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# Derivation of Incomplete gamma functions properties with applications

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In this study, we investigate novel identities involving the incomplete gamma function through the application of probabilistic techniques. By examining the distribution of order statistics derived from the gamma distribution, we establish integral identities that incorporate expressions of the incomplete gamma function. Incomplete Gamma function identities can be derived by integrating out order statistic densities. These findings provide deeper insight into the function's analytical structure and hold practical relevance. Notably, we leverage these results to construct bivariate gamma distributions, demonstrating their utility in statistical modeling.

**keywords:** Gamma distributions, order statistics, hypergeometric function, Whittaker function, bivariate distributions, bivariate Gamma distributions .

## 1 Introduction

The gamma function and its variants play a fundamental role in engineering, physics, and other disciplines where special functions are applied. They also appear frequently in discrete mathematics, number theory, and various branches of the sciences. In particular, the incomplete gamma functions have found wide applications in fields such as physics, reliability engineering, probability theory, and statistics. For example, in Alahmad and Abdelhadi (2019) and AlAhmad (2021), the gamma function was used to define the fractional derivative of analytic functions.

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The study of these functions dates back to 1877, when Prym first examined them; as a result, the function $(a, x)$ is sometimes referred to as Prym's function. However, many assertions concerning these functions are made without supporting evidence in Olver et al. (2010).

## 1.1 Incomplete gamma functions

Throughout the 20th century, the incomplete gamma function was instrumental in advancing the theory of special functions and their applications across physics, engineering, and finance. Prominent researchers such as Karl Pearson (see Pearson (1900) ) and Harold Jeffreys (see Jeffreys (1939)) contributed to this development. Pearson applied the incomplete gamma function to model measurement errors, while Jeffreys used it to study radioactive decay. Today, the function remains an essential tool across diverse areas including statistical analysis, quantum mechanics, and image processing and many other fields. For example, in R. AlAhmad (2022), Al-Ahmad et al. (2018), AlAhmad et al. (2024), AlAhmad (2025), Alahmad (2023), and Al-Ahmad et al. (2020), the incomplete gamma function was used to find the derivative and the fractional derivative of functions of certain forms.

The incomplete gamma function also plays a significant role in modeling complex phenomena. For example, it has been used in the study of power-law relaxation times in complex physical systems (Sornette, 1998), logarithmic oscillations in protein relaxation dynamics (Metzler et al., 1999), and Gaussian and exponential orbitals in quantum chemistry (Shavitt, 1963; Shavitt and Karplus, 1965).

Moreover, integrals involving products and powers of incomplete gamma functions are of considerable importance in both scientific research and engineering practice. For recent studies and extended properties, see Dimov et al. (2021), AlAhmad (2016a), and AlAhmad (2016b). In this section, will evaluate complex integrals and sums involving gamma functions and incomplete gamma functions by applying properties of order Statistics.

For $\Re(\alpha) > 0$, the gamma function denoted by $\Gamma(\alpha)$, is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\, dz. \tag{1}$$

Now, for $\Re(\alpha) > 0$, the *upper incomplete gamma function*, denoted by $\Gamma(\alpha, t)$, is defined for $t \geq 0$ as

$$\Gamma(\alpha, t) = \int_t^\infty z^{\alpha-1} e^{-z}\, dz. \tag{2}$$

In addition, for $\Re(\alpha) > 0$, the *lower incomplete gamma function*, denoted $\gamma(\alpha, t)$, is defined for $t \geq 0$ as

$$\gamma(\alpha, t) = \int_0^t z^{\alpha-1} e^{-z}\, dz. \tag{3}$$

It follows from (1), (2), and (3) that

$$\gamma(\alpha, t) + \Gamma(\alpha, t) = \Gamma(\alpha). \tag{4}$$

If $\alpha$ is a positive integer, then the upper incomplete function is given by:

$$\Gamma(\alpha,t)=e^{-t}(\alpha-1)!\sum_{m=0}^{\alpha-1}\frac{t^m}{m!}. \tag{5}$$

The following recurrence relation can be proved by integration by parts

$$\Gamma(\alpha+1,t)=e^{-t}t^{\alpha}+\alpha\Gamma(\alpha,t). \tag{6}$$

For any $\Re(\alpha)>0$,

$$\Gamma(\alpha)\Gamma\left(\alpha+\frac{1}{2}\right)=\sqrt{\pi}2^{1-2\alpha}\Gamma(2\alpha). \tag{7}$$

This identity is called the Legendre's duplication formula.
The following result is an interesting relation between gamma function and incomplete gamma function which will simplify integrals involving products of incomplete gamma functions.

**Lemma 1.1.** *If $n$ is a positive integer, then*

$$\int_0^{\infty} e^{-t}t^{\alpha-1}(\Gamma(\alpha,t))^{n-1}\,dt=\frac{(\Gamma(\alpha))^n}{n}. \tag{8}$$

*Proof.* Note that

$$\frac{d}{dt}(\Gamma(\alpha,t))^n=n\left(-e^{-t}\right)t^{\alpha-1}(\Gamma(\alpha,t))^{n-1}.$$

Therefore, since

$$\lim_{t\to\infty}(\Gamma(\alpha,t))^n=0 \quad \text{and } (\Gamma(\alpha,0))^n=\Gamma(\alpha)^n,$$

then

$$\begin{aligned}\int_0^{\infty} e^{-t}t^{\alpha}\Gamma(\alpha,t)\,dt&=-\frac{1}{n}\int_0^{\infty}\frac{d}{dt}(\Gamma(\alpha,t))^n dt\\ &=-\frac{1}{n}\Big(\lim_{t\to\infty}(\Gamma(\alpha,t))^n-(\Gamma(\alpha,0))^n\Big)=\frac{(\Gamma(\alpha))^n}{n}.\end{aligned}$$

□

## 1.2 Incomplete beta function

For $\Re(a)>0$, $\Re(b)>0$, and $0\le x\le 1$, the lower incomplete beta function is defined by

$$\beta(a,b;x)=\int_0^x t^{a-1}(1-t)^{b-1}\,dt.$$

Clearly, $\beta(a,b;1)=\beta(a,b)$ .
The upper incomplete Beta function is defined by

$$B(a,b;x)=\int_x^1 t^{a-1}(1-t)^{b-1}\,dt.$$

Clearly, $B(a,b;0)=\beta(a,b)$. For more applications of incomplete beta functions, see Alahmad and Almefleh (2020).

### 1.3 The Whittaker function and hypergeometric functions

For $\Re\left(k-\frac{1}{2}-m\right) \leq 0$ and $k-\frac{1}{2}-m$ is not an integer, The Whittaker function is defined

$$W_{k,m}(s) = \frac{e^{\frac{-s}{2}} s^k}{\Gamma\left(\frac{1}{2}-k+m\right)} \int_0^\infty t^{-k-\frac{1}{2}+m} \left(1+\frac{t}{s}\right)^{k-\frac{1}{2}+m} e^{-t} dt. \tag{9}$$

For example, the following integral is need later

$$\int_0^1 v^{d-a-b-1}(1-v)^{h-1} \exp\left(-\frac{x}{\mu v}\right) dv = \left(\frac{x}{\mu}\right)^{a+b-d} e^{-\frac{x}{2\mu}} W_{a+b-d,\, \frac{a+b-d-h}{2}} \left(\frac{x}{\mu}\right). \tag{10}$$

Also, For $\Re(a) > 0, \Re(c-a) < 0$, the integral representations of the Confluent and Gauss hypergeometric function are given as:

$${}_1F_1(a,c;s) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1} e^{su} du. \tag{11}$$

and

$${}_2F_1(a,b,c;s) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-su)^{-b} e^{su} du. \tag{12}$$

respectively. The hypergeometric function ${}_2F_1$ satisfies the following proposition.

**Proposition 1.1.** *For positive numbers $p, c$ and $a$,*

$$\int_0^\infty e^{-pt} t^{c-1} \Gamma(a,t) dt = \frac{\Gamma(a+c)}{c} {}_2F_1(a, c; c+1; -1/p).$$

*Proof. Proof.* Recall the upper incomplete gamma function:

$$\Gamma(a,t) = \int_t^\infty e^{-x} x^{a-1}\, dx.$$

Substituting this into the integral gives:

$$I := \int_0^\infty e^{-pt} t^{c-1} \left(\int_t^\infty e^{-x} x^{a-1}\, dx\right) dt.$$

We interchange the order of integration (justified by Fubini's theorem, since the integrand is positive for $t, x > 0$):

$$I = \int_0^\infty e^{-x} x^{a-1} \left(\int_0^x e^{-pt} t^{c-1}\, dt\right) dx.$$

Let

$$J(x) := \int_0^x e^{-pt} t^{c-1}\, dt.$$

We use the substitution $t = xy$, so $dt = x\, dy$ and $0 \leq y \leq 1$:

$$J(x) = x^c \int_0^1 e^{-pxy} y^{c-1}\, dy.$$

Substituting back into the outer integral:

$$I = \int_0^\infty e^{-x} x^{a-1} x^c \left( \int_0^1 e^{-pxy} y^{c-1}\, dy \right) dx.$$

$$I = \int_0^1 y^{c-1} \left( \int_0^\infty e^{-x(1+py)} x^{a+c-1}\, dx \right) dy.$$

The inner integral is a standard gamma integral:

$$\int_0^\infty e^{-qx} x^{\lambda-1}\, dx = \frac{\Gamma(\lambda)}{q^\lambda}, \quad \Re(q) > 0.$$

Here, $q = 1 + py > 0$ and $\lambda = a + c > 0$. Thus:

$$\int_0^\infty e^{-x(1+py)} x^{a+c-1}\, dx = \frac{\Gamma(a+c)}{(1+py)^{a+c}}.$$

Substituting:

$$I = \Gamma(a+c) \int_0^1 \frac{y^{c-1}}{(1+py)^{a+c}}\, dy.$$

The final integral is known in terms of the hypergeometric function:

$$\int_0^1 \frac{y^{c-1}}{(1+py)^{a+c}}\, dy = \frac{1}{c}\, {}_2F_1(a, c; c+1; -1/p).$$

Substituting back:

$$I = \frac{\Gamma(a+c)}{c}\, {}_2F_1(a, c; c+1; -1/p).$$

□

□

In this paper, section 2 will introduce an approve to derive many properties of incomplete gamma functions using order statistic. Section 3 will be devoted to deploy properties of the special function mentioned above to derive bivariate probability distributions.

## 2 Order Statistics probability density functions

Many authors used probabilistic methods to prove identities and inequalities in combinatorial, engineering, and analysis, see (Miller and Moskowitz, 1998) and (Paris, 2003), to name the least.

$$f_X(x) = \frac{\beta^\alpha t^{\alpha-1} e^{\beta(-t)}}{\Gamma(\alpha)}, t > 0 \tag{13}$$

$$F_X(x) = 1 - \frac{\Gamma(\alpha, t\beta)}{\Gamma(\alpha)}, t > 0. \tag{14}$$

In the sequel, we assume that $\alpha > 0$, $\beta > 0$, $t > 0$, and $\gamma > 0$.

The probability density function for the $k$th order statistic corresponding to a random sample of size $n$ drawn from the distribution of a continuous random variable $X$ is given by

$$f_{X_{(k)}}(t) = \frac{n!}{(k-1)!(n-k)!}[F_X(t)]^{k-1}[1-F_X(t)]^{n-k}f_X(t). \tag{15}$$

where $f_X(t)$ and $F_X(t)$ are the probability density function (pdf) and the cumulative distribution function (cdf) of $X$, respectively. (Arnold et al., 1992). We now derive the probability density function of the kth order statistic from a Gamma distribution with $\alpha >$and $\beta = 1$ using substitution into the general formula (15). The result is:

$$f_{X_{(k)}}(t) = \sum_{j=0}^{k-1}(-1)^{k-j-1}\binom{k-1}{j}\binom{n}{k}\Gamma(\alpha)^{j-n}t^{\alpha-1}e^{-t}\Gamma(\alpha,t)^{n-j-1}, \tag{16}$$

where $0 < k \leq n, t > 0$.

The pdf of the $(k+1)$st order statistics is

$$f_{X_{(k+1)}}(t) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k-1}{j}\binom{n}{k+1}\Gamma(\alpha)^{j-n}t^{\alpha-1}e^{-t}\Gamma(\alpha,t)^{-j+n-1}, \tag{17}$$

where $0 \leq k \leq n-1$.

## 2.1 Applications of order Statistics to derive identities of the incomplete gamma functions

In this subsection, we deploy properties of order Statistics to derive identities related to incomplete gamma functions. Incomplete Gamma function identities can be derived by integrating out order statistic densities. For instance, the following identity proves an approach to evaluate $\int_0^\infty e^{-t}t^{\alpha-1}(\Gamma(\alpha+1,t))^2\,dt$.

**Identity 2.1.**

$$\int_0^\infty \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha+1,t)^2}{\alpha^2\Gamma(\alpha)^3}\,dt = \frac{2\Gamma(3\alpha+1)(3\,{}_2F_1(1,-\alpha;2\alpha+1;-2)-1)}{3^{3\alpha}\Gamma(\alpha+1)^3}+2, \tag{18}$$

*where ${}_2F_1$ is the hypergeometric function defined by* (12).

*Proof.* Letting $k = 2$ and $n = 3$ in (16), we obtain

$$f_{X(2)}(t) = \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)(\Gamma(\alpha)-\Gamma(\alpha,t))}{\Gamma(\alpha)^3} \tag{19}$$

$$= \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)}{\Gamma(\alpha)^2} - \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)^2}{\Gamma(\alpha)^3}. \tag{20}$$

The first term of (20) integrates to 3, by Lemma 1.1. That is,

$$\int_0^\infty \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)}{\Gamma(\alpha)^2}\,dt = 3. \tag{21}$$

But since

$$\int_0^\infty f_{X(2)}(t)\,dt = 1, \tag{22}$$

the second term of (20) should integrate to 2. That is,

$$\int_0^\infty \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)^2}{\Gamma(\alpha)^3}\,dt = 2. \tag{23}$$

Using (6) and expanding the expression of $f_{X(1)}(t)$, we see that

$$\begin{aligned} f_{X(1)}(t) = &-\frac{6e^{-3t}t^{3\alpha-1}}{\alpha^2\Gamma(\alpha)^3} - \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha+1,t)^2}{\alpha^2\Gamma(\alpha)^3} \\ &+ \frac{12e^{-2t}t^{2\alpha-1}\Gamma(\alpha+1,t)}{\alpha^2\Gamma(\alpha)^3} + \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)}{\Gamma(\alpha)^2}. \end{aligned} \tag{24}$$

Integrating with respect to $t$ over $(0,\infty)$, we see that

$$\int_0^\infty \frac{6e^{-3t}t^{3\alpha-1}}{\alpha^2\Gamma(\alpha)^3}\,dt = \frac{2\ 3^{1-3\alpha}\Gamma(3\alpha)}{\alpha^2\Gamma(\alpha)^3} \tag{25}$$

$$\int_0^\infty \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha,t)}{\Gamma(\alpha)^2}\,dt = 3 \tag{26}$$

$$\int_0^\infty \frac{12e^{-2t}t^{2\alpha-1}\Gamma(\alpha+1,t)}{\alpha^2\Gamma(\alpha)^3}\,dt = \frac{18\Gamma(3\alpha)\,{}_2F_1(2\alpha,3\alpha+1;2\alpha+1;-2)}{\alpha^2\Gamma(\alpha)^3}, \tag{27}$$

where the last integral follows from (6.455.1) of ((Gradshteyn and Ryzhik, 2007)).

Therefore,

$$\begin{aligned} \int_0^\infty \frac{6e^{-t}t^{\alpha-1}\Gamma(\alpha+1,t)^2}{\alpha^2\Gamma(\alpha)^3}\,dt =&\frac{18\Gamma(3\alpha)\,{}_2F_1(2\alpha,3\alpha+1;2\alpha+1;-2)}{\alpha^2\Gamma(\alpha)^3} \\ &- \frac{2\ 3^{1-3\alpha}\Gamma(3\alpha)}{\alpha^2\Gamma(\alpha)^3} + 3 - 1 \\ =&\frac{2\Gamma(3\alpha+1)(3\,{}_2F_1(1,-\alpha;2\alpha+1;-2)-1)}{3^{3\alpha}\Gamma(\alpha+1)^3} + 2. \end{aligned} \tag{28}$$

□

The following identity proves an approach to evaluate $\int_0^\infty e^{-t}t^\alpha\Gamma(\alpha,t)$.

**Identity 2.2.** *For $\alpha > 0$,*

$$\int_0^\infty e^{-t}t^\alpha\Gamma(\alpha,t)\,dt = \frac{\Gamma(\alpha+1)^2 - 2^{1-2\alpha}\alpha\Gamma(2\alpha)}{2\alpha}. \tag{29}$$

*Proof.* The $(n-1)$st order statistics from the Gamma distribution with shape parameter $\alpha+1$ has a pdf obtained from (16) as

$$f_{X(n-1)}(t) = (n-1)ne^{-t}t^\alpha\Gamma(\alpha+1)^{-n}\Gamma(\alpha+1,t)(\Gamma(\alpha+1)-\Gamma(\alpha+1,t))^{n-2}, t>0. \tag{30}$$

Letting $n = 2$ and expanding, this pdf becomes

$$f_{X(1)}(t) = \frac{2e^{-t}t^{\alpha}\Gamma(\alpha+1,t)}{\Gamma(\alpha+1)^2}. \tag{31}$$

Now using (6), we see that

$$\frac{2e^{-t}t^{\alpha}\left(e^{-t}t^{\alpha}+\alpha\Gamma(\alpha,t)\right)}{\Gamma(\alpha+1)^2} = \frac{2e^{-2t}t^{2\alpha}}{\Gamma(\alpha+1)^2} + \frac{2\alpha e^{-t}t^{\alpha}\Gamma(\alpha,t)}{\Gamma(\alpha+1)^2}. \tag{32}$$

But since

$$\int_0^{\infty} \frac{2e^{-2t}t^{2\alpha}}{\Gamma(\alpha+1)^2}\,dt = \frac{2^{1-2\alpha}\alpha\Gamma(2\alpha)}{\Gamma(\alpha+1)^2}, \tag{33}$$

we get

$$\int_0^{\infty} \frac{2\alpha e^{-t}t^{\alpha}\Gamma(\alpha,t)}{\Gamma(\alpha+1)^2}\,dt = 1 - \frac{2^{1-2\alpha}\alpha\Gamma(2\alpha)}{\Gamma(\alpha+1)^2}, \tag{34}$$

from which the identity follows. □

Using Identity (2.2) and (7), we get the following corollary.

**Corollary 2.1.** *For $\alpha > 0$,*

$${}_2F_1(\alpha+1, 2\alpha+1; \alpha+2; -1) = \frac{(\alpha+1)\Gamma(\alpha)\left(\Gamma(\alpha+1) - \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}}\right)}{2\Gamma(2\alpha+1)}. \tag{35}$$

*Proof.*

$${}_2F_1(\alpha+1, 2\alpha+1; \alpha+2; -1) = (\alpha+1)\int_0^1 \frac{x^{\alpha}}{(1+x)^{2\alpha+1}}\,dx$$

Now, using the integral tables in Prudnikov et al. (1986), we have

$$\int_0^1 \frac{x^{\alpha}}{(1+x)^{2\alpha+1}}\,dx = \frac{1}{2}\left(\beta(\alpha+1,\alpha) - \frac{\beta\left(\alpha+1,\frac{\alpha}{2}\right)}{\sqrt{\pi}}\right)$$

Therefore,

$${}_2F_1(\alpha+1, 2\alpha+1; \alpha+2; -1) = \frac{1}{2}(\alpha+1)\left(\beta(\alpha+1,\alpha) - \frac{\beta\left(\alpha+1,\frac{\alpha}{2}\right)}{\sqrt{\pi}}\right). \tag{36}$$

Substituting the beta function relation:

$$\beta(\alpha+1,\alpha) = \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)}$$

and the gamma function identity

$$\Gamma\left(\alpha+\frac{1}{2}\right) = \frac{\sqrt{\pi}2^{1-2\alpha}\Gamma(2\alpha)}{\Gamma(\alpha)}$$

into the integral result (36) to get:

$$ {}_2F_1(\alpha+1,2\alpha+1;\alpha+2;-1) = \frac{(\alpha+1)}{2}\left(\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} - \frac{1}{\sqrt{\pi}}\cdot\frac{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(2\alpha+1)}\right). $$

Factoring common terms, we arrive at:

$$ {}_2F_1(\alpha+1,2\alpha+1;\alpha+2;-1) = \frac{(\alpha+1)\Gamma(\alpha)}{2\Gamma(2\alpha+1)}\left(\Gamma(\alpha+1) - \frac{\Gamma\left(\alpha+\frac{1}{2}\right)}{\sqrt{\pi}}\right) $$

which proves the desired identity. □

The following identity gives a closed-form expression for $\Gamma(\alpha+\frac{1}{2})$ in terms of $\Gamma(\alpha+1)$ and $\Gamma(2\alpha)$.

**Identity 2.3.**

$$ \Gamma\left(\alpha+\frac{1}{2}\right) = \frac{2\alpha\sqrt{\pi}\Gamma(2\alpha)}{4^\alpha\Gamma(\alpha+1)}. \tag{37} $$

*Proof.* Using Corollary 2.1 and Proposition 1.1 with $a=\alpha, p=1$ and $c=\alpha+1$

$$ \begin{aligned} \int_0^\infty e^{-t}t^\alpha\Gamma(\alpha,t)\,dt &= \frac{\Gamma(2\alpha+1)}{\alpha+1}{}_2F_1(\alpha,\alpha+1;\alpha+2;-1). && (38)\\ &= \frac{\Gamma(\alpha)\left(\Gamma(\alpha+1)-\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}}\right)}{2} && (39)\\ &= \frac{\Gamma(\alpha+1)\left(\Gamma(\alpha+1)-\frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}}\right)}{2\alpha} && (40)\\ &= \frac{\Gamma(\alpha+1)^2}{2\alpha} - \frac{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{2\sqrt{\pi}\alpha}. && (41) \end{aligned} $$

Therefore,

$$ \int_0^\infty e^{-t}t^\alpha\Gamma(\alpha,t)\,dt = \frac{\Gamma(\alpha+1)^2}{2\alpha} - \frac{\Gamma(\alpha+1)\Gamma\left(\alpha+\frac{1}{2}\right)}{2\sqrt{\pi}\alpha}. \tag{42} $$

On the other hand, letting $n=2$ and $k=1$ in (16) and using (6), we get

$$ \begin{aligned} \int_0^\infty e^{-t}t^\alpha\Gamma(\alpha,t)\,dt &= \frac{\Gamma(\alpha+1)^2}{2\alpha} - 4^{-\alpha}\Gamma(2\alpha) && (43)\\ &= \frac{\alpha\Gamma(\alpha)^2}{2} - 4^{-\alpha}\Gamma(2\alpha) && (44)\\ & && (45) \end{aligned} $$

It follows from (43) and (42) that

$$ \Gamma\left(\alpha+\frac{1}{2}\right) = \frac{2\alpha\sqrt{\pi}\Gamma(2\alpha)}{4^\alpha\Gamma(\alpha+1)}. \tag{46} $$

□

For example, when $\alpha = 1$, $\Gamma\left(\frac{3}{2}\right) = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$. The following identity finds a summation involving gamma function.

**Identity 2.4.** *If $\alpha$ is a positive integer, then*

$$\sum_{m=0}^{\alpha-1} \frac{\Gamma(m+\alpha)}{2^m m!} = 2^{\alpha-1}\Gamma(\alpha). \tag{47}$$

*Proof.* The proof follows from (5) and the fact that

$$\int_0^\infty e^{-t} t^{\alpha-1} \Gamma(\alpha, t)\, dt = \frac{\Gamma(\alpha)^2}{2} \text{and} \tag{48}$$

$$\int_0^\infty \frac{e^{-2t} t^{\alpha+m-1}}{m!}\, dt = \frac{2^{-\alpha-m}\Gamma(m+\alpha)}{m!}. \tag{49}$$

Indeed, since $\Gamma(\alpha, t) = (\alpha-1)! \sum_{m=0}^{\alpha-1} \frac{t^m e^{-t}}{m!}$, then

$$\begin{aligned}
\frac{(\Gamma(\alpha))^2}{2} &= \int_0^\infty e^{-t} t^{\alpha-1} \Gamma(\alpha, t) dt \\
&= (\alpha-1)! \sum_{m=0}^{\alpha-1} \frac{1}{m!} \int_0^\infty e^{-2t} t^{m+\alpha-1} dt \\
&= (\alpha-1)! \sum_{m=0}^{\alpha-1} \frac{1}{m! 2^{m+\alpha}} \int_0^\infty u^{m+\alpha-1} e^{-u} du \\
&= (\alpha-1)! \sum_{m=0}^{\infty} \frac{1}{m! 2^{m+\alpha}} \Gamma(m+\alpha)
\end{aligned}$$

Therefore,

$$2^{\alpha-1}(\Gamma(\alpha))^2 = (\alpha-1)! \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha)}{m! 2^m}.$$

Now, use $\Gamma(\alpha) = (\alpha-1)\,!$ to get the result.

□

This identity arises from considering the distribution of the maximum of two independent Gamma variables with parameters $\alpha$ and $\beta$.

**Identity 2.5.**

$$\int_0^\infty e^{-t} t^{\beta-1} \Gamma(\alpha, t)\, dt + \int_0^\infty e^{-t} t^{\alpha-1} \Gamma(\beta, t)\, dt = \Gamma(\alpha)\Gamma(\beta). \tag{50}$$

*Proof.* The pdf of the maximum of $X \sim Gamma(\alpha, 1)$ and $Y \sim Gamma(\beta, 1)$ is given by

$$f_{X(2)}(t) = \frac{e^{-t}\left(\Gamma(\alpha)t^\beta - t^\beta \Gamma(\alpha, t) + t^\alpha \Gamma(\beta) - t^\alpha \Gamma(\beta, t)\right)}{t\Gamma(\alpha)\Gamma(\beta)} \tag{51}$$

$$= -\frac{e^{-t} t^{\alpha-1} \Gamma(\beta, t)}{\Gamma(\alpha)\Gamma(\beta)} - \frac{e^{-t} t^{\beta-1} \Gamma(\alpha, t)}{\Gamma(\alpha)\Gamma(\beta)} + \frac{e^{-t} t^{\alpha-1}}{\Gamma(\alpha)} + \frac{e^{-t} t^{\beta-1}}{\Gamma(\beta)}. \tag{52}$$

Each of the first and second terms of (52) integrates to 1. Therefore,

$$\int_0^{\infty} \frac{e^{-t}t^{\beta-1}\Gamma(\alpha,t)}{\Gamma(\alpha)\Gamma(\beta)}\,dt + \int_0^{\infty} \frac{e^{-t}t^{\alpha-1}\Gamma(\beta,t)}{\Gamma(\alpha)\Gamma(\beta)}\,dt = 2-1=1. \tag{53}$$

□

Identity (2.5) can be generalized in an obvious way by considering the largest order statistics of a random sample of size $m$ from the independent random variables $X_i, i = 1, 2, \ldots m$, where $X_i \sim Gamma(\alpha_i, 1)$.

**Identity 2.6.** *Assume that $\alpha_i > 0$ for $i = 1, 2, \ldots m$, where $m \geq 1$. Then*

$$\sum_{j=1}^{m} \int_0^{\infty} e^{-t}t^{\alpha_j-1}\Gamma\left(\alpha_j,t\right)^{-1}\prod_{i=1}^{m}\Gamma\left(\alpha_i,t\right)\,dt = \prod_{i=1}^{m}\Gamma\left(\alpha_i\right). \tag{54}$$

**Corollary 2.2.** *If $\alpha_1 = \alpha_2 = \cdots = \alpha_m = \alpha$, then, according to Identity 2.6,*

$$\int_0^{\infty} e^{-t}t^{\alpha-1}\Gamma(\alpha,t)^{m-1}\,dt = \Gamma(\alpha)^m/m, \tag{55}$$

which provides another proof of Lemma 1.1.

**Identity 2.7.**

$$\sum_{k=0}^{\infty}\frac{(-1)^k\Gamma(k+2\alpha+1,t)}{k!(\alpha+k+1)} = \Gamma(\alpha+1)\Gamma(\alpha,t) - \frac{1}{2}\alpha\Gamma(\alpha,t)^2 - 4^{-\alpha}\Gamma(2\alpha,2t). \tag{56}$$

*Proof.* Note that

$$e^{-t}t^{\alpha} = \sum_{k=0}^{\infty}\frac{(-1)^k t^{\alpha+k}}{k!}. \tag{57}$$

By integration by parts, we see that

$$\int e^{-t}t^{\alpha}\Gamma(\alpha,t)\,dt = 4^{-\alpha}\Gamma(2\alpha,2t) - e^{-t}t^{\alpha}\Gamma(\alpha,t) - \frac{\alpha\Gamma(\alpha,t)^2}{2}. \tag{58}$$

By integration by parts, again, we see that

$$\int \frac{(-1)^k t^{\alpha+k}\Gamma(\alpha,t)}{k!}\,dt = \frac{(-1)^k t^{\alpha+k+1}\Gamma(\alpha,t)}{k!(\alpha+k+1)} - \frac{(-1)^k\Gamma(k+2\alpha+1,t)}{k!(\alpha+k+1)}. \tag{59}$$

But

$$\sum_{k=0}^{\infty}\frac{(-1)^k t^{\alpha+k+1}\Gamma(\alpha,t)}{k!(\alpha+k+1)} = \Gamma(\alpha+1)\Gamma(\alpha,t) - \Gamma(\alpha,t)\Gamma(\alpha+1,t). \tag{60}$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+2\alpha+1,t)}{k!(\alpha+k+1)} = \Gamma(\alpha+1)\Gamma(\alpha,t) - \Gamma(\alpha,t)\Gamma(\alpha+1,t) \tag{61}$$

$$+ e^{-t} t^{\alpha} \Gamma(\alpha,t) + \frac{\alpha\Gamma(\alpha,t)^2}{2} - 4^{-\alpha}\Gamma(2\alpha,2t) \tag{62}$$

$$= \Gamma(\alpha+1)\Gamma(\alpha,t) - \frac{1}{2}\alpha\Gamma(\alpha,t)^2 - 4^{-\alpha}\Gamma(2\alpha,2t). \tag{63}$$

□

**Corollary 2.3.** *If $\alpha = 1$, then*

$$\sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k+3,t)}{(k+2)k!} = \frac{1}{4} e^{-2t} \left(4e^t - 2t - 3\right). \tag{64}$$

**Corollary 2.4.** *For any $\alpha > 0$,*

$${}_2F_1(\alpha+1, 2\alpha+1; \alpha+2; -1)\Gamma(2\alpha+1) = (\alpha+1)\left(\Gamma(\alpha)\Gamma(\alpha+1)/2 - 4^{-\alpha}\Gamma(2\alpha)\right), \tag{65}$$

*where ${}_2F_1$ is the hypergeometric function defined by* (12).

# 3 Application: Bivariate Gamma Distribution

A bivariate gamma distribution is constructed from specified gamma marginals, for example, see (AlAhmad, 2016c). In this section, we introduce some bivariate probability density functions using special functions such as the hypergeometric function and Whittaker function.

**Theorem 3.1.** *Assume that $W, V$ and $U$ are independent random variables such that $W$ is gamma distributed with shape parameter $a$ and scale parameter $\frac{1}{\mu}$, assume further that $U$ and $V$ are beta distributed with shape parameters $b, c$ and $d, h$, respectively, where $c = 1$, then the joint Probability distribution of $X = WV$ and $Y = UV$ is given as:*

$$f(x,y) = \frac{x^{a-b+d-1} y^{b-1} \mu^{\frac{b-a-d+1}{2}} \Gamma(h) W_{a+b-\frac{d}{2}-2h+1, \frac{a+b-d}{2}}\left(\frac{x}{\mu}\right)}{\Gamma(a)\beta(b,1)\beta(d,h)\exp\left(\frac{x}{2\mu}\right)}.$$

*Proof.*

$$\begin{aligned} f(w,u,v) &= \frac{w^{a-1}\exp\left(-\frac{w}{\mu}\right)}{\mu^a\Gamma(a)} \frac{u^{b-1}(1-u)^{c-1}}{\beta(b,c)} \frac{v^{d-1}(1-v)^{h-1}}{\beta(d,h)} \\ &= \frac{w^{a-1}u^{b-1}v^{d-1}(1-u)^{c-1}(1-v)^{h-1}\exp\left(\frac{-w}{\mu}\right)}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}. \end{aligned}$$

Therefore,

$$f(x,y,v)=\frac{x^{1-a}va-1y^{b-1}v^{1-b}v^{d-1}\left(1-\frac{y}{v}\right)^{(c-1)}(1-v)^{h-1}\exp\left(\frac{-v}{\mu x}\right)}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}\frac{1}{x^2}$$

Hence,

$$f(x,y)=\frac{x^{a-1}y^{b-1}}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}\int_0^1 v^{d-a-b-1}\left(1-\frac{y}{v}\right)^{c-1}(1-v)^{h-1}\exp\left(-\frac{x}{mv}\right)dv$$

Substitute $c=1$ to get

$$f(x,y)=\frac{x^{a-1}y^{b-1}}{\mu^a\Gamma(a)\beta(b,1)\beta(d,h)}\int_0^1 v^{d-a-b-1}(1-v)^{h-1}\exp\left(-\frac{x}{\mu v}\right)dv$$

Using (10)

$$f(x,y)=\frac{x^{a-1}y^{b-1}}{\mu^a\Gamma(a)\beta(b,1)\beta(d,h)}\left(\frac{x}{\mu}\right)^{a+b-d}e^{-\frac{x}{2\mu}}W_{a+b-d,\,\frac{a+b-d-h}{2}}\left(\frac{x}{\mu}\right).$$

Therefore,

$$f(x,y)=\frac{x^{2a+b-d-1}y^{b-1}\mu^{d-b-2a}\Gamma(h)W_{a+b-d,\,\frac{a+b-d-h}{2}}\left(\frac{x}{\mu}\right)}{\Gamma(a)\beta(b,1)\beta(d,h)\exp\left(\frac{x}{2\mu}\right)}.$$

□

**Theorem 3.2.** *Assume that $W, V$ and $U$ are independent random variables such that $W$ is gamma distributed with shape parameter $a$ and scale parameter $\frac{1}{\mu}$, assume further that $U$ and $V$ are beta distributed with shape parameters $b, c$ and $d, h$, respectively, where $c=1$, then the joint Probability distribution of $X=\frac{V}{W}, Y=UV$ is given as:*

$$f(x,y)=\frac{x^{-1-a}y^{b-1}\Gamma(a-b)\Gamma(h)_1F_1\left(a-b,a-b+h,\frac{-1}{\mu x}\right)}{\mu^a\Gamma(a)\Gamma(a+b-h)\beta(b,1)\beta(d,h)}$$

*Proof.*

$$\begin{aligned}f(w,u,v)&=\frac{w^{a-1}\exp\left(-\frac{w}{\mu}\right)}{\mu^a\Gamma(a)}\frac{u^{b-1}(1-u)^{c-1}}{\beta(b,c)}\frac{v^{d-1}(1-v)^{h-1}}{\beta(d,h)}\\&=\frac{w^{a-1}u^{b-1}v^{d-1}(1-u)^{c-1}(1-v)^{h-1}\exp\left(\frac{-w}{\mu}\right)}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}.\end{aligned}$$

Therefore,

$$f(x,y,v)=\frac{x^{1-a}va-1y^{b-1}v^{1-b}v^{d-1}\left(1-\frac{y}{v}\right)^{c-1}(1-v)^{h-1}\exp\left(\frac{-v}{\mu x}\right)}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}\frac{1}{x^2}$$

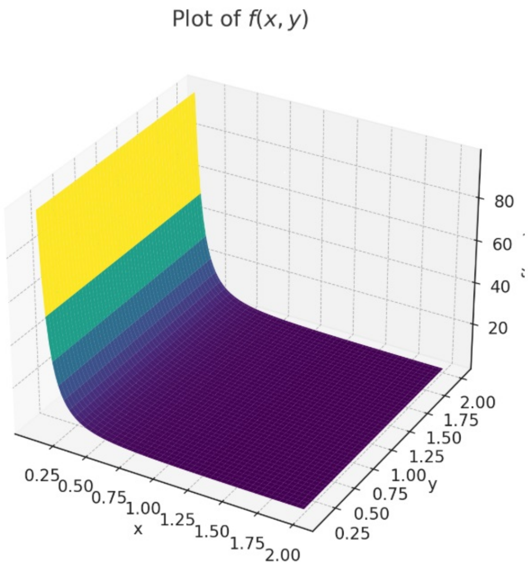


Figure 1: $f(x,y)$ for $a = 2, b = d = h = \mu = 1$

Hence,

$$f(x,y) = \frac{x^{-(a-1)}y^{b-1}}{\mu^a\Gamma(a)\beta(b,c)\beta(d,h)}\int_0^1 v^{a-b-1}\left(1-\frac{y}{v}\right)^{c-1}(1-v)^{h-1}\exp\left(-\frac{v}{\mu x}\right)dv$$

Substituting $c = 1$ to get

$$f(x,y) = \frac{x^{-(a-1)}y^{b-1}}{\mu^a\Gamma(a)\beta(b,1)\beta(d,h)}\int_0^1 v^{a-b-1}(1-v)^{h-1}\exp\left(-\frac{v}{\mu x}\right)dv$$

Now, using the integral representation of the confluent hypergeometric function to get:

$$f(x,y) = \frac{x^{-1-a}y^{b-1}\Gamma(a-b)\Gamma(h)_1F_1\left(a-b, a-b+h, \frac{-1}{\mu x}\right)}{\mu^a\Gamma(a)\Gamma(a+b-h)\beta(b,1)\beta(d,h)}$$

□

**Theorem 3.3.** *Assume that $W, V$ and $U$ are independent random variables such that* $\mathrm{W}, \mathrm{U}$ *and $V$ are beta distributed with shape parameters* $(\mathrm{a}, \mathrm{b}), (\mathrm{c}, \mathrm{d})$ *and* $(\mathrm{i}, \mathrm{j})$, *respectively, where* $\mathrm{d} = 1$, *then: Define* $\mathrm{X} = \frac{\mathrm{W}}{\mathrm{V}}, \mathrm{Y} = \frac{\mathrm{U}}{\mathrm{V}}$, *to get the joint distribution of $X$ and $Y$ is given as:*

$$f(x,y) = \frac{x^{a-1}y^{c-1}\Gamma(a+c+i-1)\Gamma(j)_2F_1(a+c+i-1, 1-b, j+a+c+i-1, x)}{\beta(a,b)\beta(c,1)\beta(i,j)\Gamma(j+a+c+i-1)}$$

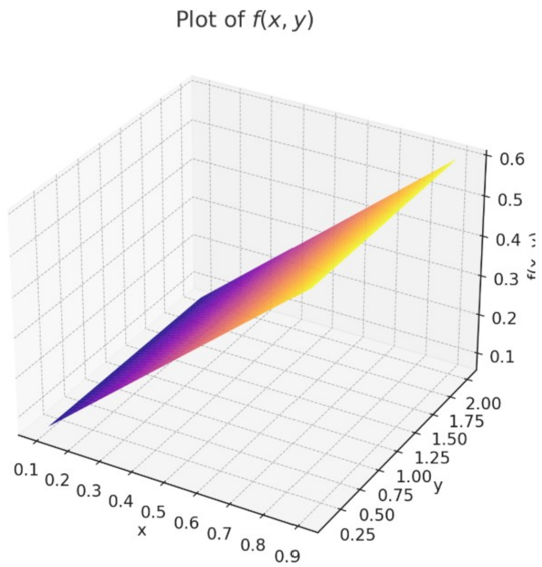


Figure 2: $f(x, y)$ for $a = 2, b = c = i = j = 1$

*Proof.*

$$\begin{aligned} f(w,u,v) &= \frac{w^{a-1}(1-w)^{b-1}}{\beta(a,b)} \frac{u^{c-1}(1-u)^{d-1}}{\beta(c,d)} \frac{v^{i-1}(1-v)^{j-1}}{\beta(i,j)} \\ &= \frac{w^{a-1}u^{c-1}v^{i-1}(1-w)^{b-1}(1-u)^{d-1}(1-v)^{j-1}}{\beta(a,b)\beta(c,d)\beta(i,j)}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x,y,v) &= \frac{x^{a-1}v^{a-1}y^{c-1}v^{c-1}v^{i-1}(1-vx)^{b-1}(1-yv)^{d-1}(1-v)^{j-1}}{\beta(a,b)\beta(c,d)\beta(i,j)} v \\ &= \frac{x^{a-1}y^{c-1}v^{a+c+i-2}(1-vx)^{b-1}(1-yv)^{d-1}(1-v)^{j-1}}{\beta(a,b)\beta(c,d)\beta(i,j)} \end{aligned}$$

Consequently,

$$f(x,y) = \frac{x^{a-1}y^{c-1}}{\beta(a,b)\beta(c,d)\beta(i,j)} \int_0^1 v^{a+c+i-2}(1-vx)^{b-1}(1-yv)^{d-1}(1-v)^{j-1}dv$$

Substituting $d = 1$ to get

$$f(x,y) = \frac{x^{a-1}y^{c-1}}{\beta(a,b)\beta(c,1)\beta(i,j)} \int_0^1 v^{a+c+i-2}(1-vx)^{b-1}(1-v)^{j-1}dv$$

Now, the integral representation of the Gauss hypergeometric function implies

$$f(x,y) = \frac{x^{a-1}y^{c-1}\Gamma(a+c+i-1)\Gamma(j)_2F_1(a+c+i-1, 1-b, j+a+c+i-1, x)}{\beta(a,b)\beta(c,1)\beta(i,j)\Gamma(j+a+c+i-1)}$$

□

## References


Al-Ahmad, S., Mamat, M., and AlAhmad, R. (2020). Finding differential transform using difference equations. *IAENG International Journal of Applied Mathematics*, 50(1):IJAM_50_1_19.

AlAhmad, R. (2016a). Products of incomplete gamma functions. *Analysis (Oldenbourg Wissenschaftsverlag)*, 36:199–203.

AlAhmad, R. (2016b). Products of incomplete gamma functions integral representations. *Mathematical Sciences and Applications E-Notes*, 4(2):47–51. b.

AlAhmad, R. (2016c). Some bivariate gamma distributions. *International Journal of Computational and Theoretical Statistics*, 3(2):49–53.

AlAhmad, R. (2021). Grünwald-leitnikov fractional derivative for a product of two functions. *International Journal of Mathematics and Computer Science*, 16(2):767–774.

Alahmad, R. (2023). The differential transform of the caputo-fabrizio fractional derivative. *Journal of Mathematics and Computer Science*, 33(2):137–145. Online publication date: 2023-12-06.

AlAhmad, R. (2025). *A New Approach to Solve Nonlinear Caputo-Fabrizo Fractional Differential Equations*, pages 895–902. Springer Nature Switzerland.

Alahmad, R. and Abdelhadi, A. (2019). Introduction to fractional calculus. In *2019 Advances in Science and Engineering Technology International Conferences (ASET)*, pages 1–5.

AlAhmad, R., Al-Khaleel, M., and Almefleh, H. (2024). On solutions of linear and nonlinear fractional differential equations with application to fractional order rc type circuits. *Journal of Computational and Applied Mathematics*, 438:115507.

Alahmad, R. and Almefleh, H. (2020). Antiderivatives and integrals involving incomplete beta functions with applications. *Aust. J. Math. Anal. Appl.*, 17(2):Art. 11, 7 pp.

Al-Ahmad, S., Mamat, M., AlAhmad, R., Sulaiman, I. M., Ghazali, P. L., and Mohamed, M. A. (2018). On new properties of differential transform via difference equations. *International Journal of Engineering & Technology (IJET)*, 7(3.28):321–324.

Arnold, B., Balakrishnan, N., and Nagaraja, H. (1992). *A First Course in Order Statistics*. Wiley, New York.

Dimov, H., Iliev, I., Radomirov, M., Rashkov, R., and Vetsov, T. (2021). Holographic fisher information metric in schrödinger spacetime. *The European Physical Journal Plus*, 136(11):1–33.

Gradshteyn, I. and Ryzhik, I. (2007). *Table of Integrals, Series, and Products*. Academic Press, 7th edition.

Jeffreys, H. (1939). *Theory of Probability*. Oxford University Press, Oxford, UK, 1 edition.

Metzler, R., Klafter, J., and Jortner, J. (1999). Hierarchies and logarithmic oscillations in the temporal relaxation patterns of proteins and other complex systems. *Proceedings* *of the National Academy of Sciences*, 96:11085–11089.

Miller, A. R. and Moskowitz, I. S. (1998). On certain generalized incomplete gamma functions. *Journal of Computational and Applied Mathematics*, 91(2):179–190.

Olver, F. W., Lozier, D. W., Boisvert, R. F., and Clark, C. W. (2010). *NIST Handbook of Mathematical Functions*. Cambridge University Press, USA, 1st edition.

Paris, R. (2003). The asymptotic expansion of a generalised incomplete gamma function. *Journal of Computational and Applied Mathematics*, 151:297–306.

Pearson, K. (1900). On the criterion that a given system of deviations from the probable in the case of a correlated system of variables is such that it can be reasonably supposed to have arisen from random sampling. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 50(302):157–175.

Prudnikov, A., Brychkov, Y., and Marichev, O. (1986). *Integrals and Series, Volume 2: Special Functions*. Gordon and Breach Science Publishers, New York.

R. AlAhmad, Q. AlAhmad, A. A. (2022). Solution of fractional autonomous ordinary differential equations. 27(1):59–64.

Shavitt, I. (1963). The gaussian function in calculations of statistical mechanics and quantum mechanics. In *Methods of Computational Physics*, volume 2, pages 1–45. Academic Press.

Shavitt, I. and Karplus, M. (1965). Gaussian-transform method for molecular integrals. i. formulation for energy integrals. *The Journal of Chemical Physics*, 43(1):398.

Sornette, D. (1998). Multiplicative processes and power laws. *Physical Review E*, 57:4811–4813.