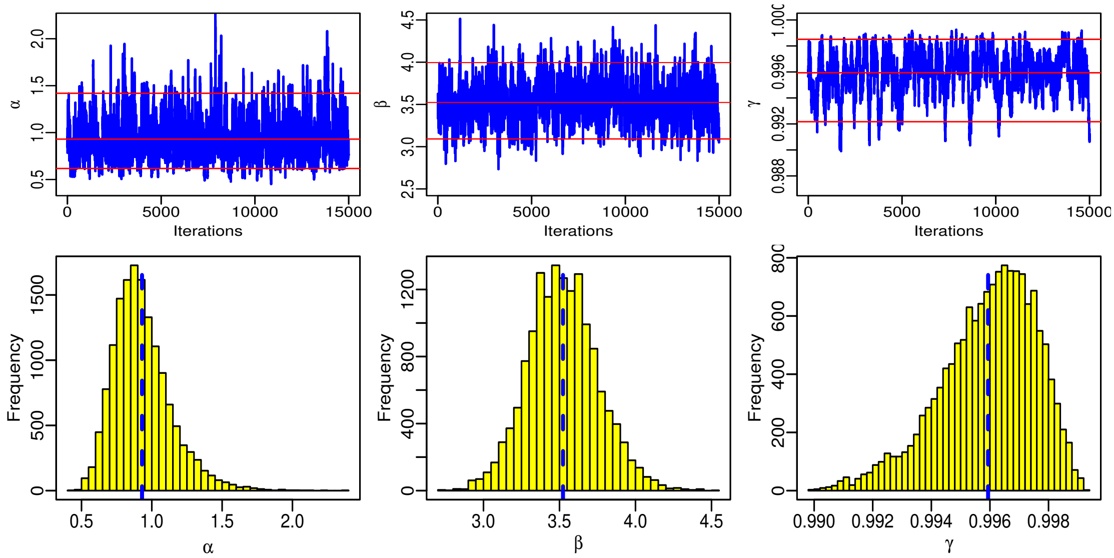

Figure 5: Trace and histogram plots of parameters for scheme I using INIP via LF

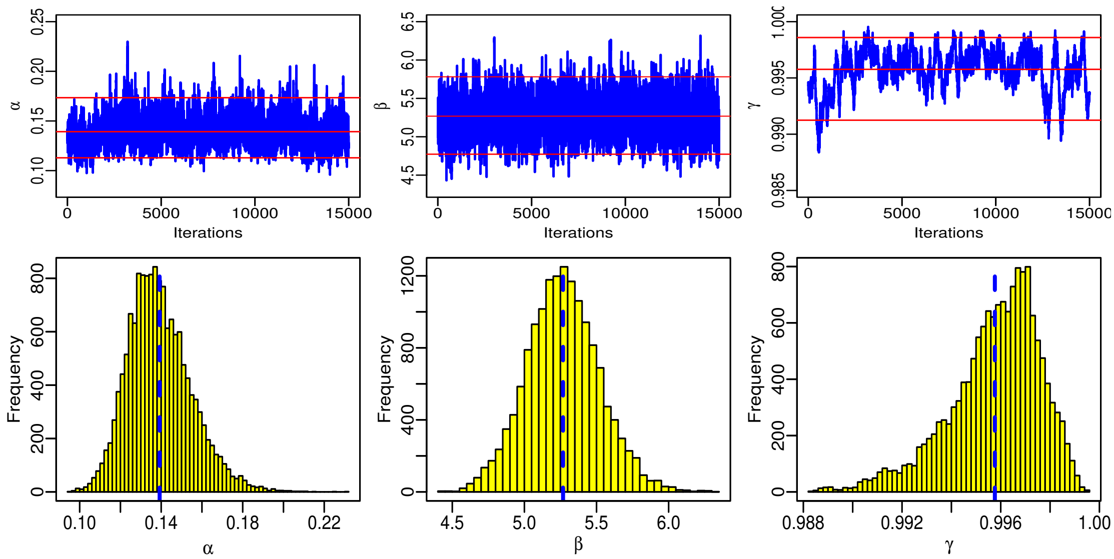

Figure 6: Trace and histogram plots of parameters for scheme II using INIP via LF

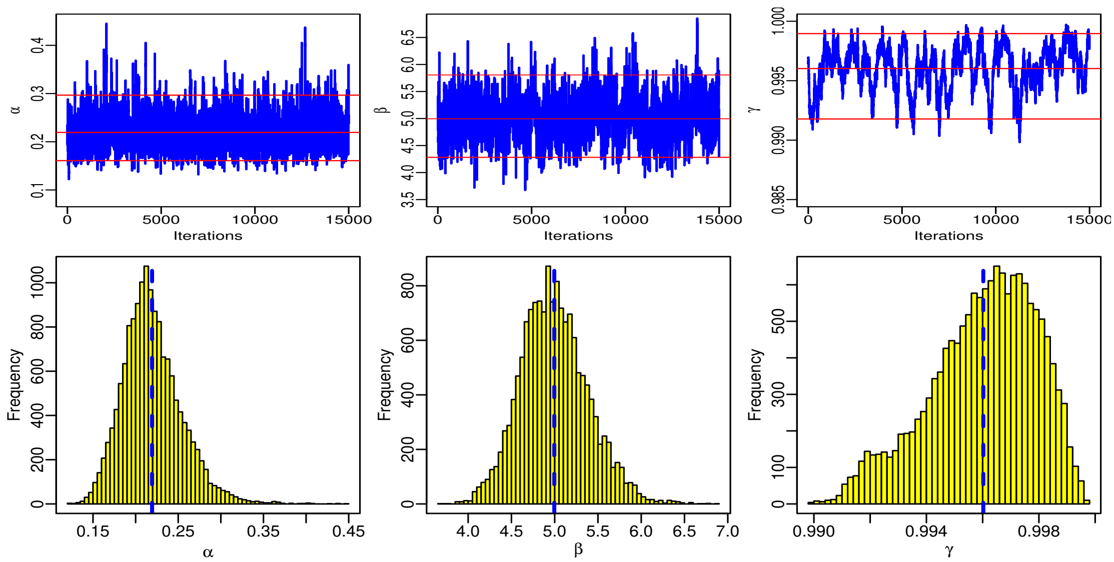

Figure 7: Trace and histogram plots of parameters for scheme III using INIP via LF

8 Conclusion

This study introduced a new three-parameter Fréchet distribution known as the Modified Lehmann Type-II Fréchet (MLIIFr) distribution via the modified Lehmann Type-II class of distribution. Some essential properties of the proposed distribution are studied. The model parameters have been estimated under a Type-II progressive censoring scheme using multiple estimation techniques, including MLE, MPSE, and Bayesian estimation with INIP via likelihood and product spacing functions. Additionally, ACIs and CIs have been obtained using MLE and INIP-based approaches. Monte Carlo simulations were employed to derive and compare the inferential results. Our findings indicate that the Bayesian estimation method via product spacing yielded the lowest MSEs across the various estimation techniques. Regarding interval estimation, the CIs showed shorter lengths than the ACIs, while CPs were satisfactory for both. The applicability of the MLIIFr distribution has been demonstrated using real-life data on the mortality rates of Japanese children under five years of age. Furthermore, the goodness of fit of the model was compared with five other distributions, and the MLIIFr distribution provided a superior fit to the data. This indicates that the proposed distribution can be effectively used to analyze real-world data sets. As a direction for future research, the inferential methodologies proposed in this study can be extended to other lifetime models by applying different censoring schemes, such as block censoring and unified progressive hybrid censoring. These alternative schemes may provide further insights and enhance the applicability of the proposed approaches across various statistical models and real-world data sets.

Funding

This research received no external funding.

Data Availability

Data used in this study are publicly available and cited in this manuscript.

Conflict of Interest

The authors declare no conflict of interest.

Acknowledgment

The authors would like to sincerely thank the anonymous reviewers and the editor for their valuable comments and constructive suggestions, which have greatly helped in improving the quality and clarity of this manuscript.

References

Ahsanullah, M., Shakil, M., and Golam Kibria, B. M. G. (2013). A characterization of the power function distribution based on lower records. In *ProbStat Forum*, volume 6, pages 68–72.

Al-Babtain, A. A., Elbatal, I., and Yousof, H. M. (2020). A new three parameter fréchet model with mathematical properties and applications. *Journal of Taibah University for Science*, 14(1):265–278.

Anatolyev, S. and Kosenok, G. (2005). An alternative to maximum likelihood based on spacings. *Econometric Theory*, 21(2):472–476.

Arshad, M. Z., Iqbal, M. Z., Anees, A., Ahmad, Z., and Balogun, O. S. (2021). A new bathtub shaped failure rate model: Properties, and applications to engineering sector. *Pakistan Journal of Statistics*, 37(1):57–80.

Awodutire, P. O., Nduka, E. C., and Ijomah, M. A. (2020). Lehmann type ii generalized half logistic distribution: Properties and application. *Mathematical Theory and Modeling*, 10(2):103–115.

Badmus, N. I., Bamiduro, T. A., and Ogunobi, S. G. (2014). Lehmann type ii weighted weibull distribution. *International Journal of Physical Sciences*, 9(4):71–78.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Springer Science & Business Media.

Balakrishnan, N. and Sandhu, R. A. (1995). A simple simulational algorithm for generating progressive type-ii censored samples. *The American Statistician*, 49(2):229–230.

Balogun, O. S., Arshad, M. Z., Iqbal, M. Z., and Ghamkhar, M. (2021). A new modified lehmann type-ii g class of distributions: Exponential distribution with theory, simulation, and applications to engineering sector. *F1000Research*, 10:483.

Barreto-Souza, W., Cordeiro, G. M., and Simas, A. B. (2011). Some results for beta fréchet distribution. *Communications in Statistics–Theory and Methods*, 40(5):798–811.

Chakraborty, S., Handique, L., Altun, E., and Yousof, H. M. (2019). A new statistical model for extreme values: Properties and applications. *International Journal of Open Problems in Computer Mathematics*, 12(1).

Chang, S. K. (2007). Characterizations of the power function distribution by the independence of record values. *Journal of the Chungcheong Mathematical Society*, 20(2):139–139.

Cheng, R. C. H. and Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B (Methodological)*, 45(3):394–403.

Cohen, A. C. (1963). Progressively censored samples in life testing. *Technometrics*, 5(3):327–339.

Cordeiro, G. M. and De Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 81(7):883–898.

Csiszár, I. and Körner, J. (2011). *Information Theory: Coding Theorems for Discrete Memoryless Systems*. Cambridge University Press.

Dallas, A. C. (1976). Characterizing the pareto and power distributions. *Annals of the Institute of Statistical Mathematics*, 28:491–497.

Elshahhat, A., Bhattacharya, R., and Mohammed, H. S. (2022). Survival analysis of type-ii lehmann fréchet parameters via progressive type-ii censoring with applications. *Axioms*, 11(12):700.

Gupta, R. C., Gupta, P. L., and Gupta, R. D. (1998). Modeling failure time data by lehman alternatives. *Communications in Statistics–Theory and Methods*, 27(4):887–904.

Isa, A. M., Kaigama, A., Adepoju, A. A., and Bashiru, S. O. (2023). Lehmann type ii–lomax distribution: Properties and application to real data set. *Communication in Physical Sciences*, 9(1).

Kundu, D. (2008). Bayesian inference and life testing plan for the weibull distribution in presence of progressive censoring. *Technometrics*, 50(2):144–154.

Lehmann, E. L. (1953). The power of rank tests. *Annals of Mathematical Statistics*, pages 23–43.

Mahto, A. K., Tripathi, Y. M., and Kızılaslan, F. (2020). Estimation of reliability in a multicomponent stress–strength model for a general class of inverted exponentiated distributions under progressive censoring. *Journal of Statistical Theory and Practice*, 14:1–35.

Maurya, R. K., Tripathi, Y. M., Sen, T., and Rastogi, M. K. (2019). Inference for an inverted exponentiated pareto distribution under progressive censoring. *Journal of Statistical Theory and Practice*, 13:1–32.

Meniconi, M. and Barry, D. M. (1996). The power function distribution: A useful and simple distribution to assess electrical component reliability. *Microelectronics Reliability*, 36(9):1207–1212.

Naveed-Shahzad, M., Asghar, Z., Shehzad, F., and Shahzadi, M. (2015). Parameter estimation of power function distribution with tl-moments. *Revista Colombiana de Estadística*, 38(2):321–334.

Ng, H. K. T., Luo, L., Hu, Y., and Duan, F. (2012). Parameter estimation of three-parameter weibull distribution based on progressively type-ii censored samples. *Journal of Statistical Computation and Simulation*, 82(11):1661–1678.

Ogunde, A. A., Fayose, S. T., Ajayi, B., and Omosigho, D. O. (2020). Extended gumbel type-2 distribution: Properties and applications. *Journal of Applied Mathematics*, 2020(1):2798327.

Prakash, A., Maurya, R. K., Alsadat, N., and Obulezi, O. J. (2024). Parameter estimation for reduced type-i heavy-tailed weibull distribution under progressive type-ii censoring scheme. *Alexandria Engineering Journal*, 109:935–949.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pages 547–562, Berkeley, CA. University of California Press.

Salah, M., El-Morshedy, M., Eliwa, M. S., and Yousof, H. M. (2020). Expanded fréchet model: Mathematical properties, copula, different estimation methods, applications and validation testing. *Mathematics*, 8(11):1949.

Shahbaz, M. Q., Shahbaz, S., and Butt, N. S. (2012). The kumaraswamy-inverse weibull distribution. *Pakistan Journal of Statistics and Operation Research*, 8(3):479–489.

Shannon, C. E. (1948). A mathematical theory of communication. *The Bell System Technical Journal*, 27(3):379–423.

Tomazella, V. L., Ramos, P. L., Ferreira, P. H., Mota, A. L., and Louzada, F. (2022). The lehmann type ii inverse weibull distribution in the presence of censored data. *Communications in Statistics–Simulation and Computation*, 51(12):7057–7073.

Yousof, H. M., Jahanshahi, S. M. A., Ramires, T. G., Aryal, G. R., and Hamedani, G. G. (2018). A new distribution for extreme values: Regression model, characterizations and applications. *Journal of Data Science*, 16(4):677–706.