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A new unit distribution: properties, inference, and applications

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We propose a new bounded distribution called the Marshall–Olkin reduced-Kies distribution, which is a competitive model to the generalized beta, Kumaraswamy and beta distributions. It is able to model both negative and positive skewed data. Eight classical estimation methods are used to estimate its parameters. A simulation study is conducted to compare the performance of the different estimators. The performance ordering of these estimators is explored using partial and overall ranks to determine the best estimation method. Two COVID-19 data sets on to recovering and death rates in Spain are analyzed to show the flexibility of the new distribution to model such data. The expected values of the first and last order statistics are used to estimate the minimum and maximum recovery and death rates.

keywords: COVID-19 data, death rate, Marshall–Olkin family, maximum likelihood estimation, recovery , reduced-Kies distribution.

1 Introduction

In the last few decades, many authors introduced different methods to add shape parameter(s) to traditional distributions to provide much flexibility in modelling real life

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applications. The extra parameter(s) of a good generator produces new distributions with different shapes to the probability density function (PDF) as well as the hazard rate function (HRF), which are of deep interest to statisticians and data modelers. One of the most important methods used extensively in the literature is the Marshall and Olkin's (MO) (1997) method Marshall (1977) which adds an extra shape parameter to a baseline distribution. Let $G(x)$ and $g(x)$ be the cumulative distribution function (CDF) and PDF of a baseline random variable W. Then, the CDF and PDF of the MO-G family are

$$F(x;\varphi) = \frac{G(x)}{\varphi + (1-\varphi)G(x)}, \quad \varphi > 0, \tag{1}$$

and

$$f(x;\varphi) = \frac{\varphi g(x)}{[\varphi + (1-\varphi)G(x)]^2}, \tag{2}$$

respectively.

Marshall and Olkin Marshall (1977) used the exponential and Weibull as baseline distributions and studied their properties. For $\varphi = 1$, $f(x)$ is equal to $g(x)$ and, for various values of φ, $f(x)$ can be more flexible than $g(x)$. Furthermore, the MO method is one of the most popular methods of adding parameters to extend classical distributions and it has been adopted extensively in the literature. For example, the MO extended Weibull Ghitany (Al-Hussaini and AlJarallah), MO extended Lomax Ghitany (Al-Awadhi and Alkhalfan), MO Nadarajah–Haghighi Lemonte (Cordeiro and Moreno-Arenas), MO alpha power exponential Nassar et al. (2019), MO logistic-exponential Mansoor et al. (2019), and MO power generalized Weibull Afify, Kumar and Elbatal (2020). More recently, the MO Burr III-G and MO Burr-R families by Afify et al. (2021) and Al-Babtain et al. (2021), respectively. For more extended generalized model, the interested reader can see Mahmoudi and Sepahdar (2013), **?**, Torabi (Bagheri and Mahmoudi), and Gómez (Bolfarine and Gómez).

As a special case of the Kies distribution, Kumar and Dharmaja Kumar (2013) introduced the reduced Kies (RKi) distribution in the interval $(0, 1)$ to compete with the beta distribution. The CDF and PDF of the RKi distribution (for $0 < x < 1$) are

$$G(x;\delta) = 1 - \exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]$$

and

$$g(x;\delta) = \delta x^{\delta-1}(1-x)^{-\delta-1}\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right],$$

respectively, where $\delta > 0$ is a shape parameter. Kumar and Dharmaja Kumar (2013)) argued that the RKi distribution performs better than some well known distributions for proportional data. An exponentiated version of the RKi distribution was introduced and studied by Kumar (2017). Dey et al. Dey (Nassar and Kumar) studied the recurrence relations of the RKi distribution based on progressive type-II censoring scheme. They also adopted the maximum likelihood and Bayesian methods to estimate the RKi parameters. Al-Babtain et al. Al-Babtain et al. (2020) introduced the modified Kies family based on the T-X method for combining distributions. A special member of this family, called the modified Kies exponential distribution, was considered and studied in more detail.

We propose a new flexible two-parameter model so-called Marshall-Olkin reduced Kies (MORKi) distribution, which has some desirable properties and can be used effectively to model recovery and death rates due to COVID-19 pandemic. The new distribution provides symmetrical, asymmetrical (left-skewed or right skewed), J-shaped, and reversed J-shaped densities, and bathtub and increasing hazard rates. The analysis of the two COVID-19 data sets show that the MORKi distribution can be used as a suitable model for fitting skewed data which are encountered in many applied areas including medicine, survival analysis, engineering, and reliability. The CDF and PDF of the MORKi distribution have closed forms which can be adopted for analyzing censored data.

Furthermore, we estimate the two parameters of the proposed distribution using eight classical methods and show how these estimators behave in some scenarios. We develop a guideline for choosing the best estimation method based on partial and overall ranks and an extensive simulation study.

The rest of the article is outlined as follows. In Section 2, we define the new bounded MORKi distribution. In Section 3, we derive some of its mathematical properties. In Section 4, we discuss eight classical estimation approaches to estimate its parameters. Section 5 is devoted to compare the performance of the proposed estimators via extensive simulations. Two COVID-19 data sets from Spain are analyzed to show the usefulness and flexibility of the MORKi distribution in Section 6. Finally, we provide some concluding remarks in Section 7.

2 The MORKi Distribution

By inserting the CDF of the RKi model in Equation (1), the CDF of the two-parameter bounded MORKi distribution (for $0 < x < 1$) follows as

$$F(x;\varphi,\delta) = \frac{1-\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]}{1-(1-\varphi)\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]}, \quad \delta > 0, \varphi > 0. \tag{3}$$

Henceforth, let X be a random variable with CDF (3). The PDF of X reduces to

$$f(x;\varphi,\delta) = \frac{\varphi\delta\, x^{\delta-1}\,(1-x)^{-\delta-1}\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]}{\left\{1-(1-\varphi)\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]\right\}^{2}}. \tag{4}$$

The HRF of X takes the form

$$h(x; \varphi, \delta) = \frac{\delta\, x^{\delta-1}\,(1-x)^{-\delta-1}}{\left\{1 - (1-\varphi)\exp\left[-\left(\frac{x}{1-x}\right)^{\delta}\right]\right\}}.$$

Figures 1 and 2 show the effects of different values of φ and δ on the density and hazard functions of X. These plots reveal that the MORKi distribution provides symmetrical, asymmetrical (left-skewed or right skewed), J-shaped, and reversed J-shaped densities. The HRF of X can be bathtub and increasing.

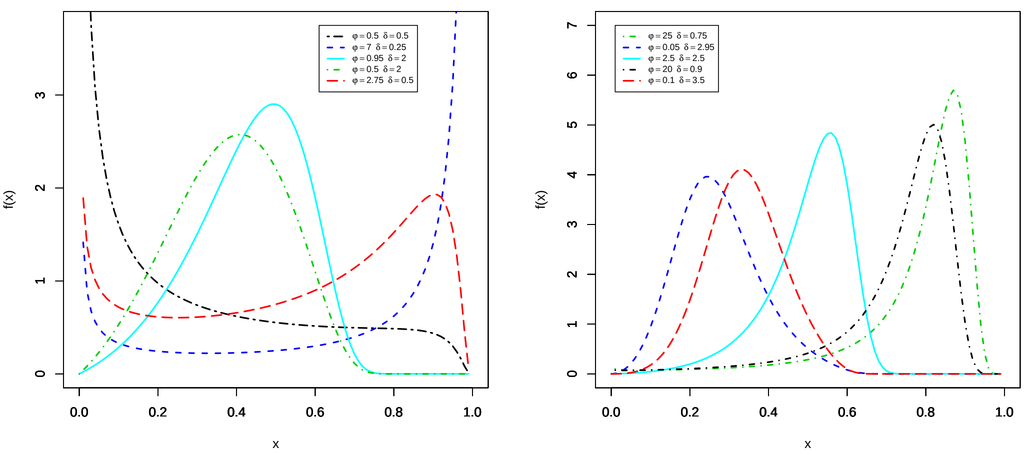

Figure 1: Some possible shapes for the PDF of the MORKi distribution for different values of φ and δ.

3 Mathematical Properties

In this section, we provide some mathematical properties of the MORKi model.

3.1 Moments

For obtaining the moments of the MORKi distribution, we require the following Lemma:

Lemma 1. *Let* X be a random variable with PDF (4) for $a > 0$ and $b > 0$ and let

$$\Delta_1(a, b, p) = \int_0^1 x^p\, x^{a-1}(1-x)^{-(a+1)} \frac{\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}}{\left[1-(1-b)\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}\right]^2} dx. \tag{5}$$

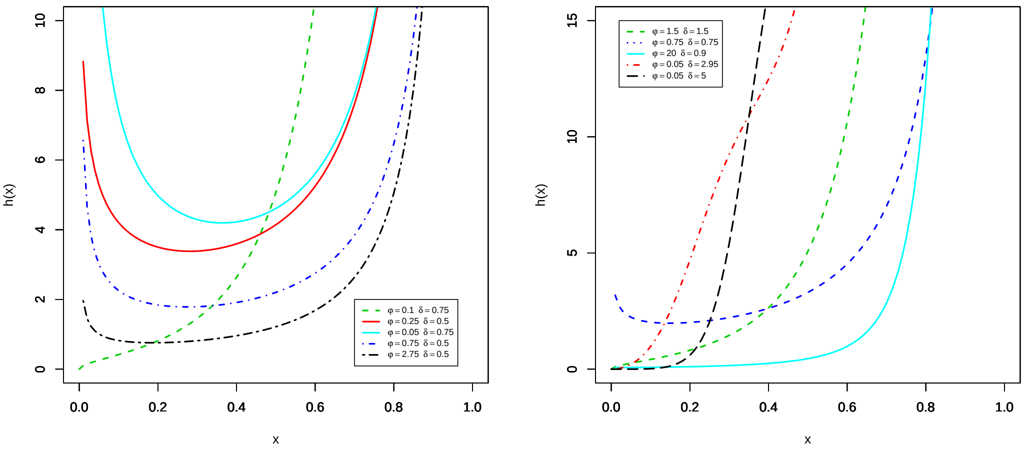

Figure 2: Some possible shapes for the HRF of the MORKi distribution for different values of φ and δ.

Then,

$$\begin{aligned}\Delta_1(a,\,b,\,p) \;&=\; \sum_{k=0}^{\infty}\left[(k+1)(1-b)^k \sum_{l=0}^{\infty}\left\{\frac{(-1)^l\,(p)_l\,\gamma\left(\frac{p+l}{a}+1,\,k+1\right)}{a\,l!\,(k+1)^{\frac{p+l}{a}+1}}\right\}\right] \\ &+\; \sum_{k=0}^{\infty}\left[(k+1)(1-b)^k \sum_{l=0}^{\infty}\left\{\frac{(-1)^l\,(p)_l}{a\,l!}\,(k+1)^{\frac{l}{2a}-1}\,\exp\left(-\frac{l}{2}\right)W_{-\frac{l}{2a},\,\frac{a-l}{2a}}(k+1)\right\}\right].\end{aligned}$$

Here, the lower and upper incomplete gamma functions are

$$\gamma(a,z)=\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\frac{z^{a+z}}{a+j} \quad \text{and} \quad \Gamma(a,z)=\Gamma(a)-\gamma(a,z),$$

$W_{a_1,a_2}(u)$ is the Whittaker function defined by

$$W_{a_1,a_2}(u)=\frac{\Gamma(-2a_2)}{\Gamma\left(\frac{1}{2}-a_2-a_1\right)}M_{a_1,a_2}(u)+\frac{\Gamma(2a_2)}{\Gamma\left(\frac{1}{2}+a_2-a_1\right)}M_{a_1,-a_2}(u), \tag{6}$$

where

$$M_{a_1,a_2}(u)=\exp\left(-\frac{u}{2}\right)u^{a_2+\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}-a_1+a_2\right)_j}{(1+2a_2)_j}\frac{u^j}{j!}, \tag{7}$$

and $(e)_k=e(e+1)\cdots(e+k-1)$ (for $k\geq 1$ and $(e)_0=1$) denotes the ascending factorial. Equation (7) converges for all finite values of u.

Proof. We can write from (5)

$$\Delta_1(a,\,b,\,p) = \int_0^1 x^p\, x^{a-1}(1-x)^{-(a+1)} \frac{\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}}{\left[1-(1-b)\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}\right]^2} dx. \tag{8}$$

By substituting $z = \left(\frac{x}{1-x}\right)^a$ in Equation (8), we have

$$\mu_r' = E(X^r) \;=\; \int_0^\infty \left(\frac{z^{\frac{1}{a}}}{1+z^{\frac{1}{a}}}\right)^p \frac{e^{-z}}{a[1-(1-b)e^{-z}]^2} dz.$$

By applying the generalized binomial expansion $(1-z)^{-2} = \sum_{k=0}^{\infty}(k+1)\,z^k$, $|z|<1$, we have

$$\mu_r' \;=\; \sum_{k=0}^{\infty}\left[(k+1)(1-b)^k \int_0^\infty z^{\frac{p}{a}}\, \frac{e^{-(k+1)z}}{a(1+z^{\frac{1}{a}})^p} dz\right].$$

Splitting the integral and expanding $(1+z^{\frac{1}{a}})^p$, we obtain

$$\begin{aligned}\Delta_1(a,\,b,\,p) \;&=\; \sum_{k=0}^{\infty}\left[(k+1)(1-b)^k \sum_{l=0}^{\infty}\left\{\frac{(-1)^l\,(p)_l}{a\,l!}\int_0^1 z^{\frac{p+l}{a}}\, e^{-(k+1)z} dz\right\}\right] \\ &+\; \sum_{k=0}^{\infty}\left[(k+1)(1-b)^k \sum_{l=0}^{\infty}\left\{\frac{(-1)^l\,(p)_l}{a\,l!}\int_1^\infty z^{-\frac{l}{a}}\, e^{-(k+1)z} dz\right\}\right]. \end{aligned} \tag{9}$$

The result follows from Equations (3.381.1) and (3.381.6) in Gradshteyn and Rhyzik (2007) (pp. 346) to calculate the integral in (9). The proof is complete.

By using Lemma 1, the rth moment of X takes the form

$$\mu_r' = \delta\,\varphi\;\Delta_1(\delta,\varphi,r).$$

Hence, the mean and variance of X are $E(X) = \delta\,\varphi\,\Delta_1(\delta,\varphi,1)$ and $V(X) = \delta\,\varphi\,[\Delta_1(\delta,\varphi,2) - (\Delta_1(\delta,\varphi,1))^2]$, respectively.

3.2 Quantile Function

The quantile function (qf) is very important to describe the characteristics of a distribution and also for data analysis Nair (Sankaran). The qf of X has the form

$$x_q = F^{-1}(q) = \frac{\left[-\log\left\{\frac{1-q}{1-q(1-\varphi)}\right\}\right]^{\frac{1}{\delta}}}{\left[1+\left\{-\log\left(\frac{1-q}{1-q(1-\varphi)}\right)\right\}^{\frac{1}{\delta}}\right]},\; 0<q<1. \tag{10}$$

In particular, the median of X follows from (10) with $q = 1/2$.

Then, a random sample from the MORKi(δ, φ) distribution can be generated from a uniform variate Y in $(0, 1)$ as

$$X = \frac{\left[-\log\left\{\frac{1-Y}{1-Y(1-\varphi)}\right\}\right]^{\frac{1}{\delta}}}{\left[1+\left\{-\log\left(\frac{1-Y}{1-Y(1-\varphi)}\right)\right\}^{\frac{1}{\delta}}\right]}.$$

Further, the first three quartiles, Q_1, Q_2 and Q_3, are obtained by setting $q = 0.25$, $q = 0.5$ and $q = 0.75$ in Equation (10), respectively.

The Galton's and Moors's quantiles measures for calculating the skewness G and M of X are

$$G = \frac{Q_{0.25} + Q_{0.75} - 2Q_{0.5}}{Q_{0.75} - Q_{0.25}}$$

and

$$M = \frac{Q_{0.875} - Q_{0.625} + Q_{0.375} - Q_{0.125}}{Q_{0.75} - Q_{0.25}}.$$

Some shapes for these two measures are displayed in Figure 3 for different values of φ and δ.

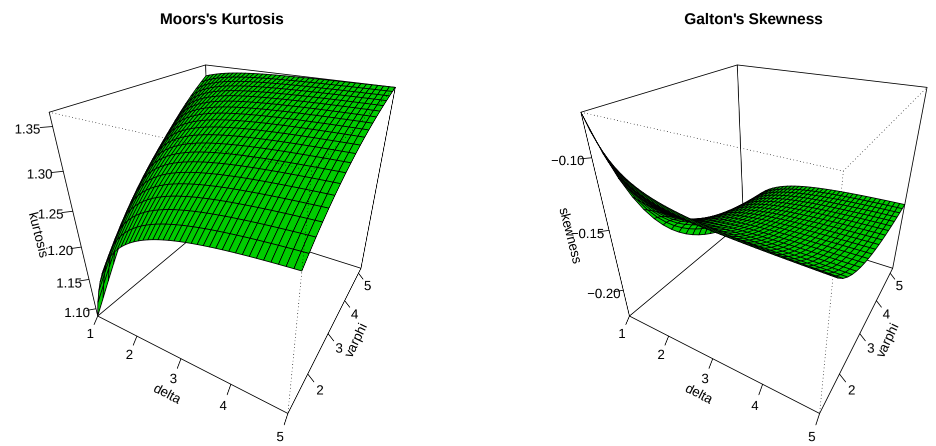

Figure 3: Some possible shapes for the Galton's skewness and Moors's kurtosis of the MORKi distribution for different values of φ and δ.

3.3 Conditional Moments

Here, we first calculate the conditional moments based on the Lemma:

Lemma 2. *If* X is a random variable with PDF (4) (for $a > 0$, $b > 0$), let

$$\Delta_2(a,\, b,\, p,\, t) = \int_t^{\infty} x^p\, x^{a-1}(1-x)^{-(a+1)} \frac{\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}}{\left[1-(1-b)\exp\left\{-\left(\frac{x}{1-x}\right)^a\right\}\right]^2} dx.$$

Hence,

$$\begin{aligned}\Delta_2(a,\, b,\, p,\, t) &= \Delta_k(b) \sum_{l=0}^{\infty} \left[\frac{(-1)^l\,(p)_l\,\left\{\Gamma\left(S\right) - \gamma\left(S,\,(k+1)\eta(t)\right) - \Gamma\left(S,\,(k+1)\right)\right\}}{a\,l!\,(k+1)^S}\right] \\ &= \Delta_k(b) \sum_{l=0}^{\infty} \frac{(-1)^l\,(p)_l}{a\,l!}\,(k+1)^{\frac{l}{2a}-1}\,\exp\left(-\frac{l}{2}\right)\,W_{-\frac{l}{2a},\,\frac{a-l}{2a}}(k+1),\end{aligned}$$

where $S = \left(\frac{p+l}{a}+1\right)$ $\eta(t) = \left(\frac{t}{1-t}\right)^a$, $\Delta_k(b) = \sum_{k=0}^{\infty}(k+1)(1-b)^k$, and $W_{a_1,\,a_2}(u)$ is the Whittaker function defined in (6).

Proof. The proof of Lemma 2 is similar to that one of Lemma 1.

Based on Lemma 2, the rth conditional moment of X can be expressed as

$$\mu_r(t) = E[X^r|x > t] = \frac{\delta\,\varphi}{[1-F(x)]}\;\Delta_2(\delta,\,\varphi,\,r,\,t).$$

An application of the conditional moments is the mean residual life (MRL), which is the expected remaining life, $X - x$, given that the item has survived to time x. We have

$$\mu_1(t) = E[X|x > t] = \frac{\delta\,\varphi}{[1-F(x)]}\;\Delta_2(\delta,\,\varphi,\,1,\,t).$$

Other applications of the conditional moments are the mean deviations about the mean and the median, the Bonferroni and Lorenz curves, and the Bonferroni and Gini indices.

3.4 Order Statistics

Let $X_1, \ldots, X_n$ be a random sample from the MORKi distribution. Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics from this random sample. Then, the PDF $f_{r:n}(x)$ of the rth order statistic (for $r = 1, 2, \ldots, n$) can be expressed as

$$\begin{aligned}f_{r:n}(x) &= \frac{\delta\,\varphi\,n!}{(r-1)!(n-r)!}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k}x^{\delta-1}\,(1-x)^{-\delta-1}\,\left[1-\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}\right]^{r+k-1} \\ &\times\;\frac{\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}}{\left[1-(1-\varphi)\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}\right]^{r+k+1}}.\end{aligned} \tag{11}$$

The PDFs of the largest order statistic ($X_{n:n}$) and the smallest order statistic ($X_{1:n}$) follow from (11) with $r = n$ and $r = 1$, respectively.

The pth ordinary moment of $X_{r:n}$ can be expressed as

$$\begin{aligned}
E[X_{r:n}^p] &= \frac{\delta\,\varphi\; n!}{(r-1)!(n-r)!}\sum_{k=0}^{n-r}(-1)^k\binom{n-r}{k} \\
&\times \int_0^1 x^{p+\delta-1}\,(1-x)^{-\delta-1}\left[1-\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}\right]^{r+k-1} \\
&\times \frac{\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}}{\left[1-(1-\varphi)\exp\left\{-\left(\frac{x}{1-x}\right)^{\delta}\right\}\right]^{r+k+1}}dx \\
&= \frac{\varphi\; n!}{(r-1)!(n-r)!}\sum_{k=0}^{n-r}\sum_{l=0}^{r+k-1}\sum_{i=0}^{\infty}(-1)^{i+k+l}\binom{n-r}{k}\binom{r+k-1}{l}\binom{r+k+i}{i}(1-\varphi)^i \\
&\times \sum_{j=0}^{\infty}\frac{(-1)^j(p)_j}{j!}\left[\frac{\gamma\left(\frac{p+j}{\delta}+1,\,l+i+1\right)}{(l+i+1)^{\frac{p+j}{\delta}+1}}+(l+i+1)^{\frac{j}{2\delta}-1}e^{-j/2}\,W_{-\frac{j}{2\delta},\,\frac{\delta-j}{2\delta}}(l+i+1)\right].
\end{aligned}$$

4 Estimation Methods

In this section, eight classical methods are presented to estimate the unknown parameters of the MORKi distribution. The **A**dequacyModel package for the `R` statistical computing environment provides a comprehensive and efficient general meta-heuristic optimization method for maximizing or minimizing an arbitrary objective function, which can be adopted to find the estimates of $\boldsymbol{\theta} = (\varphi, \delta)$ in the methods below. Details are available at https://rdrr.io/cran/AdequacyModel/.

Let $x_1, \cdots, x_m$ be a random sample of size m from the MORKi model with PDF (4). Then, the log-likelihood function for the model parameters follows as

$$\begin{aligned}
\ell(\varphi,\delta) &= m\log(\varphi\delta)+(\delta-1)\sum_{k=1}^{m}\log(x_k)-(\delta+1)\sum_{k=1}^{m}\log(1-x_k)-\sum_{k=1}^{m}\vartheta_k^{\delta} \\
&- 2\sum_{k=1}^{m}\log\left[1-(1-\varphi)e^{-\vartheta_k^{\delta}}\right], \qquad (12)
\end{aligned}$$

where $\vartheta_k = \frac{x_k}{1-x_k}, k = 1, \ldots, m$. The maximum likelihood estimates (MLEs) of φ and δ can only be obtained numerically by maximizing (12) with respect to φ and δ.

The maximum product of spacing estimates (MPSEs) Cheng and Amin (1979) and Cheng and Amin (1983) of φ and δ are obtained by maximizing the following function with respect to them

$$M(\varphi,\delta) = \frac{1}{m+1}\sum_{k=1}^{m+1}\log\left\{\frac{1-e^{-\vartheta_k^{*\delta}}}{1-(1-\varphi)e^{-\vartheta_k^{*\delta}}}-\frac{1-e^{-\vartheta_{k-1}^{*\delta}}}{1-(1-\varphi)e^{-\vartheta_{k-1}^{*\delta}}}\right\},$$

where $\vartheta_k^* = \frac{y_k}{1-y_k}$ and the y_k's are the ordered observations.

Swain et al. Swain (Venkatraman and Wilson) proposed the use of least squares and weighted least squares methods to estimate the beta distribution parameters. The least squares estimates (LSEs) and weighted least squares estimates (WLSEs) of φ and δ can be obtained by minimizing the function

$$S(\varphi,\delta) = \sum_{k=1}^{m}\omega_k\left\{\frac{1-e^{-\vartheta_k^{*\delta}}}{1-(1-\varphi)e^{-\vartheta_k^{*\delta}}}-\frac{k}{m+1}\right\}^2,$$

where $\omega_k = \frac{(m+1)^2(m+2)}{k(m-k+1)}$ in the case of weighted least squares and one otherwise.

Let $\hat{\varphi}_{CM}$ and $\hat{\delta}_{CM}$ denote the Cramér-von Mises estimates (CVMEs) of the unknown parameters of the MORKi model. To obtain these estimates, the following function should be minimized with respect to φ and δ

$$C(\varphi,\delta) = \sum_{k=1}^{m}\left\{\frac{1-e^{-\vartheta_k^{*\delta}}}{1-(1-\varphi)e^{-\vartheta_k^{*\delta}}}-\frac{2k-1}{2m}\right\}^2. \quad (13)$$

Since the MORKi model has an explicit qf, one can use the method of percentile Kao (1958) and Kao (1959) to estimate the unknown parameters. The percentile estimates (PCEs) of φ and δ, say $\hat{\varphi}_{PE}$ and $\hat{\delta}_{PE}$, can be obtained by minimizing

$$P(\varphi,\delta) = \sum_{k=1}^{m}\left\{y_k-\frac{\left[-\log\left\{\frac{1-q_k}{1-q_k(1-\varphi)}\right\}\right]^{\frac{1}{\delta}}}{\left[1+\left\{-\log\left(\frac{1-q_k}{1-q_k(1-\varphi)}\right)\right\}^{\frac{1}{\delta}}\right]}\right\}^2, \quad (14)$$

where $q_k = k/m+1$.

The Anderson-Darling method is a type of minimum distance estimator obtained by minimizing an Anderson-Darling statistic. For the MORKi model, the Anderson-Darling estimates (ADEs), say $\hat{\varphi}_{AD}$ and $\hat{\delta}_{AD}$, are found by minimizing

$$AD(\varphi,\delta) = -m-\frac{1}{m}\sum_{k=1}^{m}(2k-1)[\log(F(y_k|\varphi,\delta))+\log(\bar{F}(y_{m-k+1}|\varphi,\delta))],$$

with respect to both parameters, where $F(y_k;\varphi,\delta)$ is given by (3) and $\bar{F}(\cdot) = 1-F(\cdot)$.

Similarly, the right-tail Anderson-Darling estimates (RADEs), say $\hat{\varphi}_{RAD}$ and $\hat{\delta}_{RAD}$, can be obtained by minimizing

$$RAD(\varphi,\delta) = \frac{m}{2}-2\sum_{k=1}^{n}F\left(y_k \mid \varphi,\delta\right)-\frac{1}{m}\sum_{k=1}^{n}(2k-1)\log\overline{F}\left(y_{m-k+1} \mid \varphi,\delta\right),$$

with respect to φ, and δ.

5 Simulation Results

We compare the eight estimation methods, namely the MLEs, MPSEs, LSEs, CVMEs, WLSEs, PCEs, ADEs, and RADEs, using extensive numerical simulations in terms of mean squared errors (MSEs) given by $MSEs = \frac{1}{N}\sum_{i=1}^{N}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta})^2$, average values of absolute biases ($|Bias|$) ($|Bias(\widehat{\boldsymbol{\theta}})|$), $|Bias(\widehat{\boldsymbol{\theta}})| = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|$, and average of mean relative errors (MREs), say $MREs = \frac{1}{N}\sum_{i=1}^{N}|\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}|/\boldsymbol{\theta}$. The simulation results are reported to develop a guideline for choosing the best estimation approach which gives good estimates for the MORKi parameters.

The **R** software is used to generate $5,000$ random samples from the MORKi distribution for some sample sizes $n = 20$, 50, 100, 250 and 400 and different parameter values. The simulation results in Tables 1–4 include MSE, $|Bias|$, and MRE for the eight estimators of all parameters. Furthermore, the figures in these tables show the rank of each estimator among all estimators in each row; the superscripts refer to the indicators, and the $\sum Ranks$ refers to the partial sum of the ranks for each column and each sample size.

The MSEs, MREs, and the biases decrease when n increases, thus showing that these estimators are consistent. So, these estimates are asymptotically unbiased estimators.

The partial and overall ranks of the eight estimators are reported in Table 5 for all parameter combinations. The performance ordering of the eight estimators, based on Table 5, from best to worst is: MPSEs, MLEs, WLSEs, ADEs, RADEs, PCEs, LSEs, and CVMEs for all the studied cases. From the numbers of Table 5, we can conclude that the the MPSEs outperform all the other estimators (overall score of 43.5). Therefore, based on our study, we can consider that the MPSEs are the best estimators for the MORKi parameters. Hence, based on the simulation study, we show the superiority of the MPSEs and MLEs with overall scores of 43.5 and 56.5.

Table 1: Simulation results of the eight estimation methods for $\varphi = 1.50$ and $\delta = 0.50$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
20	MSEs	$\hat{\varphi}$	$0.17344^{\{8\}}$	$0.11548^{\{1\}}$	$0.12812^{\{2\}}$	$0.16169^{\{6\}}$	$0.13228^{\{3\}}$	$0.13782^{\{5\}}$	$0.13494^{\{4\}}$	$0.16342^{\{7\}}$
		$\hat{\delta}$	$0.00280^{\{1\}}$	$0.00416^{\{4\}}$	$0.00482^{\{7\}}$	$0.00464^{\{6\}}$	$0.00422^{\{5\}}$	$0.00497^{\{8\}}$	$0.00371^{\{3\}}$	$0.00360^{\{2\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.41646^{\{8\}}$	$0.33983^{\{1\}}$	$0.35794^{\{2\}}$	$0.40211^{\{6\}}$	$0.36370^{\{3\}}$	$0.37124^{\{5\}}$	$0.36734^{\{4\}}$	$0.40425^{\{7\}}$
		$\hat{\delta}$	$0.05295^{\{1\}}$	$0.06452^{\{4\}}$	$0.06941^{\{7\}}$	$0.06811^{\{6\}}$	$0.06495^{\{5\}}$	$0.07053^{\{8\}}$	$0.0609^{\{3\}}$	$0.05997^{\{2\}}$
	MREs	$\hat{\varphi}$	$0.27764^{\{8\}}$	$0.22655^{\{1\}}$	$0.23863^{\{2\}}$	$0.26807^{\{6\}}$	$0.24247^{\{3\}}$	$0.24750^{\{5\}}$	$0.24490^{\{4\}}$	$0.26950^{\{7\}}$
		$\hat{\delta}$	$0.10590^{\{1\}}$	$0.12904^{\{4\}}$	$0.13881^{\{7\}}$	$0.13621^{\{6\}}$	$0.12990^{\{5\}}$	$0.14105^{\{8\}}$	$0.12181^{\{3\}}$	$0.11995^{\{2\}}$
	$\sum Ranks$		$27^{\{5\}}$	$15^{\{1\}}$	$27^{\{5\}}$	$36^{\{7\}}$	$24^{\{3\}}$	$39^{\{8\}}$	$21^{\{2\}}$	$27^{\{5\}}$
50	MSEs	$\hat{\varphi}$	$0.07031^{\{8\}}$	$0.05564^{\{1\}}$	$0.05893^{\{3\}}$	$0.06398^{\{5\}}$	$0.05953^{\{4\}}$	$0.06749^{\{7\}}$	$0.05681^{\{2\}}$	$0.06587^{\{6\}}$
		$\hat{\delta}$	$0.00118^{\{1\}}$	$0.00146^{\{5\}}$	$0.00184^{\{7\}}$	$0.00182^{\{6\}}$	$0.00144^{\{4\}}$	$0.00217^{\{8\}}$	$0.00134^{\{3\}}$	$0.00125^{\{2\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.26517^{\{8\}}$	$0.23589^{\{1\}}$	$0.24276^{\{3\}}$	$0.25294^{\{5\}}$	$0.24398^{\{4\}}$	$0.25979^{\{7\}}$	$0.23835^{\{2\}}$	$0.25666^{\{6\}}$
		$\hat{\delta}$	$0.03436^{\{1\}}$	$0.03819^{\{5\}}$	$0.04286^{\{7\}}$	$0.04261^{\{6\}}$	$0.03801^{\{4\}}$	$0.04656^{\{8\}}$	$0.03656^{\{3\}}$	$0.03531^{\{2\}}$
	MREs	$\hat{\varphi}$	$0.17678^{\{7\}}$	$0.22655^{\{8\}}$	$0.16184^{\{2\}}$	$0.16863^{\{4\}}$	$0.16265^{\{3\}}$	$0.17319^{\{6\}}$	$0.15890^{\{1\}}$	$0.17111^{\{5\}}$
		$\hat{\delta}$	$0.06872^{\{1\}}$	$0.12904^{\{8\}}$	$0.08571^{\{6\}}$	$0.08521^{\{5\}}$	$0.07602^{\{4\}}$	$0.09313^{\{7\}}$	$0.07313^{\{3\}}$	$0.07062^{\{2\}}$
	$\sum Ranks$		$26^{\{4\}}$	$28^{\{5.5\}}$	$28^{\{5.5\}}$	$31^{\{2\}}$	$23^{\{2.5\}}$	$43^{\{8\}}$	$14^{\{1\}}$	$23^{\{2.5\}}$
100	MSEs	$\hat{\varphi}$	$0.03500^{\{8\}}$	$0.02842^{\{1\}}$	$0.02873^{\{2\}}$	$0.03110^{\{5\}}$	$0.02911^{\{3\}}$	$0.02959^{\{4\}}$	$0.03386^{\{7\}}$	$0.03306^{\{6\}}$
		$\hat{\delta}$	$0.00059^{\{1\}}$	$0.00071^{\{4\}}$	$0.00097^{\{7\}}$	$0.00092^{\{6\}}$	$0.00072^{\{5\}}$	$0.00105^{\{8\}}$	$0.00063^{\{2\}}$	$0.00067^{\{3\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.18707^{\{8\}}$	$0.16860^{\{1\}}$	$0.16949^{\{2\}}$	$0.17634^{\{5\}}$	$0.17061^{\{3\}}$	$0.17203^{\{4\}}$	$0.18400^{\{7\}}$	$0.18183^{\{6\}}$
		$\hat{\delta}$	$0.02432^{\{1\}}$	$0.02672^{\{4\}}$	$0.03118^{\{7\}}$	$0.03036^{\{6\}}$	$0.02680^{\{5\}}$	$0.03237^{\{8\}}$	$0.02517^{\{2\}}$	$0.02596^{\{3\}}$
	MREs	$\hat{\varphi}$	$0.12471^{\{8\}}$	$0.11240^{\{1\}}$	$0.11300^{\{2\}}$	$0.11756^{\{5\}}$	$0.11374^{\{3\}}$	$0.11469^{\{4\}}$	$0.12267^{\{7\}}$	$0.12122^{\{6\}}$
		$\hat{\delta}$	$0.04863^{\{1\}}$	$0.05343^{\{4\}}$	$0.06236^{\{7\}}$	$0.06072^{\{6\}}$	$0.05361^{\{5\}}$	$0.06474^{\{8\}}$	$0.05035^{\{2\}}$	$0.05192^{\{3\}}$
	$\sum Ranks$		$27^{\{4.5\}}$	$15^{\{1\}}$	$27^{\{4.5\}}$	$33^{\{7\}}$	$24^{\{2\}}$	$36^{\{8\}}$	$27^{\{4.5\}}$	$27^{\{4.5\}}$
250	MSEs	$\hat{\varphi}$	$0.01285^{\{4\}}$	$0.01119^{\{1\}}$	$0.01270^{\{3\}}$	$0.01292^{\{5\}}$	$0.01293^{\{6\}}$	$0.01301^{\{7\}}$	$0.01192^{\{2\}}$	$0.01307^{\{8\}}$
		$\hat{\delta}$	$0.00023^{\{1\}}$	$0.00026^{\{1.5\}}$	$0.00039^{\{6\}}$	$0.00040^{\{7\}}$	$0.00026^{\{1.5\}}$	$0.00041^{\{8\}}$	$0.00028^{\{4.5\}}$	$0.00028^{\{4.5\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.11337^{\{4\}}$	$0.10577^{\{1\}}$	$0.11270^{\{3\}}$	$0.11368^{\{5\}}$	$0.11370^{\{6\}}$	$0.11407^{\{7\}}$	$0.10919^{\{2\}}$	$0.11431^{\{8\}}$
		$\hat{\delta}$	$0.01529^{\{1\}}$	$0.01602^{\{2\}}$	$0.01979^{\{6\}}$	$0.02003^{\{7\}}$	$0.01626^{\{3\}}$	$0.02015^{\{8\}}$	$0.01659^{\{4\}}$	$0.01660^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.07558^{\{4\}}$	$0.07051^{\{1\}}$	$0.07514^{\{3\}}$	$0.07579^{\{5\}}$	$0.07580^{\{6\}}$	$0.07605^{\{7\}}$	$0.07280^{\{2\}}$	$0.07621^{\{8\}}$
		$\hat{\delta}$	$0.03057^{\{1\}}$	$0.03205^{\{2\}}$	$0.03959^{\{6\}}$	$0.04006^{\{7\}}$	$0.03252^{\{3\}}$	$0.04029^{\{8\}}$	$0.03319^{\{4\}}$	$0.03321^{\{5\}}$
	$\sum Ranks$		$15^{\{2\}}$	$8.5^{\{1\}}$	$27^{\{5\}}$	$36^{\{6\}}$	$25.5^{\{4\}}$	$45^{\{8\}}$	$18.5^{\{3\}}$	$38.5^{\{7\}}$
400	MSEs	$\hat{\varphi}$	$0.00743^{\{2\}}$	$0.00667^{\{1\}}$	$0.00805^{\{7\}}$	$0.00769^{\{5\}}$	$0.00771^{\{6\}}$	$0.00744^{\{3\}}$	$0.00762^{\{4\}}$	$0.00833^{\{8\}}$
		$\hat{\delta}$	$0.00014^{\{1\}}$	$0.00016^{\{2\}}$	$0.00024^{\{6.5\}}$	$0.00024^{\{6.5\}}$	$0.00017^{\{3.5\}}$	$0.00027^{\{8\}}$	$0.00019^{\{5\}}$	$0.00017^{\{3.5\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.08621^{\{2\}}$	$0.08167^{\{1\}}$	$0.08970^{\{6\}}$	$0.08771^{\{6\}}$	$0.08778^{\{7\}}$	$0.08624^{\{3\}}$	$0.08729^{\{4\}}$	$0.09125^{\{8\}}$
		$\hat{\delta}$	$0.01187^{\{1\}}$	$0.01247^{\{2\}}$	$0.01550^{\{7\}}$	$0.01554^{\{7\}}$	$0.01315^{\{4\}}$	$0.01639^{\{8\}}$	$0.01385^{\{5\}}$	$0.01307^{\{3\}}$
	MREs	$\hat{\varphi}$	$0.05748^{\{2\}}$	$0.05444^{\{1\}}$	$0.05980^{\{5\}}$	$0.05847^{\{5\}}$	$0.05852^{\{6\}}$	$0.05749^{\{3\}}$	$0.05819^{\{4\}}$	$0.06084^{\{8\}}$
		$\hat{\delta}$	$0.02374^{\{1\}}$	$0.02493^{\{2\}}$	$0.03099^{\{7\}}$	$0.03108^{\{7\}}$	$0.02629^{\{4\}}$	$0.03279^{\{8\}}$	$0.02769^{\{5\}}$	$0.02614^{\{3\}}$
	$\sum Ranks$		$9^{\{1.5\}}$	$9^{\{1.5\}}$	$36.5^{\{7.5\}}$	$36.5^{\{7.5\}}$	$30.5^{\{4\}}$	$33^{\{5\}}$	$27^{\{3\}}$	$33.5^{\{6\}}$

Table 2: Simulation results of the eight estimation methods for $\varphi = 0.75$ and $\delta = 1.5$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
20	MSEs	$\hat{\varphi}$	$0.04360^{\{7\}}$	$0.03550^{\{1\}}$	$0.04021^{\{4\}}$	$0.04427^{\{8\}}$	$0.04196^{\{5\}}$	$0.03736^{\{2\}}$	$0.04292^{\{6\}}$	$0.03975^{\{3\}}$
		$\hat{\delta}$	$0.03700^{\{1\}}$	$0.04519^{\{4\}}$	$0.05574^{\{7\}}$	$0.05671^{\{8\}}$	$0.04655^{\{6\}}$	$0.04593^{\{5\}}$	$0.04255^{\{3\}}$	$0.04225^{\{2\}}$
	$\|Bias\|$	$\hat{\varphi}$	$0.20881^{\{7\}}$	$0.18842^{\{1\}}$	$0.20052^{\{4\}}$	$0.21041^{\{8\}}$	$0.20483^{\{5\}}$	$0.19330^{\{2\}}$	$0.20718^{\{6\}}$	$0.19937^{\{3\}}$
		$\hat{\delta}$	$0.19236^{\{1\}}$	$0.21257^{\{4\}}$	$0.23609^{\{7\}}$	$0.23814^{\{8\}}$	$0.21576^{\{6\}}$	$0.21431^{\{5\}}$	$0.20628^{\{3\}}$	$0.20555^{\{2\}}$
	MREs	$\hat{\varphi}$	$0.27842^{\{7\}}$	$0.25123^{\{1\}}$	$0.26736^{\{4\}}$	$0.28054^{\{8\}}$	$0.27311^{\{5\}}$	$0.25773^{\{2\}}$	$0.27624^{\{6\}}$	$0.26583^{\{3\}}$
		$\hat{\delta}$	$0.12824^{\{1\}}$	$0.14171^{\{4\}}$	$0.15739^{\{7\}}$	$0.15876^{\{8\}}$	$0.14384^{\{6\}}$	$0.14287^{\{5\}}$	$0.13752^{\{3\}}$	$0.13703^{\{2\}}$
	$\sum Ranks$		$24^{\{4\}}$	$15^{\{1.5\}}$	$33^{\{6.5\}}$	$48^{\{8\}}$	$33^{\{6.5\}}$	$21^{\{3\}}$	$27^{\{5\}}$	$15^{\{1.5\}}$
50	MMSEs	$\hat{\varphi}$	$0.01664^{\{3\}}$	$0.01506^{\{1\}}$	$0.01800^{\{6\}}$	$0.01816^{\{7\}}$	$0.01939^{\{8\}}$	$0.01777^{\{5\}}$	$0.01591^{\{2\}}$	$0.01723^{\{4\}}$
		$\hat{\delta}$	$0.01264^{\{1\}}$	$0.01654^{\{4\}}$	$0.02199^{\{8\}}$	$0.02038^{\{7\}}$	$0.01904^{\{6\}}$	$0.01513^{\{3\}}$	$0.01484^{\{2\}}$	$0.01715^{\{5\}}$
	$\|Bias\|$	$\hat{\varphi}$	$0.12900^{\{3\}}$	$0.12270^{\{1\}}$	$0.13415^{\{6\}}$	$0.13475^{\{7\}}$	$0.13926^{\{8\}}$	$0.13330^{\{5\}}$	$0.12615^{\{2\}}$	$0.13127^{\{4\}}$
		$\hat{\delta}$	$0.11243^{\{1\}}$	$0.12861^{\{4\}}$	$0.14829^{\{8\}}$	$0.14276^{\{7\}}$	$0.13798^{\{6\}}$	$0.12301^{\{3\}}$	$0.12181^{\{2\}}$	$0.13096^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.17200^{\{3\}}$	$0.16360^{\{1\}}$	$0.17886^{\{6\}}$	$0.17966^{\{7\}}$	$0.18567^{\{8\}}$	$0.17773^{\{5\}}$	$0.16820^{\{2\}}$	$0.17503^{\{4\}}$
		$\hat{\delta}$	$0.07496^{\{1\}}$	$0.08574^{\{4\}}$	$0.09886^{\{8\}}$	$0.09518^{\{7\}}$	$0.09198^{\{6\}}$	$0.08201^{\{3\}}$	$0.08121^{\{2\}}$	$0.08730^{\{5\}}$
	$\sum Ranks$		$12^{\{1.5\}}$	$15^{\{3\}}$	$42^{\{7\}}$	$42^{\{7\}}$	$42^{\{7\}}$	$24^{\{4\}}$	$12^{\{1.5\}}$	$27^{\{5\}}$
100	MSEs	$\hat{\varphi}$	$0.00867^{\{4\}}$	$0.00866^{\{3\}}$	$0.00892^{\{7\}}$	$0.00877^{\{6\}}$	$0.00932^{\{8\}}$	$0.00807^{\{1\}}$	$0.00875^{\{5\}}$	$0.00817^{\{2\}}$
		$\hat{\delta}$	$0.00599^{\{1\}}$	$0.00778^{\{3\}}$	$0.01131^{\{8\}}$	$0.01021^{\{7\}}$	$0.00786^{\{4\}}$	$0.00726^{\{2\}}$	$0.00790^{\{5\}}$	$0.00843^{\{6\}}$
	$\|Bias\|$	$\hat{\varphi}$	$0.09309^{\{4\}}$	$0.09306^{\{3\}}$	$0.09445^{\{7\}}$	$0.09367^{\{6\}}$	$0.09654^{\{8\}}$	$0.08982^{\{1\}}$	$0.09355^{\{5\}}$	$0.09038^{\{2\}}$
		$\hat{\delta}$	$0.07739^{\{1\}}$	$0.08823^{\{3\}}$	$0.10634^{\{8\}}$	$0.10107^{\{7\}}$	$0.08868^{\{4\}}$	$0.08519^{\{2\}}$	$0.08886^{\{5\}}$	$0.09183^{\{6\}}$
	MREs	$\hat{\varphi}$	$0.12413^{\{4\}}$	$0.12408^{\{3\}}$	$0.12594^{\{7\}}$	$0.12490^{\{6\}}$	$0.12871^{\{8\}}$	$0.11976^{\{1\}}$	$0.12474^{\{5\}}$	$0.12051^{\{2\}}$
		$\hat{\delta}$	$0.05160^{\{1\}}$	$0.05882^{\{3\}}$	$0.07089^{\{8\}}$	$0.06738^{\{7\}}$	$0.05912^{\{4\}}$	$0.05679^{\{2\}}$	$0.05924^{\{5\}}$	$0.06122^{\{6\}}$
	$\sum Ranks$		$15^{\{2\}}$	$18^{\{3\}}$	$45^{\{8\}}$	$39^{\{7\}}$	$36^{\{6\}}$	$9^{\{1\}}$	$30^{\{5\}}$	$24^{\{4\}}$
250	MSEs	$\hat{\varphi}$	$0.00353^{\{6\}}$	$0.00318^{\{1\}}$	$0.00354^{\{7\}}$	$0.00339^{\{3\}}$	$0.00346^{\{5\}}$	$0.00340^{\{4\}}$	$0.00356^{\{8\}}$	$0.00330^{\{2\}}$
		$\hat{\delta}$	$0.00259^{\{1\}}$	$0.00305^{\{2\}}$	$0.00421^{\{8\}}$	$0.00390^{\{7\}}$	$0.00308^{\{3\}}$	$0.00310^{\{4\}}$	$0.00324^{\{5\}}$	$0.00340^{\{6\}}$
	$\|Bias\|$	$\hat{\varphi}$	$0.05943^{\{6\}}$	$0.05637^{\{1\}}$	$0.05948^{\{7\}}$	$0.05826^{\{3\}}$	$0.05885^{\{5\}}$	$0.05829^{\{4\}}$	$0.05968^{\{8\}}$	$0.05742^{\{2\}}$
		$\hat{\delta}$	$0.05091^{\{2\}}$	$0.05522^{\{3\}}$	$0.06488^{\{8\}}$	$0.06241^{\{7\}}$	$0.03700^{\{1\}}$	$0.05568^{\{4\}}$	$0.05689^{\{5\}}$	$0.05831^{\{6\}}$
	MREs	$\hat{\varphi}$	$0.07924^{\{6\}}$	$0.07516^{\{1\}}$	$0.07930^{\{7\}}$	$0.07768^{\{3\}}$	$0.07846^{\{5\}}$	$0.07772^{\{4\}}$	$0.07957^{\{8\}}$	$0.07656^{\{2\}}$
		$\hat{\delta}$	$0.03394^{\{1\}}$	$0.03681^{\{2\}}$	$0.04325^{\{8\}}$	$0.04161^{\{7\}}$	$0.03700^{\{3\}}$	$0.03712^{\{4\}}$	$0.03793^{\{5\}}$	$0.03888^{\{6\}}$
	$\sum Ranks$		$22^{\{2.5\}}$	$10^{\{1\}}$	$45^{\{8\}}$	$30^{\{6\}}$	$22^{\{2.5\}}$	$24^{\{4.5\}}$	$39^{\{7\}}$	$24^{\{4.5\}}$
400	MSEs	$\hat{\varphi}$	$0.00214^{\{4\}}$	$0.00205^{\{3\}}$	$0.00222^{\{6.5\}}$	$0.00235^{\{8\}}$	$0.00218^{\{5\}}$	$0.00185^{\{1\}}$	$0.00222^{\{6.5\}}$	$0.00201^{\{2\}}$
		$\hat{\delta}$	$0.00156^{\{1\}}$	$0.00183^{\{2\}}$	$0.00273^{\{8\}}$	$0.00248^{\{7\}}$	$0.00190^{\{3\}}$	$0.00191^{\{4\}}$	$0.00205^{\{6\}}$	$0.00192^{\{5\}}$
	$\|Bias\|$	$\hat{\varphi}$	$0.04625^{\{4\}}$	$0.04525^{\{3\}}$	$0.04713^{\{7\}}$	$0.04843^{\{8\}}$	$0.04669^{\{6\}}$	$0.04307^{\{1\}}$	$0.04708^{\{5\}}$	$0.04488^{\{2\}}$
		$\hat{\delta}$	$0.03946^{\{1\}}$	$0.04281^{\{2\}}$	$0.05229^{\{8\}}$	$0.04977^{\{7\}}$	$0.04363^{\{3\}}$	$0.04375^{\{4\}}$	$0.04529^{\{6\}}$	$0.04387^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.06167^{\{4\}}$	$0.06033^{\{3\}}$	$0.06285^{\{7\}}$	$0.06457^{\{8\}}$	$0.06225^{\{5\}}$	$0.05742^{\{1\}}$	$0.06277^{\{6\}}$	$0.05984^{\{2\}}$
		$\hat{\delta}$	$0.02631^{\{1\}}$	$0.02854^{\{2\}}$	$0.03486^{\{8\}}$	$0.03318^{\{7\}}$	$0.02909^{\{3\}}$	$0.02917^{\{4\}}$	$0.03019^{\{6\}}$	$0.02925^{\{5\}}$
	$\sum Ranks$		$15^{\{2\}}$	$15^{\{2\}}$	$44.5^{\{7\}}$	$45^{\{8\}}$	$25^{\{5\}}$	$15^{\{2\}}$	$35.5^{\{6\}}$	$21^{\{4\}}$

Table 3: Simulation results of the eight estimation methods for $\varphi = 0.75$ and $\delta = 3.0$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
20	MSEs	$\hat{\varphi}$	$0.04665^{\{8\}}$	$0.03640^{\{1\}}$	$0.03943^{\{2\}}$	$0.04638^{\{7\}}$	$0.04195^{\{6\}}$	$0.04000^{\{3\}}$	$0.04189^{\{5\}}$	$0.04117^{\{4\}}$
		$\hat{\delta}$	$0.14603^{\{1\}}$	$0.20197^{\{6\}}$	$0.23270^{\{8\}}$	$0.21915^{\{7\}}$	$0.17579^{\{4\}}$	$0.20106^{\{5\}}$	$0.17020^{\{2\}}$	$0.17255^{\{3\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.21598^{\{8\}}$	$0.19080^{\{2\}}$	$0.19857^{\{1\}}$	$0.21537^{\{7\}}$	$0.20482^{\{6\}}$	$0.20000^{\{3\}}$	$0.20468^{\{5\}}$	$0.20291^{\{4\}}$
		$\hat{\delta}$	$0.38213^{\{1\}}$	$0.44941^{\{6\}}$	$0.48239^{\{8\}}$	$0.46814^{\{7\}}$	$0.41927^{\{4\}}$	$0.44840^{\{5\}}$	$0.41255^{\{2\}}$	$0.41540^{\{3\}}$
	MREs	$\hat{\varphi}$	$0.28797^{\{8\}}$	$0.25440^{\{1\}}$	$0.26476^{\{2\}}$	$0.28716^{\{7\}}$	$0.27309^{\{6\}}$	$0.26666^{\{3\}}$	$0.27290^{\{5\}}$	$0.27055^{\{4\}}$
		$\hat{\delta}$	$0.12738^{\{1\}}$	$0.14980^{\{6\}}$	$0.16080^{\{8\}}$	$0.15605^{\{7\}}$	$0.13976^{\{4\}}$	$0.14947^{\{5\}}$	$0.13752^{\{2\}}$	$0.13847^{\{3\}}$
	$\sum Ranks$		$27^{\{5\}}$	$22^{\{3\}}$	$29^{\{6\}}$	$42^{\{8\}}$	$30^{\{7\}}$	$24^{\{4\}}$	$21^{\{1.5\}}$	$21^{\{1.5\}}$
50	MMSEs	$\hat{\varphi}$	$0.01769^{\{5\}}$	$0.01515^{\{1\}}$	$0.01835^{\{8\}}$	$0.01813^{\{7\}}$	$0.01664^{\{3\}}$	$0.01518^{\{2\}}$	$0.01809^{\{6\}}$	$0.01736^{\{4\}}$
		$\hat{\delta}$	$0.05545^{\{1\}}$	$0.06597^{\{3\}}$	$0.08045^{\{7\}}$	$0.08115^{\{8\}}$	$0.06811^{\{4\}}$	$0.07660^{\{6\}}$	$0.06535^{\{2\}}$	$0.06980^{\{5\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.13302^{\{5\}}$	$0.12310^{\{1\}}$	$0.13546^{\{8\}}$	$0.13466^{\{7\}}$	$0.12901^{\{3\}}$	$0.12322^{\{2\}}$	$0.13451^{\{6\}}$	$0.13175^{\{4\}}$
		$\hat{\delta}$	$0.23547^{\{1\}}$	$0.25685^{\{3\}}$	$0.28363^{\{7\}}$	$0.28487^{\{8\}}$	$0.26098^{\{4\}}$	$0.27678^{\{6\}}$	$0.25563^{\{2\}}$	$0.26420^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.17736^{\{5\}}$	$0.16413^{\{1\}}$	$0.18061^{\{8\}}$	$0.17955^{\{7\}}$	$0.17202^{\{3\}}$	$0.16429^{\{2\}}$	$0.17934^{\{6\}}$	$0.17567^{\{4\}}$
		$\hat{\delta}$	$0.07849^{\{1\}}$	$0.08562^{\{3\}}$	$0.09454^{\{7\}}$	$0.09496^{\{8\}}$	$0.08699^{\{4\}}$	$0.09226^{\{6\}}$	$0.08521^{\{2\}}$	$0.08807^{\{5\}}$
	$\sum Ranks$		$18^{\{2\}}$	$12^{\{1\}}$	$45^{\{7.5\}}$	$45^{\{7.5\}}$	$21^{\{3\}}$	$24^{\{4.5\}}$	$24^{\{4.5\}}$	$27^{\{6\}}$
100	MSEs	$\hat{\varphi}$	$0.00927^{\{6\}}$	$0.00811^{\{2\}}$	$0.00943^{\{7\}}$	$0.00978^{\{8\}}$	$0.00885^{\{3\}}$	$0.00784^{\{1\}}$	$0.00899^{\{5\}}$	$0.00889^{\{4\}}$
		$\hat{\delta}$	$0.02309^{\{1\}}$	$0.03467^{\{5\}}$	$0.03816^{\{7\}}$	$0.04484^{\{8\}}$	$0.03268^{\{2\}}$	$0.03725^{\{6\}}$	$0.03287^{\{3\}}$	$0.03389^{\{4\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.09627^{\{6\}}$	$0.09006^{\{2\}}$	$0.09711^{\{7\}}$	$0.09892^{\{8\}}$	$0.09408^{\{3\}}$	$0.08854^{\{1\}}$	$0.09481^{\{5\}}$	$0.09430^{\{4\}}$
		$\hat{\delta}$	$0.15196^{\{1\}}$	$0.18620^{\{5\}}$	$0.19534^{\{7\}}$	$0.21176^{\{8\}}$	$0.18077^{\{2\}}$	$0.19301^{\{6\}}$	$0.18129^{\{3\}}$	$0.18410^{\{4\}}$
	MREs	$\hat{\varphi}$	$0.12836^{\{6\}}$	$0.12008^{\{2\}}$	$0.12948^{\{7\}}$	$0.13189^{\{8\}}$	$0.12544^{\{3\}}$	$0.11806^{\{1\}}$	$0.12641^{\{5\}}$	$0.12573^{\{4\}}$
		$\hat{\delta}$	$0.05065^{\{1\}}$	$0.06207^{\{5\}}$	$0.06511^{\{7\}}$	$0.07059^{\{8\}}$	$0.06026^{\{2\}}$	$0.06434^{\{6\}}$	$0.06043^{\{3\}}$	$0.06137^{\{4\}}$
	$\sum Ranks$		$21^{\{3\}}$	$21^{\{3\}}$	$42^{\{7\}}$	$48^{\{8\}}$	$15^{\{1\}}$	$21^{\{3\}}$	$24^{\{5.5\}}$	$24^{\{5.5\}}$
250	MSEs	$\hat{\varphi}$	$0.00358^{\{7\}}$	$0.00332^{\{1\}}$	$0.00334^{\{2\}}$	$0.00354^{\{6\}}$	$0.00344^{\{4\}}$	$0.00341^{\{3\}}$	$0.00359^{\{8\}}$	$0.00348^{\{5\}}$
		$\hat{\delta}$	$0.01038^{\{1\}}$	$0.01259^{\{2\}}$	$0.01694^{\{8\}}$	$0.01677^{\{7\}}$	$0.01348^{\{3\}}$	$0.01496^{\{6\}}$	$0.01358^{\{4\}}$	$0.01368^{\{5\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.05986^{\{7\}}$	$0.05758^{\{1\}}$	$0.05781^{\{2\}}$	$0.05951^{\{6\}}$	$0.05864^{\{4\}}$	$0.05841^{\{3\}}$	$0.05994^{\{8\}}$	$0.05903^{\{5\}}$
		$\hat{\delta}$	$0.10189^{\{1\}}$	$0.11220^{\{2\}}$	$0.13017^{\{8\}}$	$0.12948^{\{7\}}$	$0.11609^{\{3\}}$	$0.12232^{\{6\}}$	$0.11655^{\{4\}}$	$0.11696^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.07982^{\{7\}}$	$0.07678^{\{1\}}$	$0.07708^{\{2\}}$	$0.07935^{\{6\}}$	$0.07819^{\{4\}}$	$0.07788^{\{3\}}$	$0.07992^{\{8\}}$	$0.07871^{\{5\}}$
		$\hat{\delta}$	$0.03396^{\{1\}}$	$0.03740^{\{2\}}$	$0.04339^{\{8\}}$	$0.04316^{\{7\}}$	$0.03870^{\{3\}}$	$0.04077^{\{6\}}$	$0.03885^{\{4\}}$	$0.03899^{\{5\}}$
	$\sum Ranks$		$24^{\{3\}}$	$9^{\{1\}}$	$30^{\{5.5\}}$	$39^{\{8\}}$	$21^{\{2\}}$	$27^{\{4\}}$	$36^{\{7\}}$	$30^{\{5.5\}}$
400	MSEs	$\hat{\varphi}$	$0.00207^{\{5.5\}}$	$0.00206^{\{3.5\}}$	$0.00237^{\{8\}}$	$0.00199^{\{1\}}$	$0.00206^{\{3.5\}}$	$0.00207^{\{5.5\}}$	$0.00215^{\{7\}}$	$0.00202^{\{2\}}$
		$\hat{\delta}$	$0.00594^{\{1\}}$	$0.00729^{\{2\}}$	$0.01111^{\{8\}}$	$0.01100^{\{7\}}$	$0.00848^{\{5\}}$	$0.00853^{\{6\}}$	$0.00832^{\{3\}}$	$0.00834^{\{4\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.04547^{\{5\}}$	$0.04535^{\{3\}}$	$0.04869^{\{8\}}$	$0.04463^{\{1\}}$	$0.04536^{\{4\}}$	$0.04552^{\{6\}}$	$0.04638^{\{7\}}$	$0.04498^{\{2\}}$
		$\hat{\delta}$	$0.07708^{\{1\}}$	$0.08536^{\{2\}}$	$0.10540^{\{8\}}$	$0.10489^{\{7\}}$	$0.09209^{\{5\}}$	$0.09236^{\{6\}}$	$0.09123^{\{3\}}$	$0.09135^{\{4\}}$
	MREs	$\hat{\varphi}$	$0.06062^{\{5\}}$	$0.06046^{\{3\}}$	$0.06492^{\{8\}}$	$0.05950^{\{1\}}$	$0.06048^{\{4\}}$	$0.06069^{\{6\}}$	$0.06183^{\{7\}}$	$0.05997^{\{2\}}$
		$\hat{\delta}$	$0.02569^{\{1\}}$	$0.02845^{\{2\}}$	$0.03513^{\{8\}}$	$0.03496^{\{7\}}$	$0.03070^{\{5\}}$	$0.03079^{\{6\}}$	$0.03041^{\{3\}}$	$0.03045^{\{4\}}$
	$\sum Ranks$		$18.5^{\{3\}}$	$15.5^{\{1\}}$	$48^{\{8\}}$	$24^{\{4\}}$	$26.5^{\{5\}}$	$35.5^{\{7\}}$	$30^{\{6\}}$	$18^{\{2\}}$

Table 4: Simulation results of the eight estimation methods for $\varphi = 3.0$ and $\delta = 3.0$.

n	Est.	Est. Par.	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
20	MSEs	$\hat{\varphi}$	$0.69932^{\{6\}}$	$0.58339^{\{1\}}$	$0.59957^{\{4\}}$	$0.70444^{\{7\}}$	$0.61783^{\{5\}}$	$0.59872^{\{3\}}$	$0.59328^{\{2\}}$	$0.76197^{\{8\}}$
		$\hat{\delta}$	$0.08646^{\{1\}}$	$0.10554^{\{4\}}$	$0.13598^{\{7\}}$	$0.13363^{\{6\}}$	$0.11658^{\{5\}}$	$0.17431^{\{8\}}$	$0.09260^{\{2\}}$	$0.09564^{\{3\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.83625^{\{6\}}$	$0.76380^{\{1\}}$	$0.77432^{\{4\}}$	$0.83931^{\{7\}}$	$0.78602^{\{5\}}$	$0.77377^{\{3\}}$	$0.77025^{\{2\}}$	$0.87291^{\{8\}}$
		$\hat{\delta}$	$0.29404^{\{1\}}$	$0.32488^{\{4\}}$	$0.36876^{\{7\}}$	$0.36556^{\{6\}}$	$0.34143^{\{5\}}$	$0.41750^{\{8\}}$	$0.30429^{\{2\}}$	$0.30925^{\{3\}}$
	MREs	$\hat{\varphi}$	$0.27875^{\{6\}}$	$0.25460^{\{1\}}$	$0.25811^{\{4\}}$	$0.27977^{\{7\}}$	$0.26201^{\{5\}}$	$0.25792^{\{3\}}$	$0.25675^{\{2\}}$	$0.29097^{\{8\}}$
		$\hat{\delta}$	$0.09801^{\{1\}}$	$0.10829^{\{4\}}$	$0.12292^{\{7\}}$	$0.12185^{\{6\}}$	$0.11381^{\{5\}}$	$0.13917^{\{8\}}$	$0.10143^{\{2\}}$	$0.10308^{\{3\}}$
	$\sum Ranks$		$21^{\{3\}}$	$15^{\{2\}}$	$33^{\{6\}}$	$39^{\{8\}}$	$30^{\{4\}}$	$33^{\{6\}}$	$12^{\{1\}}$	$33^{\{6\}}$
50	MSEs	$\hat{\varphi}$	$0.27589^{\{7\}}$	$0.26043^{\{3\}}$	$0.25594^{\{2\}}$	$0.27351^{\{5\}}$	$0.28609^{\{8\}}$	$0.26914^{\{4\}}$	$0.24822^{\{1\}}$	$0.27572^{\{6\}}$
		$\hat{\delta}$	$0.02914^{\{1\}}$	$0.03991^{\{5\}}$	$0.04696^{\{6\}}$	$0.05039^{\{7\}}$	$0.03503^{\{2\}}$	$0.06930^{\{8\}}$	$0.03845^{\{3\}}$	$0.03971^{\{4\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.52525^{\{7\}}$	$0.51033^{\{3\}}$	$0.50590^{\{2\}}$	$0.52298^{\{5\}}$	$0.53487^{\{8\}}$	$0.51879^{\{4\}}$	$0.49821^{\{1\}}$	$0.52509^{\{6\}}$
		$\hat{\delta}$	$0.17069^{\{1\}}$	$0.19976^{\{5\}}$	$0.21670^{\{6\}}$	$0.22448^{\{7\}}$	$0.18715^{\{2\}}$	$0.26325^{\{8\}}$	$0.19609^{\{3\}}$	$0.19927^{\{4\}}$
	MREs	$\hat{\varphi}$	$0.17508^{\{7\}}$	$0.17011^{\{3\}}$	$0.16863^{\{2\}}$	$0.17433^{\{5\}}$	$0.17829^{\{8\}}$	$0.17293^{\{4\}}$	$0.16607^{\{1\}}$	$0.17503^{\{6\}}$
		$\hat{\delta}$	$0.05690^{\{1\}}$	$0.06659^{\{5\}}$	$0.07223^{\{6\}}$	$0.07483^{\{7\}}$	$0.06238^{\{2\}}$	$0.08775^{\{8\}}$	$0.06536^{\{3\}}$	$0.06642^{\{4\}}$
	$\sum Ranks$		$24^{\{3\}}$	$24^{\{3\}}$	$24^{\{3\}}$	$36^{\{7.5\}}$	$30^{\{5.5\}}$	$36^{\{7.5\}}$	$12^{\{1\}}$	$30^{\{5.5\}}$
100	MSEs	$\hat{\varphi}$	$0.13279^{\{6\}}$	$0.12754^{\{2\}}$	$0.12388^{\{1\}}$	$0.13230^{\{4\}}$	$0.12930^{\{3\}}$	$0.13724^{\{7\}}$	$0.13260^{\{5\}}$	$0.14156^{\{8\}}$
		$\hat{\delta}$	$0.01443^{\{1\}}$	$0.01874^{\{3\}}$	$0.02418^{\{7\}}$	$0.02389^{\{6\}}$	$0.01933^{\{4\}}$	$0.03564^{\{8\}}$	$0.01985^{\{5\}}$	$0.01832^{\{2\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.36440^{\{6\}}$	$0.35713^{\{2\}}$	$0.35197^{\{1\}}$	$0.36372^{\{4\}}$	$0.35959^{\{3\}}$	$0.37046^{\{7\}}$	$0.36414^{\{5\}}$	$0.37625^{\{8\}}$
		$\hat{\delta}$	$0.12011^{\{1\}}$	$0.13690^{\{3\}}$	$0.15551^{\{7\}}$	$0.15456^{\{6\}}$	$0.13901^{\{4\}}$	$0.18878^{\{8\}}$	$0.14088^{\{5\}}$	$0.13534^{\{2\}}$
	MREs	$\hat{\varphi}$	$0.12147^{\{6\}}$	$0.11904^{\{2\}}$	$0.11732^{\{1\}}$	$0.12124^{\{4\}}$	$0.11986^{\{3\}}$	$0.12349^{\{7\}}$	$0.12138^{\{5\}}$	$0.12542^{\{8\}}$
		$\hat{\delta}$	$0.04004^{\{1\}}$	$0.04563^{\{3\}}$	$0.05184^{\{7\}}$	$0.05152^{\{6\}}$	$0.04634^{\{4\}}$	$0.06293^{\{8\}}$	$0.04696^{\{5\}}$	$0.04511^{\{2\}}$
	$\sum Ranks$		$21^{\{2.5\}}$	$15^{\{1\}}$	$24^{\{4\}}$	$30^{\{6\}}$	$21^{\{2.5\}}$	$45^{\{8\}}$	$30^{\{6\}}$	$30^{\{6\}}$
250	MSEs	$\hat{\varphi}$	$0.05110^{\{1\}}$	$0.05726^{\{6\}}$	$0.05617^{\{4\}}$	$0.05699^{\{5\}}$	$0.05393^{\{2\}}$	$0.05822^{\{7\}}$	$0.05504^{\{3\}}$	$0.05889^{\{8\}}$
		$\hat{\delta}$	$0.00638^{\{2\}}$	$0.00637^{\{1\}}$	$0.00973^{\{7\}}$	$0.00932^{\{6\}}$	$0.00684^{\{3\}}$	$0.01231^{\{8\}}$	$0.00725^{\{4\}}$	$0.00759^{\{5\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.22605^{\{1\}}$	$0.23930^{\{6\}}$	$0.23700^{\{4\}}$	$0.23872^{\{5\}}$	$0.23223^{\{2\}}$	$0.24128^{\{7\}}$	$0.23460^{\{3\}}$	$0.24268^{\{8\}}$
		$\hat{\delta}$	$0.07990^{\{2\}}$	$0.07979^{\{1\}}$	$0.09865^{\{7\}}$	$0.09655^{\{6\}}$	$0.08268^{\{3\}}$	$0.11096^{\{8\}}$	$0.08517^{\{4\}}$	$0.08711^{\{5\}}$
	MREs	$\hat{\varphi}$	$0.07535^{\{1\}}$	$0.07977^{\{6\}}$	$0.07900^{\{4\}}$	$0.07957^{\{5\}}$	$0.07741^{\{2\}}$	$0.08043^{\{7\}}$	$0.07820^{\{3\}}$	$0.08089^{\{8\}}$
		$\hat{\delta}$	$0.02663^{\{2\}}$	$0.02660^{\{1\}}$	$0.03288^{\{7\}}$	$0.03218^{\{6\}}$	$0.02756^{\{3\}}$	$0.03699^{\{8\}}$	$0.02839^{\{4\}}$	$0.02904^{\{5\}}$
	$\sum Ranks$		$9^{\{1\}}$	$21^{\{3.5\}}$	$33^{\{5.5\}}$	$33^{\{5.5\}}$	$15^{\{2\}}$	$45^{\{8\}}$	$21^{\{3.5\}}$	$39^{\{7\}}$
400	MSEs	$\hat{\varphi}$	$0.03325^{\{3\}}$	$0.03408^{\{5\}}$	$0.03449^{\{6\}}$	$0.03540^{\{7\}}$	$0.03204^{\{1\}}$	$0.03858^{\{8\}}$	$0.03381^{\{4\}}$	$0.03234^{\{2\}}$
		$\hat{\delta}$	$0.00378^{\{1\}}$	$0.00413^{\{2\}}$	$0.00594^{\{6\}}$	$0.00606^{\{7\}}$	$0.00427^{\{3\}}$	$0.00812^{\{8\}}$	$0.00487^{\{5\}}$	$0.00462^{\{4\}}$
	$\lvert Bias \rvert$	$\hat{\varphi}$	$0.18235^{\{3\}}$	$0.18460^{\{5\}}$	$0.18572^{\{6\}}$	$0.18814^{\{7\}}$	$0.17898^{\{1\}}$	$0.19641^{\{8\}}$	$0.18387^{\{4\}}$	$0.17984^{\{2\}}$
		$\hat{\delta}$	$0.06148^{\{1\}}$	$0.06425^{\{2\}}$	$0.07704^{\{6\}}$	$0.07782^{\{7\}}$	$0.06538^{\{3\}}$	$0.09009^{\{8\}}$	$0.06981^{\{5\}}$	$0.06797^{\{4\}}$
	MREs	$\hat{\varphi}$	$0.06078^{\{3\}}$	$0.07977^{\{5\}}$	$0.06191^{\{6\}}$	$0.06271^{\{7\}}$	$0.05966^{\{1\}}$	$0.06547^{\{8\}}$	$0.06129^{\{4\}}$	$0.05995^{\{2\}}$
		$\hat{\delta}$	$0.02049^{\{1\}}$	$0.02660^{\{2\}}$	$0.02568^{\{5\}}$	$0.02594^{\{6\}}$	$0.02179^{\{2\}}$	$0.03003^{\{8\}}$	$0.02327^{\{4\}}$	$0.02266^{\{3\}}$
	$\sum Ranks$		$12^{\{2\}}$	$26^{\{4.5\}}$	$35^{\{6\}}$	$41^{\{7\}}$	$11^{\{1\}}$	$48^{\{8\}}$	$26^{\{4.5\}}$	$17^{\{3\}}$

Table 5: Partial and overall ranks of all estimation methods for some combinations of φ and δ.

Parameters	n	MLEs	MPSEs	LSEs	CVMEs	WLSEs	PCEs	ADEs	RADEs
	20	5	1	5	7	3	8	2	5
	50	4	5.5	5.5	2	2.5	8	1	2.5
$(\varphi = 1.5, \delta = 0.5)$	100	4.5	1	4.5	7	2	8	4.5	4.5
	250	2	1	5	6	4	8	3	7
	400	1.5	1.5	7.5	7.5	4	5	3	6
	20	4	1.5	6.5	8	6.5	3	5	1.5
	50	1.5	3	7	7	7	4	1.5	5
$(\varphi = 0.75, \delta = 1.5)$	100	2	3	8	7	6	1	5	4
	250	2.5	1	8	6	2.5	4.5	7	4.5
	400	2	2	7	8	5	2	6	4
	20	5	3	6	8	7	4	1.5	1.5
	50	2	1	7.5	7.5	3	4.5	4.5	6
$(\varphi = 0.75, \delta = 3.0)$	100	3	3	7	8	1	3	5.5	5.5
	250	3	1	5.5	8	2	4	7	5.5
	400	3	1	8	4	5	7	6	2
	20	3	2	6	8	4	6	1	6
	50	3	3	3	7.5	5.5	7.5	1	5.5
$(\varphi = 3.0, \delta = 3.0)$	100	2.5	1	4	6	2.5	8	6	6
	250	1	3.5	5.5	5.5	2	8	3.5	7
	400	2	4.5	6	7	1	8	4.5	3
$\sum ranks$		56.5	43.5	122.5	135	75.5	111.5	78.5	92
Overallrank		2	1	7	8	3	6	4	5

6 Analyzing COVID-19 Data Sets

We show the flexibility of the MORKi distribution in modeling COVID-19 data and compare its fit with those from the generalized beta (G-beta), Kumaraswamy (Kw), beta, and reduced-Kies (RKi) distributions. The unknown parameters of the fitted models are estimated via maximum likelihood. The (K-S) statistic (K-S (stat)) with its associated p-value (K-S (p-value)) are taken to compare the MORKi model with other fitted distributions.

The two data sets refer to the recovery and death rates in Spain (from 3th March to 7th May, 2020) due to COVID-19 infections. Each data set contains 66 observations and the two data sets are reported in Tables 6 and 7. Further, some descriptive measures for these data are listed in Table 8.

Table 8: Descriptive statistics for recovery and death rates data due to COVID-19 infections in Spain.

Data	Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Skewness	Kurtosis
Recover Data	0.4228	0.6474	0.7533	0.7240	0.7975	0.8628	-0.7214	2.8106
Deathes Data	0.1372	0.2024	0.2467	0.2760	0.3526	0.5714	0.7214	2.8106

Table 6: Recovery rates due to COVID-19 infections in Spain from 3 March to 7 May, 2020.

0.6670	0.5000	0.5000	0.4286	0.7500	0.6531	0.5161	0.7895	0.7689	0.6873	0.5200	0.7251	0.6375	0.6078
0.6289	0.5712	0.5923	0.6061	0.5924	0.5921	0.5592	0.5954	0.6164	0.6455	0.6725	0.6838	0.6850	0.6947
0.7210	0.7315	0.7412	0.7508	0.7519	0.7547	0.7645	0.7715	0.7759	0.7807	0.7838	0.7847	0.7871	0.7902
0.7934	0.7913	0.7962	0.7971	0.7977	0.8007	0.8038	0.8289	0.8322	0.8354	0.8371	0.8387	0.8456	0.8490
0.8535	0.8547	0.8564	0.8580	0.8604	0.8628	0.6586	0.7070	0.7963	0.8516				

Table 7: Death rates due to COVID-19 infections in Spain from 3 March to 7 May, 2020.

0.3330	0.5000	0.5000	0.5714	0.2500	0.3469	0.4839	0.2105	0.2311	0.3127	0.4800	0.2749	0.3625	0.3922
0.3414	0.3711	0.4288	0.4077	0.3939	0.4076	0.4079	0.4408	0.4046	0.3836	0.3545	0.3275	0.3162	0.3150
0.3053	0.2930	0.2790	0.2685	0.2588	0.2492	0.2481	0.2453	0.2355	0.2285	0.2241	0.2193	0.2162	0.2153
0.2129	0.2098	0.2037	0.2066	0.2087	0.2038	0.2029	0.2023	0.1993	0.1962	0.1711	0.1678	0.1646	0.1629
0.1613	0.1544	0.1510	0.1484	0.1465	0.1453	0.1436	0.1420	0.1396	0.1372				

The parameter estimates, their standard errors (SEs), the K-S values and their associated p-values for all fitted distributions are reported in Tables 9 and 10 for the recovery and death data sets, respectively. The fits of the fitted MORKi, G-beta, Kw, beta and RKi distributions are also compared via K-S statistics in these tables. The MORKi distribution has the lowest value of this statistic and the largest value of p-value among all fitted models, i.e., it is the best model for the two COVID-19 data sets. The fitted density and the estimated CDF, SF and Probability-Probability (PP) plots of the MORKi distribution are displayed in Figures 4 and 6 for the two COVID-19 data sets. The PP plots for all fitted distributions are depicted in Figures 5 and 7 for both COVID-19 data sets. The results in Tables 9 and 10 and the plots in Figures 4-7 illustrate that the MORKi distribution provides close fits to the current data and then it can be adopted to model left-skewed and right-skewed data effectively. The HRF plots of the MORKi model for both data sets along with TTT plots are displayed in Figures 8 and 9. The TTT plots for the two data are concave which means that both COVID-19 data sets have increasing HRFs as shown through the HRF plots. We conclude that the MORKi is suitable for modeling data with increasing hazard functions.

Table 9: MLEs with their associated SEs, K-S, and p-values for recovery rates.

Model	Estimates	SEs	Estimates	SEs	Estimates	SEs	K-S (stat)	K-S (p-value)
MORKi	$\hat{\delta}$ =0.99128	0.05039	$\hat{\varphi}$ =19.73989	5.65890			0.07854	0.81025
G-beta	$\hat{\alpha}$ =0.06524	0.00944	$\hat{\beta}$ =183372.8	9012.6	$\hat{p}$ =83.28392	3.54697	0.09152	0.63814
Kw	$\hat{\alpha}$ =8.07822	0.94697	$\hat{\beta}$ =7.73824	2.01859			0.09972	0.52780
Beta	$\hat{\alpha}$ =12.79427	2.22909	$\hat{\beta}$ =4.89941	0.82697			0.11479	0.34943
RKi	$\hat{\delta}$ =0.66468	0.05104					0.61697	0.00000

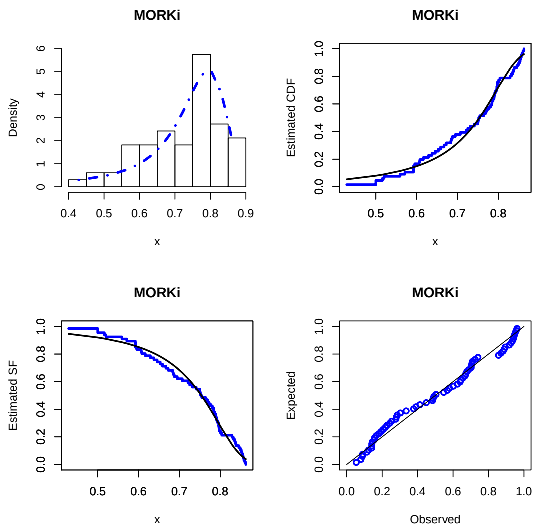

Figure 4: Estimated density, CDF, SF, and PP plots of the MORKi model for recovery COVID-19 data.

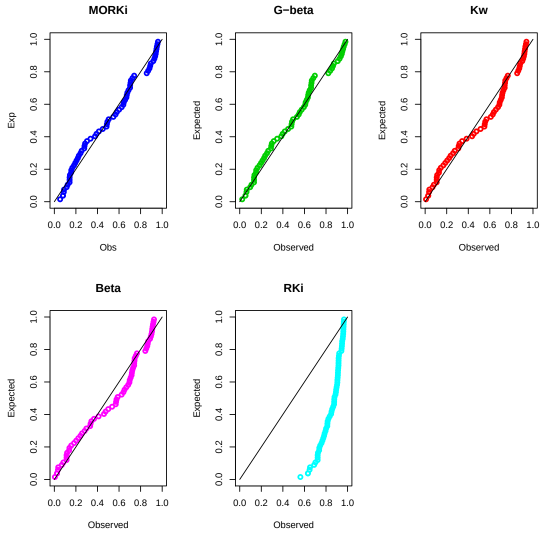

Figure 5: The PP plots of the MORKi model and other models for recovery COVID-19 data.

Table 10: MLEs with their associated SEs, K-S, and p-values for death rates.

Model	Estimates	SEs	Estimates	SEs	Estimates	SEs	K-S (stat)	K-S (p-value)
MORKi	$\hat{\delta}$ =2.95919	0.31984	$\hat{\varphi}$ =0.04649	0.01947			0.08736	0.69494
G-beta	$\hat{\alpha}$ =56.2586	167.50559	$\hat{\beta}$ =11.39122	1.96253	$\hat{p}$ =0.1365727	0.37561	0.1094831	0.40747
Kw	$\hat{\alpha}$ =2.68469	0.26646	$\hat{\beta}$ =22.12626	6.68456			0.12713	0.23647
Beta	$\hat{\alpha}$ =4.89941	0.82697	$\hat{\beta}$ =12.79427	2.22910			0.11479	0.34943
RKi	$\hat{\delta}$ =1.25025	0.13045					0.42783	0.00000

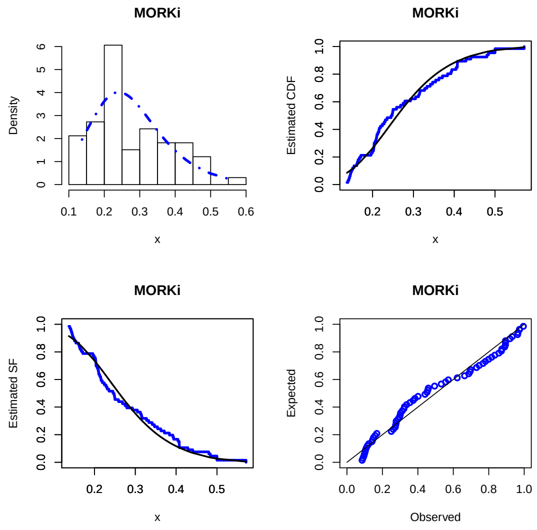

Figure 6: Estimated density, CDF, SF, and PP plots of the MORKi model for death COVID-19 data.

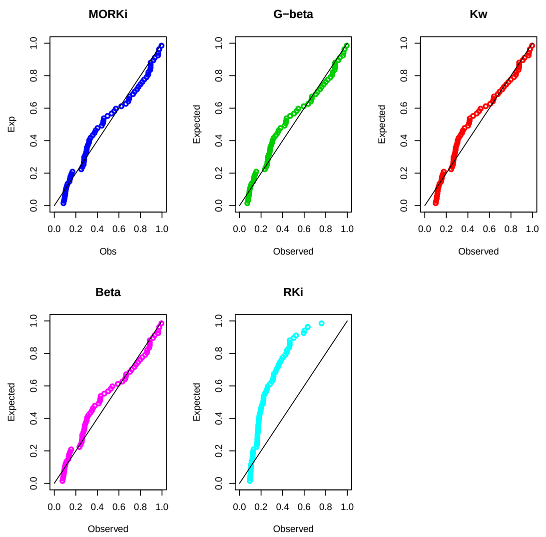

Figure 7: The PP plots of the MORKi model and other models for death COVID-19 data.

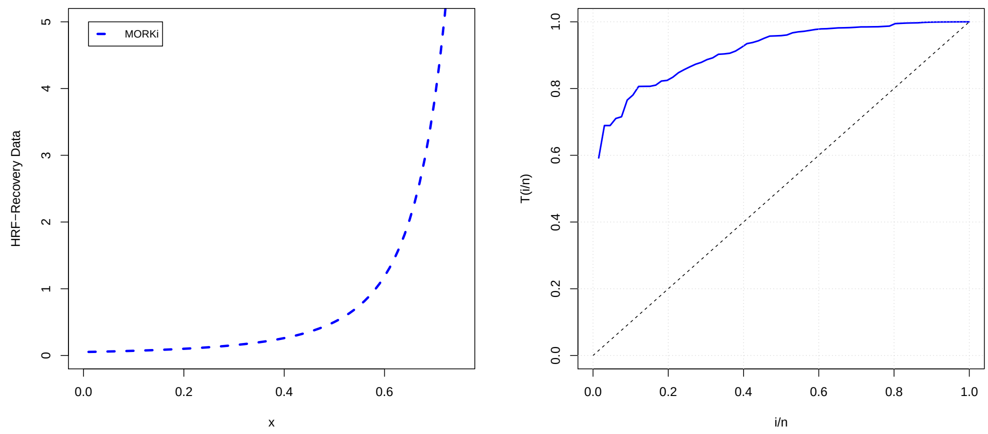

Figure 8: The HRF plots of the MORKi distribution and TTT plot for recovery COVID-19 data.

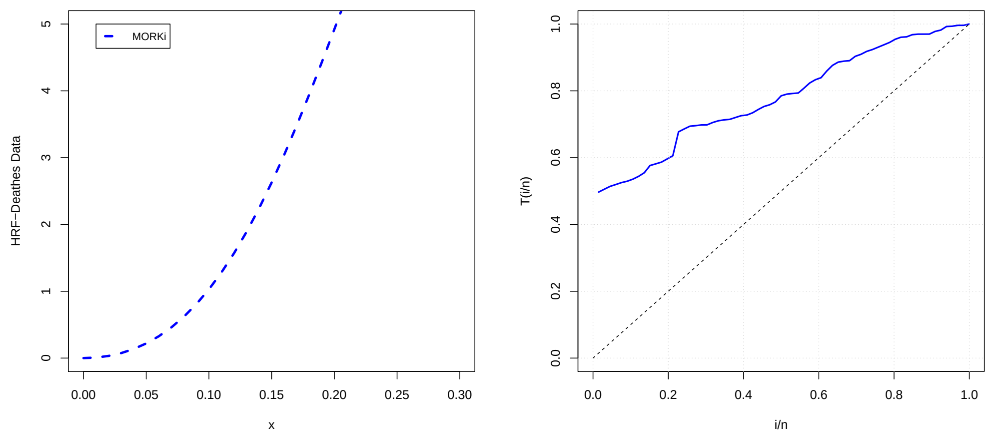

Figure 9: The HRF plots of the MORKi distribution and TTT plot for death COVID-19 data.

The MPS method, which is the best estimation method according to the study in the previous section, is also adopted to estimate the parameters of the MORKi model from both data sets. The estimates of the MORKi parameters, the values of W^* (Cramér-von Mises), A^* (Anderson-Darling), K-S and their p-values are reported in Table 11 for the two COVID-19 data sets. Using the estimates in Table 11, we predict the minimum and maximum death and recovery rates on average in every n days. These rates can be estimated by $E(X_{1:n})$ and $E(X_{n:n})$, which represent the expected minimum and maximum order statistics, respectively. Table 12 gives the predictions of $E(X_{1:n})$ and $E(X_{n:n})$ for $n = 10(10)170$. Based on the results in this table using a sample of 170 days, it is expected to have 0.049 and 0.581 as minimum and maximum death rates. Similarly, it is expected to have 0.086 and 0.899 as minimum and maximum recovery rates.

Table 11: The MPS estimates of the MORKi parameters, K-S and p-value for both COVID-19 data.

Data Set	Estimates		W^*	A^*	K-S (stat)	K-S (p-value)
Recovery Data	$\hat{\delta}$ =0.97897	$\hat{\varphi}$ =19.73984	0.11372	0.86262	0.07357	0.84765
Death Data	$\hat{\delta}$ =2.86874	$\hat{\varphi}$ =0.049208	0.16153	0.96724	0.09444	0.57528

Table 12: Estimates of $E(X_{1:n})$ and $E(X_{n:n})$ for death and recovery rates.

n	Death Rate		Recovery Rate	
	$E(X_{1:n})$	$E(X_{n:n})$	$E(X_{1:n})$	$E(X_{n:n})$
10	0.123	0.447	0.434	0.853
20	0.099	0.489	0.331	0.868
30	0.087	0.511	0.273	0.876
40	0.079	0.525	0.234	0.880
50	0.073	0.535	0.205	0.884
60	0.069	0.543	0.183	0.886
70	0.066	0.550	0.166	0.889
80	0.063	0.555	0.151	0.890
90	0.061	0.560	0.139	0.892
100	0.059	0.564	0.129	0.893
110	0.057	0.567	0.121	0.894
120	0.055	0.570	0.113	0.895
130	0.054	0.573	0.106	0.896
140	0.052	0.575	0.100	0.897
150	0.051	0.577	0.095	0.898
160	0.050	0.580	0.090	0.899
170	0.049	0.581	0.086	0.899

7 Concluding Remarks

We introduced a two-parameter bounded model, called the Marshall-Olkin reduced-Kies (MORKi) distribution, which make it suitable for modeling skewed data. Some of its mathematical properties are derived. The model parameters are estimated by eight classical methods. A detailed simulation study is conducted to explore and compare the performance of these estimators. We also provide a guideline to choose the best method for estimating its parameters by exploring the ordering performance of these estimation methods, based on partial and overall ranks, which is of great interest to applied statisticians and engineers. The simulation showed that the maximum product of spacings method outperforms the other estimation methods. Hence, we can confirm the superiority of this estimation method for the MORKi distribution. Finally, two real COVID-19 data sets from Spain are modeled to illustrate the applicability of the proposed model.

References

Afify, A. Z., Cordeiro, G. M., Ibrahim, N. A., Jamal, F., Elgarhy, M., and Nasir, M. A. (2021). The Marshall-Olkin odd Burr III-G family: theory, estimation, and engineering applications. IEEE Access, 9, 4376–4387.

Afify, A. Z., Kumar, D. and Elbatal, I. (2020). Marshall–Olkin power generalized Weibull distribution with applications in engineering and medicine. Journal of Statistical Theory and Applications, 19, 223-237.

Al-Babtain, A. A., Sherwani, R. A. K., Afify, A. Z., Aidi, K., Nasir, M. A., Jamal, F. and Saboor, A. (2021). The extended Burr-R class: properties, applications and modified test for censored data. AIMS Mathematics, 6, 2912-2931.

Al-Babtain, A. A., Shakhatreh, M. K., Nassar, M. and Afify, A. Z. (2020). A new modified Kies family: properties, estimation under complete and type-II censored samples, and engineering applications. Mathematics, 8, 1345.

Cheng, R. and Amin, N. (1979). Maximum Product of Spacings Estimation with Application to the Lognormal Distribution (Mathematical Report 79-1); University of Wales IST: Cardiff, UK.

Cheng, R. and Amin, N. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. J. R. Stat. Soc. Ser. B, 45, 394–403.

Dey, S., Nassar, M. and Kumar, D. (2019). Moments and estimation of reduced Kies distribution based on progressive type-II right censored order statistics. Hacettepe Journal of Mathematics and Statistics, 48, 332-350.

Ghitany, M. E., Al-Hussaini, E. K. and AlJarallah, R. A. (2005) Marshall–Olkin extendedWeibull distribution and its application to censored data. J Appl Stat, 32, 1025–1034.

Ghitany, M. E., Al-Awadhi, F. A. and Alkhalfan, L. A. (2007) Marshall–Olkin extended Lomax distribution and its application to censored data. Commun Stat Theory Methods, 36, 1855-1866.

Gómez, Y. M., Bolfarine, H. and Gómez, H. W. (2019). Gumbel distribution with heavy tails and applications to environmental data. Mathematics and Computers in Simulation, 157, 115-129.

Gomez2018 Gómez-Déniz, E. (2018). Adding a parameter to the exponential and Weibull distributions with applications. Mathematics and Computers in Simulation, 144, 108-119.

Gradshteyn, I. S. and Ryhzik, I. M. (2007), Tables of Integrals, Series and Products, Academic Press, New York.

Kao, J. (1958). Computer methods for estimating Weibull parameters in reliability studies. IRE Reliab. Qual. Control, 13, 15–22.

Kao, J. (1959). A graphical estimation of mixed Weibull parameters in life testing electron tube. Technometrics, 1, 389–407.

Kumar, C. S., Dharmaja, S. H. S. (2013). On reduced Kies distribution. In Collection of Recent Statistical Methods and Applications, Kumar, C.S., Chacko, M., Sathar, E.I.A., Eds. Department of Statistics, University of Kerala Publishers: Trivandrum, India. 111–123.

Kumar, C. S. and Dharmaja, S. H. S. (2017). The exponentiated reduced Kies distribution: properties and applications. Communications in Statistics-Theory and Methods, 46, 8778-8790.

Lemonte, A. J., Cordeiro, G. M. and Moreno–Arenas, G. (2016) A new useful three-parameter extension of the exponential distribution. Statistics, 50, 312-337.

Mahmoudi, E. and Sepahdar, A. (2013). Exponentiated Weibull–Poisson distribution: Model, properties and applications. Mathematics and computers in simulation, 92, 76-97.

Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika, 84, 641–652.

Nair, N. U., Sankaran, P. G. and Balakrishnan, N. (2013). Quantile-based reliability analysis. Basel: Birkhäuser.

Nassar, M., Kumar, D., Dey, S., Cordeiro, G. M. and Afify, A. Z. (2019). The Marshall–Olkin alpha power family of distributions with applications. Journal of Computational and Applied Mathematics, 351, 41-53.

Mansoor, M., Tahir, M. H., Cordeiro, G. M., Provost, S. B. and Alzaatreh, A. (2019). The Marshall-Olkin logistic-exponential distribution. Communications in Statistics-Theory and Methods, 48, 220-234.

Swain, J., Venkatraman, S. and Wilson, J. (1988). Least squares estimation of distribution function in Johnsons translation system. J. Stat. Comput. Simul., 29, 271–297.

Torabi, H., Bagheri, F. L. and Mahmoudi, E. (2018). Estimation of parameters for the Marshall–Olkin generalized exponential distribution based on complete data. Mathematics and Computers in Simulation, 146, 177-185.