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More on the Generalized fuzzy entropy of order α and β

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Generalized fuzzy entropy is more comprehensive than standard fuzzy entropy, as it enhances the performance in theory and application.
New generalized fuzzy entropies are proposed and analyzed. The two proposed measures satisfied the axiomatic requirements of De Luca and Termini (1972), and hence the validation of the measures is established.
Real life example is studied and the performance of the new measures is noted and compared to other measures.

keywords: Fuzzy sets, Fuzzy entropy, generalized fuzzy measure.

1 Introduction

Shannon (1948) introduced the cornerstone entropy in information theory given by

$$H(P) = -\sum_{i=1}^{n} p(x_i)\log(p(x_i))$$

Later on, many generalizations had been introduced and studied; to mention the most familiar ones, Rènyi's entropy Rényi (1961) of order α

$$H_\alpha^R(P) = \frac{1}{1-\alpha}\log\left[\sum_{i=1}^{n}(p(x_i))^\alpha\right]; \qquad \alpha \neq 1, \alpha > 0.$$

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and, Tsallis entropy of order α, Tsallis (1988)

$$H_{\alpha}^{T}(P) = \frac{1}{1-\alpha}\left[1 - \sum_{i=1}^{n}(p(x_i))^{\alpha}\right]; \qquad \alpha \neq 1, \alpha > 0.$$

These entropies have their continuous versions presented by the integrals instead of summation symbol. Both are directly related to the Shannon entropy through the limits, as α goes to 1, these entropies will end up with Shannon entropy itself.
Varma (1966) generalized Shannon entropy to order α and β as

$$H_{\alpha,\beta}^{V}(X) = \frac{1}{\beta-\alpha}\left[\int_{0}^{\infty} f^{\alpha+\beta-1}(x)\right]; \qquad \beta \neq \alpha, \beta \geq 1, \beta - 1 < \alpha < \beta.$$

where X is a non-negative continuous random variable with a probability density function (pdf)$f(x)$. This entropy reduces to Shannon entropy when $\beta = 1$ and $\alpha \to 1$.

Broadly speaking, one-parameter and two-parameter generalizations provides a better entropy measure, Kumar and Singh (2018) stated that Varma's entropy measure is much more flexible due to more parameters; enabling several measurements of uncertainty within a given distribution and increase the scope of application. Amigó et al. (2018) emphasizes that the parametric weighting of the probabilities grants data analysis with additional flexibility. More on generalized entropy measures and applications we refer the reader to Cover and Thomas (2006), Amigó et al. (2018), Furuichi et al. (2012), Ciavolino and Calcagnì (2016); Ciavolino et al. (2014).

Zadeh (1965) introduced the concepts of fuzzy sets, in which fuzziness is considered as a measure of uncertainty. In which an object is not definitely a member of the set or not . The following definition presents the fuzzy set.
Definition A fuzzy set A on a universe $X = (x_1, x_2, ..., x_n)$ is given by

$$A = \{\langle x_i, \mu_A(x_i)\rangle | x_i \in X, \forall i = 1, ..., n\},$$

where $0 \leq \mu_A(x_i) \leq 1$ is the membership degree of x_i to belong to the set A.
The complement of a fuzzy set is defined by changing the membership function $\mu_A(x_i)$ by $1 - \mu_A(x_i)$, i.e.,

$$A^c = \{\langle x_i, 1 - \mu_A(x_i)\rangle | x_i \in X, \forall i = 1, ..., n\},$$

Afterwards, Zadeh (1968) proposed an entropy measure based on fuzzy set

$$H(P) = -\sum_{i=1}^{n} \mu_A(x_i) p(x_i) \log(p(x_i)),$$

which is the Shannon entropy of the fuzzy sets and often referred to as (weighed) fuzzy entropy. Later on, De Luca and Termini (1972) sat up four axioms to a measure in order to be considered as entropy of a fuzzy set based on Shannons function. Generally, fuzzy entropy expresses the amount of average uncertainty or the difficulty in making sure that an element should belong to a set or not. A measure of fuzziness in a fuzzy set should have at least the following axioms:

- **P1 (Sharpness):** $H(A)$ is minimum if and only if A is a crisp set; i.e. $\mu_A(x_i) = 0$ or 1, $\forall i$.
- **P2 (Maximality):** $H(A)$ is maximum if and only if A is the fuzziest set; i.e. $\mu_A(x_i) = 0.5\ \forall i$.
- **P3 (Resolution):** $H(A^*) \leq H(A)$, where A^* is a crispier set in comparison with A (or the set A is fuzzier).
- **P4 (Symmetry):** $H(A) = H(A^c)$, i.e., $\mu_A(x_i) = 1 - \mu_A(x_i)$.

De Luca and Termini (1972) presented a dynamic fuzzy entropy given by

$$H(A) = -\frac{1}{n}\sum_{i=1}^{n}\Big[\mu_A(x_i)\log\mu_A(x_i) + \big(1-\mu_A(x_i)\big)\log\big(1-\mu_A(x_i)\big),\Big].$$

Bhandari and Pal (1993) extended the idea of Renyi entropy and introduced the following measure;

$$H_\alpha^{FR}(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\Big[\mu_A(x_i)^\alpha + \big(1-\mu_A(x_i)\big)^\alpha\Big]; \qquad \alpha \neq 1, \alpha > 0.$$

Al-Talib and Al-Nasser (2018) proposed a fuzzy entropy measure outperform the existing generalized measures in terms of the informative degree, the measure is as follows;

$$H_\alpha^{NT}(A) = \sum_{i=1}^{n}\left[\frac{\mu_A(x_i)^{\alpha/2}\big(1-\mu_A(x_i)\big)^{\alpha/2}}{\mu_A(x_i)e^{-\alpha\left(1-\mu_A(x_i)\right)} + \big(1-\mu_A(x_i)\big)e^{-\alpha\mu_A(x_i)}}\right]^{1/\alpha}; \quad \alpha > 0,$$

the importance of the above measure is especially used in multi criteria decision making problems, and the performance in fuzzy setting. In this article, we are interested in generalizing Al-Talib and Al-Nasser (2018) to a two-parameter fuzzy measure.

2 Generalized entropy of order α and β

As pointed out earlier, generalizing a measure guaranties flexible and more applicable measures. We propose the following measures as generalizations to Al-Talib and Al-Nasser (2018).

$$N_\alpha^\beta(A) = e^{\beta-\alpha}\cdot\sum_{i=1}^{n}\left[\frac{\mu_A(x_i)^{\frac{\alpha-\beta}{2}}\big(1-\mu_A(x_i)\big)^{\frac{\alpha-\beta}{2}}}{\mu_A(x_i)e^{(\beta-\alpha)\left(1-\mu_A(x_i)\right)} + \big(1-\mu_A(x_i)\big)e^{(\beta-\alpha)\mu_A(x_i)}}\right]^{\frac{-1}{\beta-\alpha}}, \qquad (1)$$

where, $\alpha > 0$, and $0 < \beta < 1$, $\alpha \neq \beta$.

$$Q_\alpha^\beta(A) = e^{\frac{1}{\alpha-\beta}}\cdot\sum_{i=1}^{n}\left[\frac{\mu_A(x_i)^{\frac{\alpha-\beta}{2}}\big(1-\mu_A(x_i)\big)^{\frac{\alpha-\beta}{2}}}{\mu_A(x_i)e^{(\beta-\alpha)\left(1-\mu_A(x_i)\right)} + \big(1-\mu_A(x_i)\big)e^{(\beta-\alpha)\mu_A(x_i)}}\right]^{\frac{-1}{\beta-\alpha}}, \qquad (2)$$

where, $\alpha > 0$, and $0 < \beta < 1$, $\alpha \neq \beta$.

The resemblance between the proposed measures and Al-Talib and Al-Nasser (2018) is beneficial as we have a certain degree of confident that the axioms of De Luca and Termini (1972) are satisfied. Nevertheless, the following theorems shows that both proposed measures are valid fuzzy entropy measures.

Theorem 2.1 *The measure given in equation (1) satisfies all axiomatic requirements of being a fuzzy entropy measure.*

Proof: we will prove the pre-stated axioms **P1–P4**

- **P1 (Sharpness):**

substituting the value of $\mu_A(x_i)$ by 0 or 1, the numerator of equation (1) becomes 0 and hence $N_\alpha^\beta(A)$ is 0, for all $\alpha > 0$, and $0 < \beta < 1$. Conversely, setting $N_\alpha^\beta(A)$ to zero, follows that

$$\mu_A(x_i)^{\frac{\alpha-\beta}{2}}\left(1-\mu_A(x_i)\right)^{\frac{\alpha-\beta}{2}} = 0,$$

then, $\mu_A(x_i) = 0$ or 1.

- **P2 (Maximality):**

The first derivative of $N_\alpha^\beta(A)$ with respect to $\mu_A(x_i)$ is given in APPENDIX A1, and we observe the following:

setting $0 \leq \mu_A(x_i) < 0.5$, the first derivative is positive, and when $0.5 < \mu_A(x_i) \leq 1$, the derivative is negative, $\forall \alpha > 0$ and $0 < \beta < 1$. When $\mu_A(x_i) = 0.5$, we end up with $\frac{\partial N_\alpha^\beta(\mu_A(x_i))}{\partial \mu_A(x_i)} = 0$.

The following figure presents these findings.

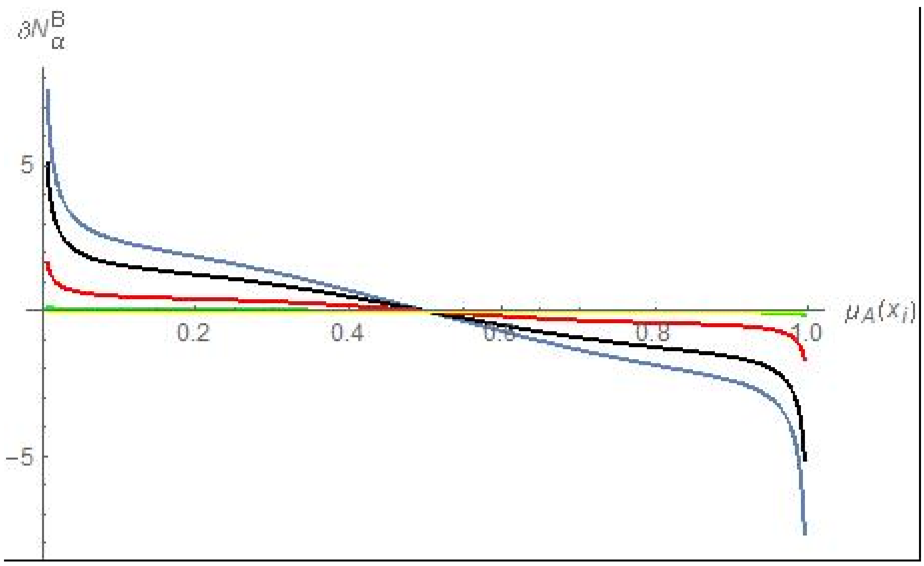

Figure 1: First derivative of $N_\alpha^\beta(A)$ at different values of α and β (blue line (0.2,0.3),black line(0.8,0.4),red line(2,0.6),green line(4,0.1), yellow line(16,0.6))

Its clear that, $N_{\alpha}^{\beta}(\mu_A(x_i))$ has its maximum at $\mu_A(x_i) = 0.5$, i.e., when the entropy measure is at it fuzziest phase. For more investigation, we evaluate the second derivative (see Appendix A2).

The figure below shows the second derivative, we note that the its always negative for different values of α and β. It reaches its maximum at $\mu_A(x_i) = 0.5$.

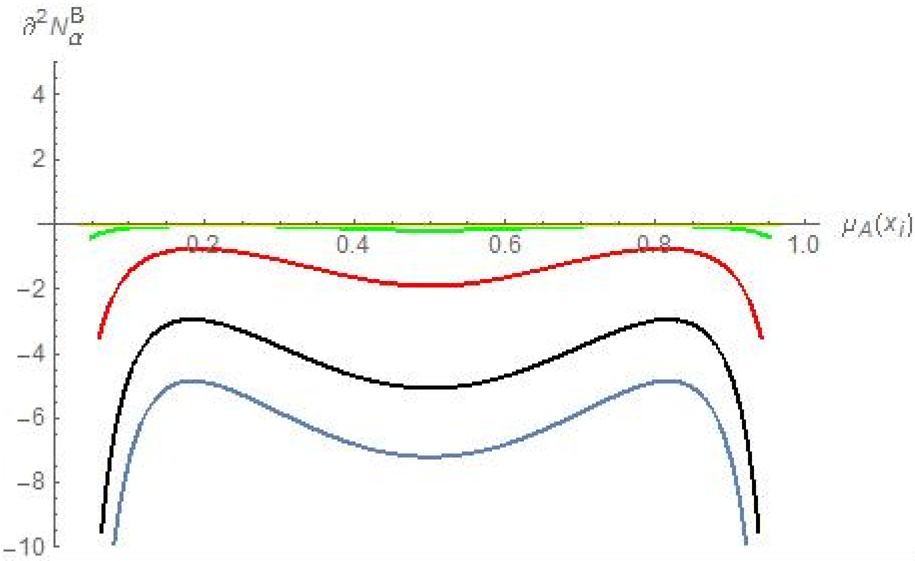

Figure 2: Second derivative of $N_{\alpha}^{\beta}(A)$ at different values of α and β (blue line (0.2,0.3),black line(0.8,0.4),red line(2,0.6),green line(4,0.1), yellow line(16,0.6))

- **P3 (Resolution):**

Table(APPENDIX A3) presents numerical values of the proposed entropy measure for different values of α and β alongside Figure 1, that the most fuzzy values appears when the membership function have a value of 0.5. Also the concavity is clear; as the values of the entropy measure are increasing when $\mu_A(x_i)$ is between 0 and 0.5, and are decreasing between 0.5 and 1. which proves Axiom **P3**; that the crispier the set (less membership function value) the lower the value of the entropy measure. It is notable that the value of the entropy measure is 0 when $\mu_A(x_i)$ equals 0 or 1, which goes along with Axiom **P1**.

- **P4 (Symmetry):**

Substituting $\mu_A(x_i)$ by $1 - \mu_A(x_i)$ in measure 1, we end up with $N_{\alpha}^{\beta}(\mu_A(x_i)) = N_{\alpha}^{\beta}(1 - \mu_A(x_i)), \forall i$. this conclusion can be obtained by noticing symmetry from Figure 2 and the results from Table APPENDIX A3.

And hence, the proof is done.

Theorem 2.2 *The measure given in equation (2) satisfies all axiomatic requirements of being a fuzzy entropy measure.*

Proof: we will prove the pre-stated axioms **P1–P4**

- **P1 (Sharpness):**

substituting the value of by 0 or 1, the numerator of equation (2) becomes 0 and hence $Q_\alpha^\beta(A)$ is 0, for all $\alpha > 0$, and $0 < \beta < 1$. Conversely, setting $Q_\alpha^\beta(A)$ to zero, follows that the numerator equals to zero, i.e.,

$$\mu_A(x_i)^{\frac{\alpha-\beta}{2}}\left(1-\mu_A(x_i)\right)^{\frac{\alpha-\beta}{2}} = 0,$$

then, $\mu_A(x_i) = 0$ or 1.

- **P2 (Maximality):**

Appendix B1. presents the first derivative of $Q_\alpha^\beta(A)$ with respect to $\mu_A(x_i)$,we notice that; when $0 \leq \mu_A(x_i) < 0.5$, the first derivative is positive, and when $0.5 < \mu_A(x_i) \leq 1$, the derivative is negative, $\forall\alpha > 0$ and $0 < \beta < 1$. When $\mu_A(x_i) = 0.5$, we end up with $\frac{\partial Q_\alpha^\beta(\mu_A(x_i))}{\partial \mu_A(x_i)} = 0$. The following figure presents these findings.

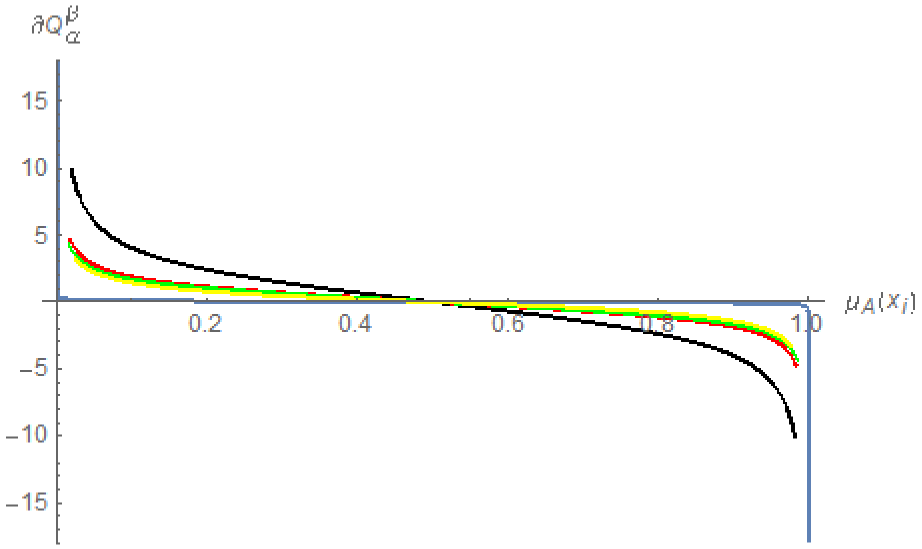

Figure 3: First derivative of $Q_\alpha^\beta(A)$ at different values of α and β (blue line (0.2,0.3),black line(0.8,0.4),red line(2,0.6),green line(4,0.1), yellow line(16,0.6))

Its clear that, $Q_\alpha^\beta(\mu_A(x_i))$ has its maximum at $\mu_A(x_i) = 0.5$, i.e., when the entropy measure is at it fuzziest phase. For more investigation we evaluate the second derivative and its presented in Appendix B2.

The figure below shows the second derivative, we note that the its negative and reaches its maximum at $\mu_A(x_i) = 0.5$.

- **P3 (Resolution):**

Table(APPENDIX B3) presents numerical values of the proposed entropy measure for different values of α and β alongside Figure 3 that the most fuzzy values appears when the membership function have a value of 0.5, also the concavity is clear; as the values of

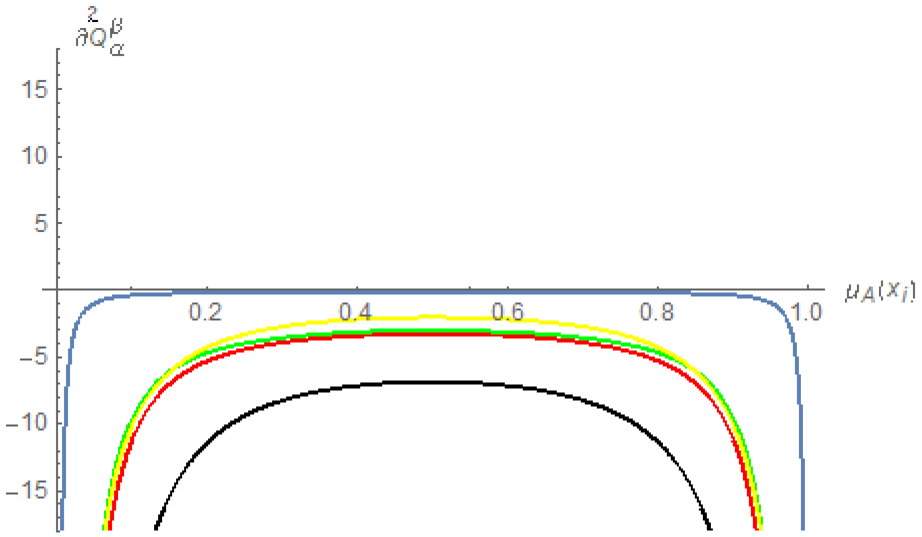

Figure 4: Second derivative of $Q_{\alpha}^{\beta}(A)$ at different values of α and β (blue line (0.2,0.3),black line(0.8,0.4),red line(2,0.6),green line(4,0.1), yellow line(16,0.6))

the entropy measure are increasing when $\mu_A(x_i)$ is between 0 and 0.5, and are decreasing between 0.5 and 1. which proves Axiom **P3**; that the crispier the set (less membership function value) the lower the value of the entropy measure. It is notable that the value of the entropy measure is 0 when $\mu_A(x_i)$ equals 0 or 1, which goes along with Axiom **P1**.

- **P4 (Symmetry):**

Substituting $\mu_A(x_i)$ by $1 - \mu_A(x_i)$ in measure 1, we end up with $N_{\alpha}^{\beta}(\mu_A(x_i)) = N_{\alpha}^{\beta}(1 - \mu_A(x_i)), \forall i$. this conclusion can be obtained by noticing symmetry from Figure 3 and the results from Table APPENDIX B3.

And hence, the proof is done.

3 Application and comparison

In linguistics, phrases such as " less", " more", "very" are used to express ambiguity or probability. These phrases posses fuzziness and uncertainty, hence they are considered as fuzzy sets. For more details see, Hung and Yang (2006).
Linguistic hedges and phrases are defined on a fuzzy set A as follows:

$$\text{Very } A{=}A^2, \text{ more or less } A = A^{1/2}$$
$$\text{Quite Very } A = A^3, \text{ Very very } A = A^4$$

where,

$$A^n = \{\langle x_i, (\mu_A(x_i))^n\rangle | x_i \in X, \forall i = 1, ..., n\},$$

the entropy should satisfy the following requirement to be considered to have a good performance:

$$H(A^{1/2}) > H(A) > H(A^2) > H(A^3) > H(A^4) \tag{3}$$

Joshi and Kumar (2018) presented a two-parameter fuzzy entropy measure to study the discriminate power of that attribute in decision- making process, they stated "the greater the value of entropy corresponding to a special attribute, which implies the smaller attribute's weight, the less is the discriminate power of that attribute in decision-making process."
Their proposed measure (denoted here as ${}_{JK}H_{\alpha}^{\beta}$) compared and was found to be superior to the well known entropy measures. In Table (1), we subject to the same data set used by Joshi and Kumar (2018).

Table 1: Results of different fuzzy entropy measures

Fuzzy set	${}_{JK}H_{0.5}^{15}$	$N_{0.5}^{15}$	$Q_{0.5}^{15}$
$A^{1/2}$	0.4672	$0.1216 * 10^{-7}$	1.7521
A	0.4134	$0.1037 * 10^{-7}$	1.6309
A^2	0.2834	$0.6720 * 10^{-8}$	1.2045
A^3	0.2202	$0.5327 * 10^{-8}$	0.9249
A^4	0.1906	$0.4756 * 10^{-8}$	0.7566

From the below table that both of our proposed measures satisfies equation (3), i.e.,

$$N_{\alpha}^{\beta}(A^{1/2}) > N_{\alpha}^{\beta}(A) > N_{\alpha}^{\beta}(A^2) > N_{\alpha}^{\beta}(A^3) > N_{\alpha}^{\beta}(A^4),$$

$$Q_{\alpha}^{\beta}(A^{1/2}) > Q_{\alpha}^{\beta}(A) > Q_{\alpha}^{\beta}(A^2) > Q_{\alpha}^{\beta}(A^3) > Q_{\alpha}^{\beta}(A^4).$$

Despite that Q_{α}^{β} has the worst results among the three measures, but we can see that the other proposed measure; N_{α}^{β} have great performance. which nominates it to be called the crispiest fuzzy measure.

4 Conclusion

In this article, we proposed two new fuzzy entropy measures of order α and β, a generalization of Al-Talib and Al-Nasser (2018). We managed to prove that both measures are indeed fuzzy entropy measures. a real life application proves the effectiveness of one of the measures. In future, we will propose an interval valued fuzzy measure based on the results of this work .

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APPENDIX A1. FIRST DERIVATIVE OF $N_{\alpha}^{\beta}\mu_A(x_i)$

$$\frac{\partial N_{\alpha}^{\beta}(\mu_A(x_i))}{\partial \mu_A(x_i)} = -e^{\beta-\alpha}\left[\frac{\big((1-\mu_A(x_i))\mu_A(x_i)\big)^{\frac{\alpha-\beta}{2}}}{e^{(\beta-\alpha)\mu_A(x_i)}(1-\mu_A(x_i))+e^{(\beta-\alpha)(1-\mu_A(x_i))}(\mu_A(x_i))}\right]\times$$

$$\left(\frac{-e^{\alpha+2\beta\mu_A(x_i)}(1-\mu_A(x_i))(\beta-\alpha-2\mu_A(x_i)-2(\beta-\alpha)\mu_A(x_i)^2}{2(\beta-\alpha)(1-\mu_A(x_i))\mu_A(x_i)\big(e^{\alpha+2\beta\mu_A(x_i)}(1-\mu_A(x_i))+e^{\beta+2\alpha\mu_A(x_i)}\mu_A(x_i)\big)}\right.$$

$$\left.+\frac{e^{\beta+2\alpha\mu_A(x_i)}\mu_A(x_i)\Big(2\mu_A(x_i)-\beta-2-2\beta(\mu_A(x_i)-2)\mu_A(x_i)+\alpha\big(1+2(\mu_A(x_i)-2)\mu_A(x_i)\big)\Big)}{2(\beta-\alpha)\mu_A(x_i)(1-\mu_A(x_i))\big(e^{\alpha+2\beta\mu_A(x_i)}(1-\mu_A(x_i))+e^{\beta+2\alpha\mu_A(x_i)}\mu_A(x_i)\big)}\right)$$

APPENDIX A2. SECOND DERIVATIVE OF $N_{\alpha}^{\beta}\mu_A(x_i)$

$$\frac{\partial^2 N_{\alpha}^{\beta}\mu_A(x_i)}{\partial^2\mu_A(x_i)} =$$

$$e^{-\alpha+\beta}\left(-\frac{(-(-1+\mu_A(x_i))(\mu_A(x_i)))^{\frac{\alpha-\beta}{2}}}{e^{(-\alpha+\beta)(\mu_A(x_i))}(-1+(\mu_A(x_i)))-e^{(\alpha-\beta)(-1+\mu_A(x_i))}(\mu_A(x_i))}\right)^{\frac{1}{\alpha-\beta}}$$

$$\times\frac{1}{\left(4(\alpha-\beta)^2(-1+(\mu_A(x_i)))^2(\mu_A(x_i))^2\left(e^{\alpha+2\beta(\mu_A(x_i))}(-1+(\mu_A(x_i)))-e^{\beta+2\alpha(\mu_A(x_i))}(\mu_A(x_i))\right)^2\right)}$$

$$\Big[16\alpha^3 e^{(\alpha+\beta)(1+2(\mu_A(x_i)))}(-1+(\mu_A(x_i)))^3(\mu_A(x_i))^3 - 2e^{(\alpha+\beta)(1+2\mu_A(x_i))}(-1+(\mu_A(x_i)))(\mu_A(x_i))(2\beta+4(-1+(\mu_A(x_i)))\mu_A(x_i)+$$

$$8\beta^3(-1+\mu_A(x_i))^2\mu_A(x_i)^2\beta^2(-1-4(-2+\mu_A(x_i))(-1+\mu_A(x_i))\mu_A(x_i)(1+(\mu_A(x_i))))) + e^{2\beta+4\alpha\mu_A(x_i)}\mu_A(x_i)^2 4(-1+\mu_A(x_i))^2$$

$$-4\beta(-1+\mu_A(x_i))(\mu_A(x_i))(-3+2(\mu_A(x_i))) + \beta^2(-1+4(-1+(\mu_A(x_i)))(\mu_A(x_i))(1+(-3+(\mu_A(x_i)))\mu_A(x_i)))$$

$$+e^{2\alpha+4\beta\mu_A(x_i)}(-1+\mu_A(x_i))^2\left(4\mu_A(x_i)^2+4\beta(-1+\mu_A(x_i))(\mu_A(x_i))(1+2(\mu_A(x_i)))+\beta^2\left(-1+4(\mu_A(x_i))\left(1-2x+(\mu_A(x_i))^3\right)\right)\right)$$

$$+\alpha^2(e^{2\beta+4\alpha(\mu_A(x_i))}\mu_A(x_i)^2(-1+4(-1+\mu_A(x_i))(\mu_A(x_i))(1+(-3+\mu_A(x_i))\mu_A(x_i))) - 2e^{(\alpha+\beta)(1+2\mu_A(x_i))}$$

$$\times\Big[(-1+\mu_A(x_i))\mu_A(x_i)(-1+4(-1+\mu_A(x_i))(\mu_A(x_i))(2+(-1+6\beta)(-1+\mu_A(x_i))\mu_A(x_i)))\Big]$$

$$+e^{2\alpha+4\beta\mu_A(x_i)}(-1+\mu_A(x_i))^2\left(-1+4\mu_A(x_i)\left(1-2\mu_A(x_i)+(\mu_A(x_i))^3\right)\right)) + 2\alpha(24\beta^2 e^{(\alpha+\beta)(1+2(\mu_A(x_i)))}(-1+\mu_A(x_i))^3\mu_A(x_i)^3$$

$$-2(-1+\mu_A(x_i))(\mu_A(x_i))\left(-e^{(\alpha+\beta)(1+2\mu_A(x_i))} - e^{2\beta+4\alpha(\mu_A(x_i))}\mu_A(x_i)^2(-3+2\mu_A(x_i)) + e^{2\alpha+4\beta\mu_A(x_i)}(-1+(\mu_A(x_i)))^2(1+2(\mu_A(x_i)))\right)$$

$$+\beta(e^{2\beta+4\alpha(\mu_A(x_i))}\left(\mu_A(x_i)^2+4\mu_A(x_i)^3-16(\mu_A(x_i))^4+16(\mu_A(x_i))^5-4\mu_A(x_i)^6\right) -$$

$$2e^{(\alpha+\beta)(1+2(\mu_A(x_i)))}(-1+(\mu_A(x_i)))\mu_A(x_i)(1+4(-2+\mu_A(x_i))(-1+\mu_A(x_i))(\mu_A(x_i))(1+(\mu_A(x_i))))$$

$$-e^{2\alpha+4\beta(\mu_A(x_i))}(-1+(\mu_A(x_i)))^2\left(-1+4(\mu_A(x_i))\left(1-2(\mu_A(x_i))+\mu_A(x_i)^3\right)\right)))\Big]$$

APPENDIX A3. DIFFERENT VALUES OF $N_\alpha^\beta \mu_A(x_i)$ FOR SEVERAL VALUES OF α and β

$\mu_A(x_i)$	α	β	$N_\alpha^\beta(\mu_A(x_i))$	α	β	$N_\alpha^\beta(\mu_A(x_i))$	α	β	$N_\alpha^\beta(\mu_A(x_i))$	α	β	$N_\alpha^\beta(\mu_A(x_i))$	α	β	$N_\alpha^\beta(\mu_A(x_i))$
0			0			0			0			0			0
0.1			0.3981			0.2639			0.0859			0.0068			$6.845 * 10^{-8}$
0.2			0.6105			0.4046			0.1313			0.0104			$1.016 * 10^{-7}$
0.3			0.7721			0.5141			0.1683			0.0134			$1.298 * 10^{-7}$
0.4			0.8753			0.5856			0.1939			0.0157			$1.546 * 10^{-7}$
0.5	0.2	0.3	0.9110	0.8	0.5	0.6107	2	0.6	0.2032	4	0.1	0.0166	16	0.6	$1.690 * 10^{-7}$
0.6			0.8753			0.5856			0.1939			0.0157			$1.546 * 10^{-7}$
0.7			0.7721			0.5141			0.1683			0.0134			$1.298 * 10^{-7}$
0.8			0.6105			0.4046			0.1313			0.0104			$1.016 * 10^{-7}$
0.9			0.3981			0.2639			0.0859			0.0068			$6.845 * 10^{-8}$
1			0			0			0			0			0

APPENDIX B1. FIRST DERIVATIVE OF $Q_{\alpha}^{\beta}\mu_A(x_i)$

$$\left(-\frac{e^{1+\alpha}\left(-(-1+\mu_A(x_i))\mu_A(x_i)\right)^{\frac{\alpha-\beta}{2}}}{e^{\alpha-\alpha\mu_A(x_i)}(-1+\mu_A(x_i))+e^{\alpha+\beta\mu_A(x_i)}(-1+\mu_A(x_i))-e^{\alpha\mu_A(x_i)}\mu_A(x_i)-e^{\alpha+\beta-\beta\mu_A(x_i)}\mu_A(x_i)}\right)^{\frac{1}{\alpha-\beta}}$$

$$(e^{(2\alpha+\beta)\mu_A(x_i)}\mu_A(x_i)(-2-\beta+2(1+\beta)\mu_A(x_i)) + ae^{(2\alpha+\beta)\mu_A(x_i)}\mu_A(x_i)(1+2(-2+\mu_A(x_i))\mu_A(x_i))$$
$$+\alpha e^{\alpha}\left(e^{(\alpha+2\beta)\mu_A(x_i)}(-1+\mu_A(x_i))(-1+2\mu_A(x_i)) + e^{\beta+\alpha\mu_A(x_i)}\left(\mu_A(x_i)-2\mu_A(x_i)^2\right) + e^{\beta\mu_A(x_i)}(-1+\mu_A(x_i))\left(-1+2\mu_A(x_i)^2\right)\right)+$$
$$e^{\alpha}(-2\left(e^{\beta\mu_A(x_i)} + e^{(\alpha+2\beta)\mu_A(x_i)} - e^{\beta+\alpha\mu_A(x_i)}\right)(-1+\mu_A(x_i))x + \beta(e^{\beta\mu_A(x_i)}(-1+(3-2\mu_A(x_i))\mu_A(x_i))$$
$$-e^{\beta+\alpha\mu_A(x_i)}\mu_A(x_i)(1+2(-2+\mu_A(x_i))\mu_A(x_i)) - e^{(\alpha+2\beta)\mu_A(x_i)}(-1+\mu_A(x_i))\left(-1+2\mu_A(x_i)^2\right))))$$

$$*\frac{1}{\left(2(\alpha-\beta)(-1+\mu_A(x_i))\mu_A(x_i)\left(e^{\alpha+\beta\mu_A(x_i)}(-1+\mu_A(x_i))+e^{\alpha+\alpha\mu_A(x_i)+2\beta\mu_A(x_i)}(-1+\mu_A(x_i))-e^{(2\alpha+\beta)\mu_A(x_i)}\mu_A(x_i)-e^{\alpha+\beta+\alpha\mu_A(x_i)}\mu_A(x_i)\right)\right)}$$

APPENDIX B2. SECOND DERIVATIVE OF $Q_{\alpha}^{\beta}\mu_A(x_i)$

$$\frac{\partial^2 Q_{\alpha}^{\beta}(\mu_A(x_i))}{\partial^2 \mu_A(x_i)} =$$

$$\frac{1}{\alpha-\beta}\left(-1+\frac{1}{\alpha-\beta}\right)e^{\frac{1}{\alpha-\beta}}(A)^{-2+\frac{1}{\alpha-\beta}}\left(\frac{(\alpha-\beta)A\mu_A(x_i)^{-1}}{2L}-\frac{(\alpha-\beta)A(\mu_A(x_i))^{-1}}{2L}-\frac{AC}{L^2}\right)^2+$$

$$\frac{1}{\alpha-\beta}e^{\frac{1}{\alpha-\beta}}\left(\frac{A}{L}\right)^{-1+\frac{1}{\alpha-\beta}}\left(\begin{array}{l}\frac{\left(-1+\frac{\alpha-\beta}{2}\right)(\alpha-\beta)A(\mu_A(x_i))^{-2}}{2L}-\frac{(\alpha-\beta)^2A\mu_A(x_i)^{-1}(1-\mu_A(x_i))}{2L}^{-1}+\\ \frac{\left(-1+\frac{\alpha-\beta}{2}\right)(\alpha-\beta)A(1-\mu_A(x_i))^{-2}}{2L}-\frac{(\alpha-\beta)A\mu_A(x_i)^{-1}C}{L^2}+\\ \frac{(\alpha-\beta)A(1-\mu_A(x_i))^{-1}C}{L^2}+\frac{2AC^2}{L^3}-\frac{AZ}{L^2}\end{array}\right)$$

where

$$A=(1-\mu_A(x_i))^{\frac{\alpha-\beta}{2}}\mu_A(x_i)^{\frac{\alpha-\beta}{2}}$$

$$L=e^{-\alpha\mu_A(x_i)}(1-\mu_A(x_i))+e^{\beta\mu_A(x_i)}(1-\mu_A(x_i))+e^{-\alpha(1-\mu_A(x_i))}\mu_A(x_i)+e^{\beta(1-\mu_A(x_i))}\mu_A(x_i)$$

$$C=e^{-\alpha(1-\mu_A(x_i))}+e^{\beta(1-\mu_A(x_i))}-e^{-\alpha\mu_A(x_i)}-e^{\beta\mu_A(x_i)}-\alpha e^{-\alpha\mu_A(x_i)}(1-\mu_A(x_i))+$$
$$\beta e^{\beta\mu_A(x_i)}(1-\mu_A(x_i))+\alpha e^{-\alpha(1-\mu_A(x_i))}\mu_A(x_i)-\beta e^{\beta(1-\mu_A(x_i))}\mu_A(x_i)$$

$$Z=2\alpha e^{-\alpha(1-\mu_A(x_i))}-2\beta e^{\beta(1-\mu_A(x_i))}+2\alpha e^{-\alpha\mu_A(x_i)}-2\beta e^{\beta\mu_A(x_i)}+\alpha^2e^{-\alpha\mu_A(x_i)}(1-\mu_A(x_i))+$$
$$\beta^2e^{\beta x}(1-\mu_A(x_i))+\alpha^2e^{-\alpha(1-\mu_A(x_i))}x+\beta^2e^{\beta(1-\mu_A(x_i))}\mu_A(x_i)$$

APPENDIX B3. DIFFERENT VALUES OF $Q_{\alpha}^{\beta}\mu_A(x_i)$ FOR SEVERAL VALUES OF α and β

$\mu_A(x_i)$	α	β	$Q_{\alpha}^{\beta}(\mu_A(x_i))$	α	β	$Q_{\alpha}^{\beta}(\mu_A(x_i))$	α	β	$Q_{\alpha}^{\beta}(\mu_A(x_i))$	α	β	$Q_{\alpha}^{\beta}(\mu_A(x_i))$	α	β	$Q_{\alpha}^{\beta}(\mu_A(x_i))$
0			0			0			0			0			0
0.1			0.3981			0.8591			0.3903			0.3423			0.3145
0.2			0.6105			1.1705			0.5365			0.4741			0.4205
0.3			0.7721			1.3609			0.6269			0.5567			0.4807
0.4			0.8753			1.4673			0.6775			0.6033			0.5129
0.5	0.2	0.3	0.9110	0.8	0.5	1.5018	2	0.6	0.6939	4	0.1	0.6184	16	0.6	0.5232
0.6			0.8753			1.4673			0.677			0.6033			0.5129
0.7			0.7721			1.3609			0.6269			0.5567			0.4807
0.8			0.6105			1.1705			0.5365			0.4741			0.4205
0.9			0.3981			0.8591			0.3903			0.3423			0.3145
1			0			0			0			0			0