
**Electronic Journal of Applied Statistical Analysis**
**EJASA, Electron. J. App. Stat. Anal.**
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v14n2p417

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Published: 20 November 2021

*Electronic Journal of Applied Statistical Analysis*
Vol. 14, Issue 02, November 2021, 417-424
DOI: 10.1285/i20705948v14n2p417

# A stochastic diffusion model of Lorenz Curve with a birth-and-death diffusion and general external effect processes

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The Lorenz curve has been used as an essential instrument in the studies of inequality of wealth and income distributions. It was further devised as an inequality research tool in many disciplines. A stochastic Lorenz Curve model is considered with a birth-and-death diffusion and general external effect processes in this paper. Also, the corresponding Stochastic Diffusion Gini-Index is obtained. The suggested functional form attains the properties of the Lorenz curve and works nicely, as shown in the numerical example.

**keywords:** Birth-Death Diffusion Process, General External Effect, Gini-Index, Lorenz Curve.

## 1 Introduction and Background

Lorenz curve is defined as a real-valued function that assigns to each value of the ordered beneficiaries cumulative distribution by share size, a value of the corresponding aggregate shares cumulative distribution according to Al-Hussainan, 2000. Any proposed functional form of the Lorenz curve should satisfy the following Lorenz curve properties. (i) $L(p) \geq 0$; $p \in [0, 1]$ (ii) $L(0) = 0$; $L(1) = 1$ (iii) $L(p) \leq p$ (iv) $L'(p) \geq 0$; $L''(p) \geq 0$.

The Lorenz curve has been devised to study the inequality of distribution of recourses on beneficiaries in many disciplines. Including; but not limited to, peace research (Høivik, 1977), Epidemiology (Lee, 1996), and decision analysis (Pham-Gia, 1995).

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ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

Several authors have proposed different functional forms for the empirical and the theoretical Lorenz curves and their corresponding inequality indices (e.g., Gini index). For example, an exponential form of the Lorenz curve has suggested by Kakwani and Podder (1973). Basmann et al. (1990) introduced a Kakwani and Podder form generalization by taking the percentile's power to be a linear combination of the exponential function's scale parameter. Villasenor and Arnold (1989) proposed a family of elliptical Lorenz curves. Al-Eideh and Al-Hussainan (2002) have recently proposed using the Quasi-Lorenz diffusion model, a quasi-functional form of the Lorenz curve. Their new form was obtained using a stochastic diffusion model for the differences between consecutive 378 observed wealth data points or income quantiles. The model was taken to be as a solution of an integrodifferential equation.

This paper focuses on studying a new model form of the stochastic Lorenz curve with a birth-and-death stochastic diffusion process. Also, the associated Gini-Index is found. The suggested functional form attains the properties of the Lorenz curve and works nicely, as shown in the numerical example (Al-Eideh and Al-Hussainan, 2002; Al-Hussainan, 2000; Atkinson et al., 1970; Basmann et al., 1990; Gihman et al., 1976; Gini and Mutabilita, 1912; Høivik, 1977; Kakwani and Podder, 1973; Lee, 1996; Pham-Gia, 1995; Gini et al., 1955; Taylor and Karlin, 1984; Theil, 1967).

## 2 The Stochastic Diffusion Model of Lorenz Curve with a Birth-and-Death Stochastic Diffusion and General External Effect Processes

Let $Xp$ be the p-th quantile function of a given random variable. Conditional on the initial value $X_0 = x$, the successive quantile differences assuming $n$ is the sample size of the observed income or wealth data are given by

$$\Delta X_p = X_{P+\frac{1}{n}} - X_p \tag{1}$$

In general, for large $n$, $\Delta X_p$ can be approximated by the differential $dX_p$. Thus we assume that the quantile differences follow a stochastic diffusion process $\{X_p; p \geq 0\}$ where the drift and diffusion coefficients $b$ and $a$ are both proportional to $X_p$ . Assume the stochastic diffusion process $X_p$ interrupts by jumps having magnitude with distribution function $H_x(.)$ and occurs at a constant rate $c$ . Then $\{X_p; p \geq 0\}$ is a Markovian process with State Space $[0, \alpha)$ and should be considered as the solution of the SD equations given the initial quantile is $X_0$:

$$dX_p = bX_p dp + aX_p dW_p - X_p - \ dZ_p \tag{2}$$

Where $\{W_p : p \geq 0\}$ is a Wiener process with mean 0 and variance $\sigma^2 p$. Also, the process $(Z_p : p \geq 0)$ is a Compound Poisson Process with jump size distribution $H_x$ and external jump rate $c > 0$ is a given by

$$Z_p = \sum_{i=1}^{N_p} Y_i \tag{3}$$

Note that the process $\{N_p\}$ is a Poisson with mean rate $c$, ( $c$ is the external jump rate), also the random variables $Y1, Y2, Y3, ...$ are independently and identically distributed with Distribution Function $H_x(.)$, with mean and variance, $\mu = E(Y_1)$ and $\nu^2 = Var(Y_1)$ respectively. Using the random sums formulas, we can determine the moments of $Z_p$ as follows:

$$\mathrm{E}[Z_p] = c\ \mu p \tag{4}$$

and

$$Var[Z_p] = c(\nu^2 + \mu^2)p \tag{5}$$

According to Gihman et al. (1976), if we assume the conditions of uniqueness and existence are satisfied. Then the solution of the above SDE in equation (2) is given by:

$$X_p = X_0\ exp\{bp + aW_p - Z_p\} \tag{6}$$

Definition: A Stochastic Diffusion Lorenz model denoted by $DL_p(n); 0 \leq p \leq 1$ is defined by $QL_p(n) = p.L_p(n)$. such that

$$L_p(n) = \frac{\int_0^p X_u du}{\int_0^1 X_u du} \tag{7}$$

where $L_p(n)$ is the empirical functional form of the Lorenz curve.

Now, to derive $DL_p(n)$, we substitute the form (6) in equation (7). Hence, we get

$$L_p(n) = \frac{\int_0^p exp\{bu + aW_u - Z_u\}du}{\int_0^1 exp\{bu + aW_u - Z_u\}du} \tag{8}$$

Now, using Al-Eideh and Al-Hussainan (2002) and Taylor and Karlin, 1984 (pp. 177), it is easy to show that

$$\int_0^p exp\{bu + aW_u - Z_u\}du = \frac{2(1-b)}{2a + a^2 - b^2} X_0\ exp\{bp + aW_p - Z_p\} \tag{9}$$

Consequently,

$$\int_0^1 exp\{bu + aW_u - Z_u\}du = \frac{2(1-b)}{2a + a^2 - b^2} X_0\ exp\{b + aW_1 - Z_1\} \tag{10}$$

Therefore, the Stochastic Diffusion Lorenz model $DL_p(n)$ at p using equation (8), (9), and (10) is then given by

$$DL_p(n) = p\ exp\{b(p-1) + a(W_p - W_1) - (Z_p - Z_1)\} \tag{11}$$

where $a > 0, 0 \leq p \leq 1$.

It is easily shown that the Lorenz curve properties are all satisfied by the Stochastic Diffusion Lorenz curve $DL_p(n)$; $DL_0(n) = 0$, $DL_1(n) = 1$, and it is monotone. Also, note that model (11) is robust against the initial value $X_0$ .

# 3 The Stochastic Diffusion Gini Index

There are several measures of income inequality. Among which are the Gini ratio (Gini and Mutabilita, 1912), the entropy index (Theil, 1967), Atkinson's index (Atkinson et al., 1970).

The Stochastic Diffusion Gini index is found to be

$$DG = 2\int_0^1 (p - DL_p(n))dp \tag{12}$$

By computing the definite integral of equation (11), over the open interval $(0, p)$,we get

$$\int_0^p DL_u(n)du = \left[\frac{2(1-b)p}{2a+a^2-b^2} - \frac{4(1-b)^2)}{(2a+a^2-b^2)^2}\right].exp\{b(p-1)+a[W_p-W_1]-[Z_p-Z_1]\}$$

Substituting the result in equation (12) above, we obtain the functional form of the corresponding Stochastic Diffusion Gini index as

$$DG = 1 - \frac{4(1-b)}{2a+a^2-b^2} + \frac{8(1-b)^2}{(2a+a^2-b^2)^2} \tag{14}$$

# 4 Numerical Example

This section considers a numerical example that shows the Stochastic Diffusion Lorenz model $DL_p(n)$ and the Stochastic Diffusion Gini Index DG obtained in sections 2 and 3 respectively, with the general external effect process.

Consider the Stochastic Diffusion Lorenz model $DL_p(n)$ subject to general external effect process with diffusion and drift parameters $a = 1$ and $b = 0, 3, 5$, assuming the general external effect rate $c = 0.02$. Then $DL_p(n)$ versus $p$ is shown in Figure 1 below

The above Figure 1 shows that the Stochastic Diffusion Lorenz Model graph $DL_p(n)$ with general external effect satisfies the Lorenz curve properties. Namely $DL_0(n) = 0$, $DL_1(n) = 1$,and that the curve is convex and monotonically increasing in $p$.

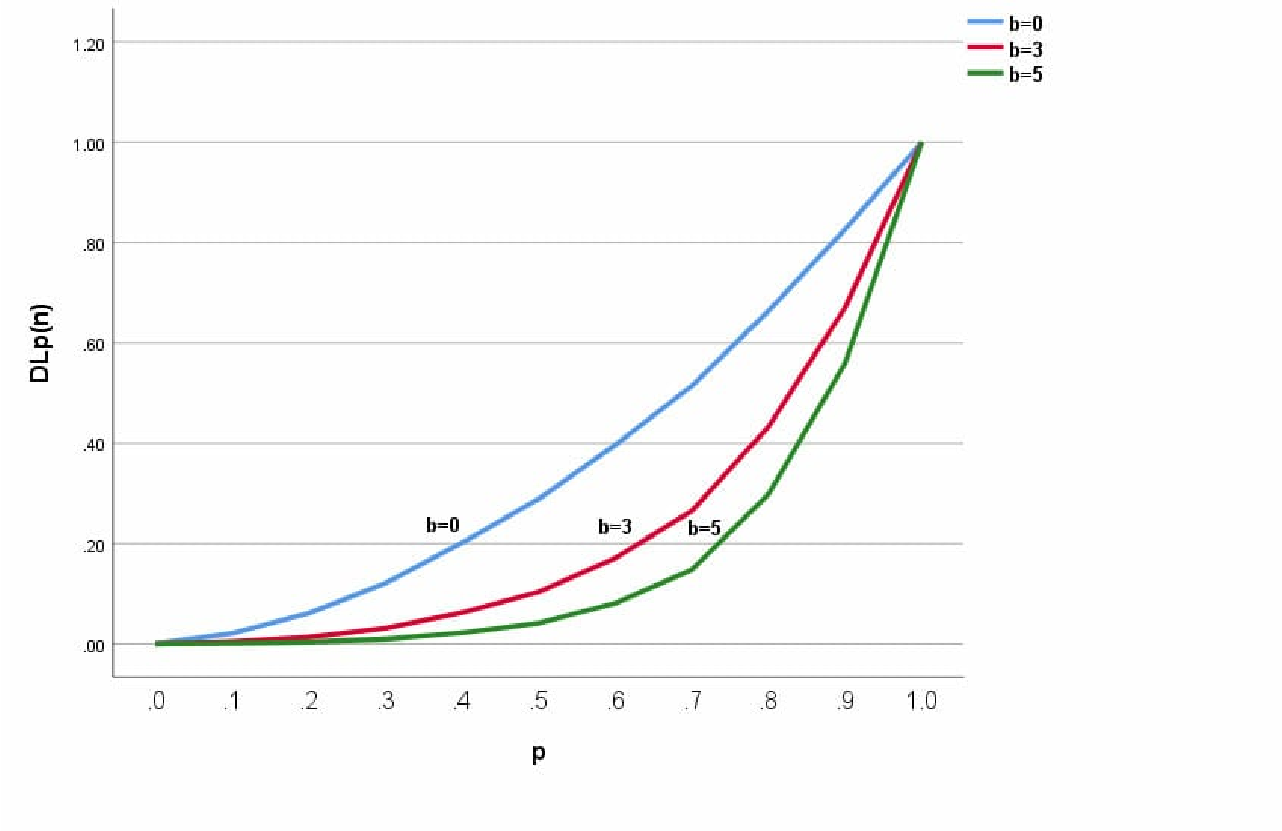


Figure 1: The Stochastic Diffusion Lorenz Model $DL_p(n)$ with General External Effect

Consider the Stochastic Diffusion Gini Index with general external effect as a function of $b$, where $a = 1$, and $c = 0.02$ . Note that total inequality of distribution (i.e.,$DG = 1$) is attained at $b = 1 - \sqrt{2}, 2, 1 + \sqrt{2}$ and $\lim\limits_{b \to \infty} DG = 1$.

It should be noted here that in this example where $a = 1$ and $c = 0.02$, the Stochastic Diffusion Gini index with the general external effect is only valid for $b \in (1 - \sqrt{2}) \bigcup (1 + \sqrt{2})$. The following Figures 2 and 3 below show these cases.

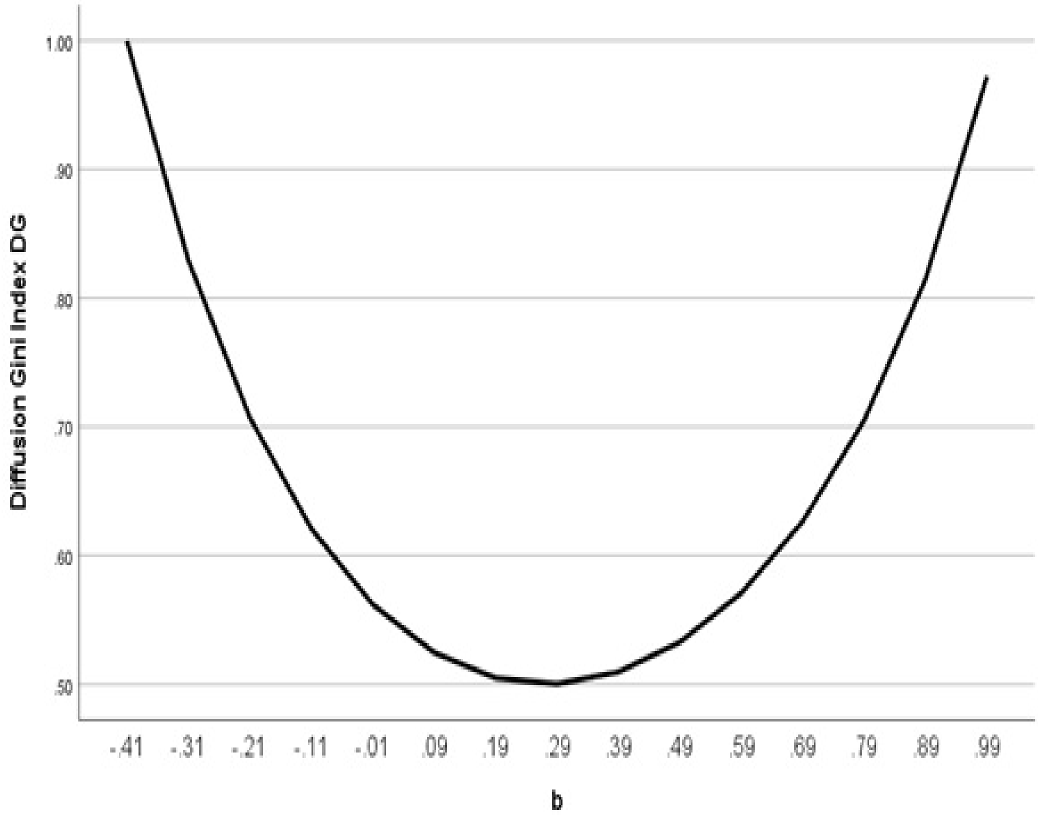


Figure 2: The Stochastic Diffusion Gini Index when $b \in (1-\sqrt{2}, 1)$ with General External Effect

# 5 Conclusion

This paper offers a new method to describe the behavior of the Lorenz curve model. This study departed from the classical techniques of the time series analysis and the before-and-after regression. A new stochastic Diffusion, the Lorenz Curve model, is developed with a birth-and-death stochastic diffusion and general downward external effect processes for the differences between successive observed wealth data points or income quantiles and the associated Stochastic Diffusion Gini-Index is obtained. The suggested functional form attains the properties of the Lorenz curve and works nicely, as shown in the above numerical example.

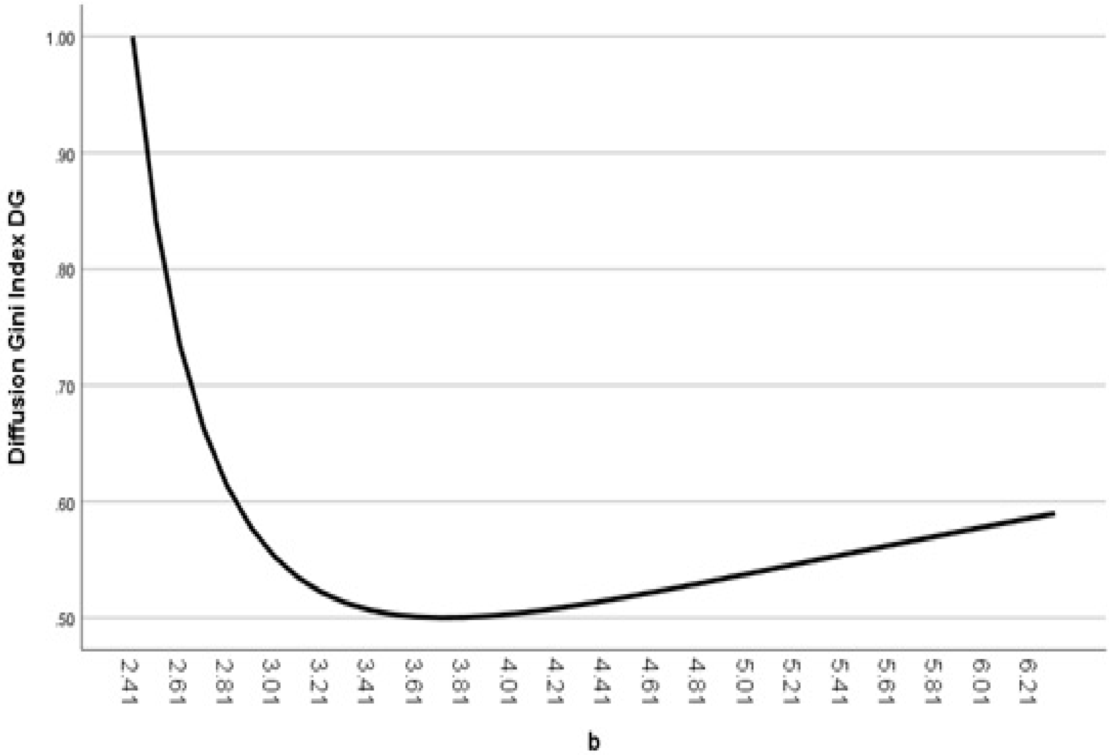


Figure 3: The Stochastic Diffusion Gini Index when $b \in (1 + \sqrt{2}, \infty)$ with General External Effect

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