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Inference on Constant Stress Accelerated Life Tests Under Exponentiated Exponential Distribution

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Accelerated life tests have become increasingly important because of higher customer expectations for better reliability, more complicated products with more components, rapidly changing technologies advances, and the clear need for rapid product development. Hence, accelerated life tests have been widely used in manufacturing industries, particularly to obtain timely information on the reliability. Maximum likelihood estimation is the starting point when it comes to estimating the parameters of the model. In this paper, besides the method of maximum likelihood, nine other frequentist estimation methods are proposed to obtain the estimates of the exponentiated exponential distribution parameters under constant stress accelerated life testing. We consider two parametric bootstrap confidence intervals based on different methods of estimation. Furthermore, we use the different estimates to predict the shape parameter and the reliability function of the distribution under the usual conditions. The performance of the ten proposed estimation methods is evaluated via an extensive simulation study. As an empirical illustration, the proposed estimation methods are applied to an accelerated life test data set.

keywords: Accelerated life testing; Anderson-Darling estimation; Cramér-von-Mises estimation; exponentiated exponential distribution; maximum likelihood method; least squares method.

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1 Introduction

In the manufacturing sector, accelerated life tests (ALTs) are frequently employed, especially to gather failure time information on the reliability of materials and components in a short time span than testing under normal use conditions. See, for example, Meeker et al. (2022), Nelson (2004) and Bagdonavicius and Nikulin (2001). Products are subjected to higher stress conditions, such as temperature, pressure, humidity, voltage, etc., in order to experience fast breakdowns. The collected data is then investigated to determine the life characteristics under normal circumstances. The failure data gathered by ALT are used to calculate reliability metrics like mean time to failure and reliability function under normal stress settings. There are different varieties of ALTs depending on the stress loading, including constant stress, step stress, ramp stress, sinusoidal cycle stress, and their combinations. In reality, the constant stress test requires more time at low-stress levels to collect enough failure information. This has prompted the industry to think about additional stress loadings. Each stress loading, however, has advantages and disadvantages of its own. Even though complex stress profiles could result in failures far faster than a constant stress test, they could nevertheless have an impact on how accurately reliability is predicted. Many authors studied the estimation problems of some lifetime distributions under constant stress ALT (CSALT). Haghighi (2014) obtained the maximum likelihood and the asymptotic confidence intervals of an extension of the exponential distribution. Han (2015) studied optimal designs of CSALT and step-stress ALT with the exponential failure data under time constraint. Wu and Huang (2017) investigated the optimization problem when the competing risks data come from a progressive type II censoring, see also for more detail Nassar et al. (2021) and Kumar et al. (2022). Meeker et al. (2022) provided a comprehensive review on methodologies and computational aspects of CSALT models.

Parameter estimation is always of essential importance for every probability distribution. In this study, we examine ten techniques for estimating the parameters of the exponentiated exponential (EEx) distribution under CSALT. These methods are the maximum likelihood estimation (MLE), percentile estimation (PCE), least square estimation (LSE), weighted least square estimation (WLSE), maximum product of spacing estimation (MPSE), minimum spacing absolute distance estimation (MSADE), minimum spacing absolute-log distance estimation (MSALDE), Cramér-von-Mises estimation (CME), Anderson-Darling estimation (ADE) and Right-tail Anderson-Darling estimation (RTADE) methods. Numerous writers have investigated the use of MLE, LSE, WLSE, and probability weighted moments (PWM) approaches to analyze entire data under CSALT. See, e.g., Singpurwalla (1974), Bhattacharyya and Soejoeti (1981), Bhattacharyya and Fries (1982), Kim and Bai (2002), Zhang and Wang (2009), Park and Bae (2010), Zhu and Elsayed (2011) and Fan and Yu (2013). Further, Abdel Ghaly et al. (2016) considered four estimation methods namely, MLE, LSE, WLSE and PWM under CSALT for the exponentiated inverse Weibull distribution. Wang and Shi (2017) obtained the estimates of generalized half-normal parameters under CSALT and obtained the confidence intervals of the model parameters by using bootstrap technique. More recently, Nassar and Dey (2018) considered six estimation methods to estimate the exponentiated Rayleigh distribution parameters and the reliability function under CSALT. Dey and Nassar (2020) considered nine estimation methods to estimate the parameters of the exponentiated Lindley distribution under CSALT. However, Bartolucci et al. (1999) stated that PWM estimators are not as accurate as the MLE method. Hence, we did not consider the PWM method of estimation in our study. Additionally, Rodrigues et al. (2018) illustrated that the ADE is the most efficient for estimating the Poisson–exponential parameters. Ramos et al. (2019) stated that the penalized MLE is the most efficient method for estimating the generalized gamma parameters in case of small samples. Al-Mofleh et al. (2020) showed that the MPSE outperforms all other estimation methods for estimating the generalized Ramos–Louzada parameters.

The primary goal of the ALT is to quickly gather failure information about a product's lifetime using stress levels that are higher than those found in normal operating settings in order to estimate how well the product will operate. Any ALT process is based on three factors: stress, model, and estimation approach. In order to estimate the EEx distribution's unknown parameters under CSALT, this research focuses on ten different estimation techniques. The various estimations methods are utilized to estimate the reliability function and shape parameter of the EEx distribution under normal circumstances employing the assumption that the invariance property is exist for the various estimation approaches. We also considered two bootstrap confidence intervals for the unknown parameters. Via an extensive simulation study, we show how the estimators of the unknown parameters and the reliability function behave for different sample sizes and for different parameter values. In addition, using CSALT, guidelines are produced for picking the best estimation technique that can be utilized to estimate the unknown parameters and the reliability function of the EEx distribution. Since the introduction of the EEx distribution by Gupta and Kundu (1999) and till now no work was carried out to compare ten estimation methods using the EEx distribution in the field of CSALT. Hence, it is the first time to consider the aforementioned ten estimation methods to estimate the EEx parameters under CSALT. We think that the new results investigated in this paper may be of a great interest to reliability engineers and applied statisticians.

The reminder of the paper is organized as follows: In Section 2, EEx distribution along with the basic assumptions of CSALT are provided. In Section 3, we describe ten estimation methods, namely, MLE, PCE, LSE, WLSE, MPSE, MSADE, MSALDE, CME, ADE and RTADE for the EEx distribution. The two bootstrap confidence intervals are considered in Section 4. The performance of these methods for the proposed model is assessed using a simulation study in Section 5. In Section 6, the usefulness of the EEx distribution is illustrated via two real data sets. Finally, Section 7 provides some concluding remarks.

2 Model and basic assumptions

Gupta and Kundu (1999) introduced the two-parameter EEx distribution as an alternative to the Weibull and gamma distributions. They observed that the EEx distribution has a good connections with the gamma and Weibull distributions and the three models have many quite similar properties. Suppose X is a random variable follows EEx distribution with shape and scale parameters γ and λ, respectively, then the probability density function (PDF) of the EEx model is given by

$$g(x;\gamma,\lambda) = \gamma\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\gamma-1};\ x > 0,\ \gamma,\lambda > 0 \tag{1}$$

and the corresponding cumulative distribution function (CDF) is

$$G(x;\gamma,\lambda) = \left(1 - e^{-\lambda x}\right)^{\gamma}. \tag{2}$$

The hazard rate function (HRF) of X is given by

$$h(x;\gamma,\lambda) = \frac{\gamma\lambda e^{-\lambda x}\left(1 - e^{-\lambda x}\right)^{\gamma-1}}{1 - \left(1 - e^{-\lambda x}\right)^{\gamma}}.$$

The following basic assumptions are considered for estimation under CSALT based on EEx distribution.

1- There are k stress levels in the experiment denoted by $S_j, j = 1, ..., k$ and the normal conditions is denoted by S_u.
2- The stress levels are gradually increased in k i.e., $S_u < S_1 < ... < S_k$.
3- There is N identical test units and divided into $n_1, n_2, ..., n_k$, where $N = \sum_{j=1}^{k} n_j$.
4- Each n_j units are put under constant stress S_j , for $j = 1, 2, ..., k$.
5- The failure times denoted by X_{ij}, $i = 1, 2, ..., n_j$ and $j = 1, 2, ..., k$, are independently and identically distributed random variables from the PDF in (1).
6- We assume log-linear function between the stress level S_j and the shape parameter γ_j of the EEx distribution, i.e., $\ln\gamma_j = \alpha + \beta S_j$, for $j = 1, 2, ..., k$, where α and β are two unknown parameters which depend on the nature of the product.

3 Estimation methods

In this section, we estimate the unknown parameters of the EEx distribution under CSALT using ten estimation methods, namely: MLE, PCE, LSE, WLSE, MPSE, MSADE, MSALDE, CME, ADE and RTADE methods.

3.1 Maximum likelihood estimation

In this subsection, the MLEs of the EEx parameters under CSALT are obtained. Based on Equation (1), the likelihood function is given by

$$L(\alpha,\beta,\lambda,x) = \prod_{j=1}^{k}\prod_{i=1}^{n_j} \lambda\, e^{\alpha+\beta S_j}\, e^{-\lambda x_{ij}}\left(1 - e^{-\lambda x_{ij}}\right)^{e^{\alpha+\beta S_j}-1}.$$

The log-likelihood function, $\ell(\alpha,\beta,\lambda)$, is given by

$$\begin{aligned}\ell(\alpha,\beta,\lambda) &= N\ln(\lambda)+\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\alpha+\beta S_j)-\sum_{j=1}^{k}\sum_{i=1}^{n_j}(\lambda x_{ij})\\ &+ \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left\{(e^{\alpha+\beta S_j}-1)\ln\left(1-e^{-\lambda x_{ij}}\right)\right\}.\end{aligned}$$

The MLEs of the unknown parameters α, β and λ are obtained by solving the following three equations simultaneously

$$\frac{\partial\,\ell(\alpha,\beta,\lambda)}{\partial\alpha}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[1+e^{\alpha+\beta S_j}\,\ln\left(1-e^{-\lambda x_{ij}}\right)\right]=0,$$

$$\frac{\partial\,\ell(\alpha,\beta,\lambda)}{\partial\beta}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[S_j+S_j\,e^{\alpha+\beta S_j}\,\ln\left(1-e^{-\lambda x_{ij}}\right)\right]=0$$

and

$$\frac{\partial\,\ell(\alpha,\beta,\lambda)}{\partial\lambda}=\frac{N}{\lambda}-\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[x_{ij}-\left(e^{\alpha+\beta S_j}-1\right)\frac{x_{ij}\,e^{-\lambda x_{ij}}}{1-e^{-\lambda x_{ij}}}\right]=0.$$

The above three equations have no closed form solution, hence a numerical technique like Quasi Newton method can be used to obtain the MLEs of α, β and λ denoted by $\hat{\alpha}_{MLE}$, $\hat{\beta}_{MLE}$ and $\hat{\lambda}_{MLE}$. After obtaining the MLEs of α and β, the MLE of the shape parameter γ_j denoted by $\hat{\gamma}_{MLEj}$ can be obtained as follows

$$\hat{\gamma}_{MLEj}=e^{\hat{\alpha}_{MLE}+\hat{\beta}_{MLE}S_j}.$$

3.2 Least squares and weighted least squares methods

In this subsection, the LSEs and WLSEs of the EEx parameters are obtained. Let $x_{11:N},...,x_{n_k k:N}$ be the order statistics of a random sample of size N from the EEx distribution under CSALT. The LSEs denoted by $\hat{\alpha}_{LSE},\hat{\beta}_{LSE}$ and $\hat{\lambda}_{LSE}$ can be obtained by minimizing

$$S(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)^2, \tag{3}$$

where, for simplicity in subsequent sections, $x_{ij}=x_{ij:N}$, for $i=1,2,...,n_k$ and $j=1,2,..,k$ and $\psi_{ij}=\left(1-e^{-\lambda x_{ij}}\right)$.

Further, the LSEs can be obtained by solving the following three non-linear equations numerically

$$\frac{\partial\, S(\alpha,\beta,\lambda)}{\partial\alpha}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)=0,$$

$$\frac{\partial\, S(\alpha,\beta,\lambda)}{\partial\beta}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)S_j\, e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)=0$$

and

$$\frac{\partial\, S(\alpha,\beta,\lambda)}{\partial\lambda}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}=0.$$

Hence, the LSE of γ_j follows as

$$\hat{\gamma}_{LSEj}=e^{\hat{\alpha}_{LSE}+\hat{\beta}_{LSE}S_j}.$$

The WLSEs of the unknown parameters α,β and λ denoted by, $\hat{\alpha}_{WLSE},\hat{\beta}_{WLSE}$ and $\hat{\lambda}_{WLSE}$, are obtained by minimizing

$$W(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\frac{(n_j+1)^2(n_j+2)}{i(n_j-i+1)}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)^2.$$

Further, these estimators follows by solving the following equations numerically

$$\frac{\partial\, W(\alpha,\beta,\lambda)}{\partial\alpha}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\frac{(n_j+1)^2(n_j+2)}{i(n_j-i+1)}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)=0,$$

$$\frac{\partial\, W(\alpha,\beta,\lambda)}{\partial\beta}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\frac{(n_j+1)^2(n_j+2)}{i(n_j-i+1)}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)S_j\, e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)=0$$

and

$$\frac{\partial\, W(\alpha,\beta,\lambda)}{\partial\lambda}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\frac{(n_j+1)^2(n_j+2)}{i(n_j-i+1)}\left(\psi_{ij}^{e^{\alpha+\beta S_j}}-\frac{i}{n_j+1}\right)x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}=0.$$

Then, the WLSE of γ_j can be obtained as follows

$$\hat{\gamma}_{WLSEj}=e^{\hat{\alpha}_{WLSE}+\hat{\beta}_{WLSE}S_j}.$$

3.3 Percentile estimation estimation

In this subsection, we estimate the unknown parameters of the EEx distribution under CSALT using PCE method. The unknown parameters can be estimated by equating the sample percentile points with the population percentile points. We can write the CDF of the EEx distribution under CSALT as follows

$$G(x_{ij};\alpha,\beta,\lambda)=\left(1-e^{-\lambda x_{ij}}\right)^{e^{\alpha+\beta S_j}}.$$

Then, we have

$$x_{ij}=\frac{-1}{\lambda}\ln\left(\xi_{ij}\right), \tag{4}$$

where $\xi_{ij}=1-p_{ij}^{e^{-\alpha-\beta S_j}}$ and $p_{ij}=[i/(n_j+1)]$. The PCEs of the parameters α, β and λ, say $\hat{\alpha}_{PCE},\hat{\beta}_{PCE}$ and $\hat{\lambda}_{PCE}$, can be obtained by minimizing

$$P(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[x_{ij}+\frac{\ln\left(\xi_{ij}\right)}{\lambda}\right]^2,$$

The PCEs can also be obtained by solving the following nonlinear equations numerically

$$\frac{\partial\, P(\alpha,\beta,\lambda)}{\partial\alpha}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[x_{ij}+\frac{\ln\left(\xi_{ij}\right)}{\lambda}\right]\frac{e^{-\alpha-\beta S_j}\,p_{ij}^{e^{-\alpha-\beta S_j}}\ln(p_{ij})}{\lambda\left(\xi_{ij}\right)}=0,$$

$$\frac{\partial\, P(\alpha,\beta,\lambda)}{\partial\beta}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[x_{ij}+\frac{\ln\left(\xi_{ij}\right)}{\lambda}\right]\frac{S_j\,e^{-\alpha-\beta S_j}\,p_{ij}^{e^{-\alpha-\beta S_j}}\ln(p_{ij})}{\lambda\left(\xi_{ij}\right)}=0$$

and

$$\frac{\partial\, P(\alpha,\beta,\lambda)}{\partial\lambda}=\sum_{j=1}^{k}\sum_{i=1}^{n_j}\left[x_{ij}+\frac{\ln\left(\xi_{ij}\right)}{\lambda}\right]\left[-\frac{\ln\left(\xi_{ij}\right)}{\lambda^2}\right]=0.$$

3.4 Maximum product of spacings estimation

In this subsection, we estimate the unknown parameters of the EEx distribution under CSALT using the MPSE method introduced by Cheng and Amin (1983). Consider the uniform spacings of a random sample from the EEx distribution under CSALT, say $D_{ij}(\alpha,\beta,\lambda)$, which are defined by

$$D_{ij}(\alpha,\beta,\lambda)=G\left(x_{ij}\mid\alpha,\beta,\lambda\right)-G\left(x_{i-1j}\mid\alpha,\beta,\lambda\right). \tag{5}$$

The MPSEs of α, β and λ are obtained by maximizing

$$M(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{\ln D_{ij}(\alpha,\beta,\lambda)}{n_j+1}.$$

The MPSEs, say $\hat{\alpha}_{MPSE}, \hat{\beta}_{MPSE}$ and $\hat{\lambda}_{MPSE}$, can be obtained by solving the following three non-linear equations numerically

$$\frac{\partial M(\alpha,\beta,\lambda)}{\partial\alpha}=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})}{(n_j+1)D_{ij}(\alpha,\beta,\lambda)}=0,$$

$$\frac{\partial M(\alpha,\beta,\lambda)}{\partial\beta}=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{S_j\,e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-S_j\,e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})}{(n_j+1)D_{ij}(\alpha,\beta,\lambda)}=0$$

and

$$\frac{\partial M(\alpha,\beta,\lambda)}{\partial\lambda}=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}-x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{i-1j}^{e^{\alpha+\beta S_j}-1}}{(n_j+1)D_{ij}(\alpha,\beta,\lambda)}=0,$$

where $D_{ij}(\alpha,\beta,\lambda)$ is given by Equation (5). Then, the MPSE of γ_j follows as

$$\hat{\gamma}_{MPSEj}=e^{\hat{\alpha}_{MPSE}+\hat{\beta}_{MPSE}S_j}.$$

3.5 Minimum spacing distance estimation

The minimum spacing distance estimators (MSDEs) of α, β and λ, denoted by $\widehat{\alpha}_{MSDE}$, $\widehat{\beta}_{MSDE}$ and $\widehat{\lambda}_{MSDE}$, are obtained by minimizing

$$T(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}h\left(D_{ij}(\alpha,\beta,\lambda),\frac{1}{n_j+1}\right), \tag{6}$$

where $h(x, y)$ is an appropriate distance. Some choices of $h(x, y)$ in Equation (6) are the absolute distance, $|x - y|$, and the absolute-log distance, $|\ln x - \ln y|$. These estimators are called MSADE and MSALDE, respectively. This method was originally proposed by Torabi (2008). The MSADE and MSALDE of the parameters α, β and λ can be obtained by minimizing the following two equations

$$T(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\left|D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}\right|$$

and

$$T(\alpha,\beta,\lambda)=\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\left|\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}\right|,$$

with respect to α, β and λ, respectively. The estimators $\hat{\alpha}_{MSADE}$, $\hat{\beta}_{MSADE}$ and $\hat{\lambda}_{MSADE}$ of α, β and λ can be obtained by solving the following nonlinear equations numerically

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\alpha}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}}{\left|D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}\right|}\\&\times\left[e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})\right]=0\end{aligned}$$

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\beta}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}}{\left|D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}\right|}\\&\times\left[S_j\,e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-S_j\,e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})\right]=0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\lambda}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}}{\left|D_{ij}(\alpha,\beta,\lambda)-\frac{1}{n_j+1}\right|}\\&\times\left[x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}-x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{i-1j}^{e^{\alpha+\beta S_j}-1}\right]=0,\end{aligned}$$

where $D_{ij}(\alpha,\beta,\lambda)\neq\frac{1}{n_j+1}$. The estimators $\hat{\alpha}_{MSALDE}$, $\hat{\beta}_{MSALDE}$ and $\hat{\lambda}_{MSALDE}$ of α, β and λ can be obtained by solving the following nonlinear equations numerically

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\alpha}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}}{\left|\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}\right|}\frac{1}{D_{ij}(\alpha,\beta,\lambda)}\\&\times\left[e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})\right]=0,\end{aligned}$$

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\beta}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}}{\left|\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}\right|}\frac{1}{D_{ij}(\alpha,\beta,\lambda)}\\&\times\left[S_j\,e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij})-S_j\,e^{\alpha+\beta S_j}\psi_{i-1j}^{e^{\alpha+\beta S_j}}\ln(\psi_{i-1j})\right]=0\end{aligned}$$

and

$$\begin{aligned}\frac{\partial T(\alpha,\beta,\lambda)}{\partial\lambda}=&\sum_{j=1}^{k}\sum_{i=1}^{n_j+1}\frac{\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}}{\left|\ln D_{ij}(\alpha,\beta,\lambda)-\ln\frac{1}{n_j+1}\right|}\frac{1}{D_{ij}(\alpha,\beta,\lambda)}\\&\times\left[x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}-x_{ij}\,e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{i-1j}^{e^{\alpha+\beta S_j}-1}\right]=0,\end{aligned}$$

where $\ln D_{ij}(\alpha,\beta,\lambda) \neq \ln \frac{1}{n_j+1}$. Now the MSADE and MSALDE of γ_j follow, respectively, as

$$\hat{\gamma}_{MSADEj} = e^{\hat{\alpha}_{MSADE}+\hat{\beta}_{MSADE}S_j}$$

and

$$\hat{\gamma}_{MSALDEj} = e^{\hat{\alpha}_{MSALDE}+\hat{\beta}_{MSALDE}S_j}.$$

3.6 Cramér-Von Mises estimation

The Cramér-Von Mises (Cramér (1928); Mises (2013)) estimators of the parameters α,β and λ denoted by $\hat{\alpha}_{CME},\hat{\beta}_{CME}$ and $\hat{\lambda}_{CME}$, of the EEx parameters under CSALT can be obtained by minimizing

$$C(\alpha,\beta,\lambda) = \frac{1}{12N} + \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}} - \frac{2i-1}{2n_j}\right)^2,$$

with respect to α,β and λ. The CMEs can also be obtained by solving the following non-linear equations numerically

$$\frac{\partial\, C(\alpha,\beta,\lambda)}{\partial\alpha} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}} - \frac{2i-1}{2n_j}\right) e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij}) = 0,$$

$$\frac{\partial\, C(\alpha,\beta,\lambda)}{\partial\beta} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}} - \frac{2i-1}{2n_j}\right) S_j\, e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln(\psi_{ij}) = 0$$

and

$$\frac{\partial\, C(\alpha,\beta,\lambda)}{\partial\lambda} = \sum_{j=1}^{k}\sum_{i=1}^{n_j}\left(\psi_{ij}^{e^{\alpha+\beta S_j}} - \frac{2i-1}{2n_j}\right) x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1} = 0.$$

Hence, the CME of γ_j follows simply as

$$\hat{\gamma}_{CMEj} = e^{\hat{\alpha}_{CME}+\hat{\beta}_{CME}S_j}.$$

3.7 Anderson-Darling and Right-tail Anderson-Darling estimation

Anderson and Darling (1952) proposed the Anderson-Darling test as an alternative to other statistical tests for detecting sample distributions departure from normality. The Anderson-Darling estimators $\widehat{\alpha}_{ADE}$, $\widehat{\beta}_{ADE}$ and $\widehat{\lambda}_{ADE}$ of α, β and $\lambda$of the EEx parameters under CSALT are obtained by minimizing the following function with respect to α, β and λ

$$A(\alpha,\beta,\lambda) = -N - \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{2i-1}{n_j}\left\{\ln G\left(x_{ij} \mid \alpha,\beta,\lambda\right) + \ln \overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)\right\},$$

where $\overline{G}(.) = 1 - G(.)$. These estimators can also be obtained by solving the following non-linear equations numerically

$$\sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{2i-1}{n_j}\left\{\frac{e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)}{G\left(x_{ij} \mid \alpha,\beta,\lambda\right)} - \frac{e^{\alpha+\beta S_j}\psi_{n+1-ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{n+1-ij}\right)}{\overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)}\right\} = 0,$$

$$\sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{2i-1}{n_j}\left\{\frac{S_j\, e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right)}{G\left(x_{ij} \mid \alpha,\beta,\lambda\right)} - \frac{S_j\, e^{\alpha+\beta S_j}\psi_{n+1-ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{n+1-ij}\right)}{\overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)}\right\} = 0$$

and

$$\sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{2i-1}{n_j}\left\{\frac{x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{ij}^{e^{\alpha+\beta S_j}-1}}{G\left(x_{ij} \mid \alpha,\beta,\lambda\right)} - \frac{x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\psi_{n+1-ij}^{e^{\alpha+\beta S_j}-1}}{\overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)}\right\} = 0.$$

The ADE of γ_j can be calculated as follows

$$\hat{\gamma}_{ADEj} = e^{\hat{\alpha}_{ADE}+\hat{\beta}_{ADE}S_j}.$$

The Right-tail Anderson-Darling estimators of the parameters α, β and λ denoted by $\widehat{\alpha}_{RTADE}$, $\widehat{\beta}_{RTADE}$ and $\widehat{\lambda}_{RTADE}$ of the EEx parameters under CSALT can be obtained by minimizing the following function with respect to α, β and λ

$$R(\alpha,\beta,\lambda) = \frac{N}{2} - 2\sum_{j=1}^{k}\sum_{i=1}^{n_j} G\left(x_{ij} \mid \alpha,\beta,\lambda\right) - \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{2i-1}{n_j}\ln \overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right).$$

Further, the RTADEs can also be obtained by solving the following non-linear equations numerically

$$-2\sum_{j=1}^{k}\sum_{i=1}^{n_j} e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right) + \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{(2i-1)}{n_j}\frac{e^{\alpha+\beta S_j}\psi_{n+1-ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{n+1-ij}\right)}{\overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)} = 0,$$

$$-2\sum_{j=1}^{k}\sum_{i=1}^{n_j} S_j\, e^{\alpha+\beta S_j}\psi_{ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{ij}\right) + \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{(2i-1)}{n_j}\frac{S_j\, e^{\alpha+\beta S_j}\psi_{n+1-ij}^{e^{\alpha+\beta S_j}}\ln\left(\psi_{n+1-ij}\right)}{\overline{G}\left(x_{n+1-ij} \mid \alpha,\beta,\lambda\right)} = 0$$

and

$$-2\sum_{j=1}^{k}\sum_{i=1}^{n_j} x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\,\psi_{ij}^{e^{\alpha+\beta S_j}-1} + \sum_{j=1}^{k}\sum_{i=1}^{n_j} \frac{(2i-1)}{n_j}\,\frac{x_{ij}\, e^{\alpha+\beta S_j-\lambda x_{ij}}\,\psi_{n+1-ij}^{e^{\alpha+\beta S_j}-1}}{\overline{G}\left(x_{n+1-ij}\mid \alpha,\beta,\lambda\right)} = 0.$$

Then, the RTADE of γ_j follows as

$$\hat{\gamma}_{RTADEj} = e^{\hat{\alpha}_{RTADE}+\hat{\beta}_{RTADE}S_j}.$$

4 Bootstrap Confidence Intervals

In this section, we propose the use of two parametric bootstrap confidence intervals for the parameters (α, β, λ). The two different parametric confidence intervals are the percentile bootstrap (Boot-P)and the bias corrected percentile bootstrap (Boot-BCP) confidence intervals, which can be applied according to the following:

A) Boot-P confidence interval

1. Calculate the estimates $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ from the original data, where $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ denoted the different estimates obtained from the different estimation methods.
2. Use the estimates $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ to generate a bootstrap sample.
3. Obtain the bootstrap estimates $\hat{\alpha}^{*b}$, $\hat{\beta}^{*b}$ and $\hat{\lambda}^{*b}$.
4. Repeat step 2-3 B times to have $\hat{\alpha}^{*b(1)}, ..., \hat{\alpha}^{*b(B)}$, $\hat{\beta}^{*b(1)}, ..., \hat{\beta}^{*b(B)}$ and $\hat{\lambda}^{*b(1)}, ..., \hat{\lambda}^{*b(B)}$.
5. Arrange the bootstrap estimates in step 4 in ascending order as $\hat{\alpha}^{*b[1]}, ..., \hat{\alpha}^{*b[B]}$, $\hat{\beta}^{*b[1]}, ..., \hat{\beta}^{*b[B]}$ and $\hat{\lambda}^{*b[1]}, ..., \hat{\lambda}^{*b[B]}$.
6. The two-sided $100(1-\tau)$ percentile bootstrap confidence intervals for the unknown parameters are given by

$$\hat{\alpha}^{*b[B\tau/2]}, \hat{\alpha}^{*b[B(1-\tau/2)]},\ \hat{\beta}^{*b[B\tau/2]}, \hat{\beta}^{*b[B(1-\tau/2)]} \text{ and } \hat{\lambda}^{*b[B\tau/2]}, \hat{\lambda}^{*b[B(1-\tau/2)]}.$$

B) Boot-BCP confidence interval

1. Calculate the estimates $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ from the original data, where $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ denoted to the different estimates obtained from the different estimation methods.
2. Use the estimates $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ to generate a bootstrap sample.
3. Obtain the bootstrap estimates $\hat{\alpha}^{*b}$, $\hat{\beta}^{*b}$ and $\hat{\lambda}^{*b}$.
4. Repeat step 2-3 B times to have $\hat{\alpha}^{*b(1)}, ..., \hat{\alpha}^{*b(B)}$, $\hat{\beta}^{*b(1)}, ..., \hat{\beta}^{*b(B)}$ and $\hat{\lambda}^{*b(1)}, ..., \hat{\lambda}^{*b(B)}$.

5. Arrange the bootstrap estimates in step 4 in ascending order as $\hat{\alpha}^{*b[1]}, ..., \hat{\alpha}^{*b[B]}$, $\hat{\beta}^{*b[1]}, ..., \hat{\beta}^{*b[B]}$ and $\hat{\lambda}^{*b[1]}, ..., \hat{\lambda}^{*b[B]}$.

6. The two-sided $100(1-\tau)$ bias corrected percentile bootstrap confidence intervals for the unknown parameters are given by

$$\hat{\alpha}^{*b[B\delta_1]}, \hat{\alpha}^{*b[B\delta_2]},\ \hat{\beta}^{*b[B\delta_1]}, \hat{\beta}^{*b[B\delta_2]} \text{ and } \hat{\lambda}^{*b[B\delta_1]}, \hat{\lambda}^{*b[B\delta_2]}.$$

   where

$$\delta_1 = \Phi(2z_0 + z_{\tau/2}) \text{ and } \delta_2 = \Phi(2z_0 + z_{1-\tau/2}),$$

   where $\Phi(.)$ is the the CDF of the standard normal distribution and $z_\tau = \Phi^{-1}(\tau)$ and z_0 can be obtained as follows

$$z_0 = \Phi^{-1}\left(\frac{\#\{\hat{\kappa_l}^{*b(i)} < \hat{\kappa_l}^*\}}{B}\right), i = 1, ..., B,\ l = 1, 2, 3,\ \kappa_l = \alpha, \kappa_2 = \beta \text{ and } \kappa_3 = \lambda.$$

5 Simulation Study

This section is devoted to perform a simulation study to compare the performance of the different estimators derived in section 3 and the two bootstrap confidence intervals discussed in section 4 in terms of mean relative estimates (MRE), mean square errors (MSE) and confidence interval lengths. The simulation study is conducted by choosing different sample sizes, stress levels and parameters values. The sample sizes are chosen as $N = (40, 100)$ and allocated equally to the two stresses. We select two stress levels, i.e. $S_1 = 0.2$ and $S_2 = 0.4$. In all the setting, the scale parameter λ is considered to be 1.5 and by choosing different values of the parameters α and β. The simulation study is conducted according to the following steps

1. Choose the parameters values of α, β, λ and determine the values of n_j and S_j, where $j = 1, 2$.

2. Generate a random sample from the EEx distribution based on CSALT and determine x_{ij}, $i = 1, ..., n_j$ and $j = 1, 2$.

3. Use the generated sample in step 2 to obtain the different estimates of $\alpha, \beta, \lambda, \gamma_1$ and γ_2, namely, MLEs, LSEs, WLSEs, PCEs, MPSEs, MSADEs, MSALDEs, CMEs, ADEs and RTADEs.

4. Obtain the two bootstrap confidence intervals for the different parameters based on the different estimation methods.

5. Obtain the predicted shape parameter and the predicted reliability function based on some selected lifetime values and by choosing the usual use condition to be 0.1.

6. Redo step 2 to 5 1000 times.

7. Compute the MRE, MSE, where $MRE = \sum_{t=1}^{1000} \frac{\hat{\kappa}_{m,t}/\kappa_m}{1000}$ and $MSE = \sum_{t=1}^{1000} \frac{(\hat{\kappa}_{m,t}-\kappa_m)^2}{1000}$, $m = 1, ..., 7$ and $\boldsymbol{\kappa} = (\alpha, \beta, \lambda, \gamma_1, \gamma_2, \gamma_u, R_u(.))$.

Using this approach, the most efficient method of estimation will have MREs close to one and MSEs close to zero. We also obtain the confidence interval length to show the performance of the two bootstrap confidence intervals. All the computation results were obtained using MATHCAD program version 15. The average values of MRE and MSE are displayed in Tables 1 and 2 while the average confidence intervals lengths are displayed in Tables 3 and 4. The MRE and MSE of the predicted shape parameter are presented in Table 5 and in Tables 6 and 7 for the predicted reliability function. Comparing the results in these Tables, we can conclude the following observations:

1. All the estimates are consistent, where the MSEs decrease as the sample size increases in all the cases for the ten estimation methods.

2. For the ten methods of estimation, the MREs tend to one in all the cases as the sample size increases, which indicate that the different estimates are asymptotically unbiased.

3. The confidence intervals lengths decrease as the sample size increases for all the methods of estimation.

4. The ADEs perform better than the other estimates followed by the RADEs in terms of MREs and MSEs.

5. The MSADEs are the most inefficient estimates in most of the cases especially when the sample size is small.

6. For the predicted shape parameter under the usual use condition, the MSEs of the ten estimation methods decrease as the sample size increases.

7. For the predicted reliability function under the usual use condition, the MSEs of the ten estimation methods decrease as the sample size increases and very close to zero, especially for the ADEs when $N = 100$.

8. The Boot-P confidence intervals perform better than the Boot-BCP confidence intervals in most of the cases in terms of minimum confidence interval lengths for the different estimation methods.

9. The confidence interval lengths for the ADEs perform better than the other estimates in terms of minimum confidence interval lengths in most of the cases.

Combining all the above results, we can recommend the use of the ADEs when estimating the unknown parameters of the EEx distribution under the CSALT. The results show that the ADEs perform better than the other estimation methods in terms of minimum MSEs and confidence interval lengths. The results show also that the ADEs provide an efficient estimates when estimating the shape parameter and the reliability function under the usual use condition.

Table 1: Average values of MREs and MSEs (in parentheses)for $n = 40$.

Initial values			Method									
α	β	Par	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	α	1.309(0.325)	0.821(0.380)	0.960(0.365)	0.529(0.130)	0.890(0.330)	0.658(0.370)	0.557(0.260)	1.374(0.451)	0.789(0.056)	0.928(0.059)
		β	1.051(2.873)	0.896(3.322)	0.963(3.277)	0.849(0.025)	1.093(2.970)	0.512(1.218)	1.121(2.082)	0.961(3.925)	0.909(0.008)	0.987(0.054)
		λ	1.078(0.110)	0.964(0.128)	0.999(0.118)	0.892(0.155)	1.114(0.135)	0.439(1.010)	0.962(0.145)	1.084(0.172)	0.943(0.098)	1.089(0.104)
		γ_1	1.134(0.297)	1.134(0.297)	1.047(0.291)	0.942(0.226)	1.026(0.235)	0.687(0.330)	0.944(0.186)	1.177(0.519)	0.974(0.114)	0.959(0.163)
		γ_2	1.134(0.290)	1.134(0.290)	1.048(0.391)	0.944(0.285)	1.027(0.217)	0.676(0.383)	0.951(0.199)	1.179(0.660)	0.973(0.141)	0.959(0.164)
0.25	0.5	α	1.217(0.289)	0.704(0.354)	0.869(0.339)	0.586(0.089)	0.801(0.300)	0.578(0.369)	0.499(0.310)	1.233(0.411)	0.851(0.039)	0.914(0.053)
		β	1.188(2.674)	1.060(3.050)	1.078(2.991)	0.816(0.215)	1.215(2.791)	0.606(1.611)	1.175(2.607)	1.146(3.603)	0.944(0.049)	0.956(0.078)
		λ	1.081(0.115)	0.966(0.128)	1.001(0.119)	0.886(0.154)	1.116(0.137)	0.464(0.989)	0.969(0.152)	1.084(0.173)	0.944(0.095)	0.954(0.109)
		γ_1	1.120(0.260)	1.120(0.260)	1.029(0.292)	0.942(0.230)	1.014(0.199)	0.698(0.500)	0.943(0.224)	1.151(0.530)	0.985(0.120)	1.014(0.188)
		γ_2	1.145(0.368)	1.145(0.368)	1.037(0.355)	0.954(0.451)	1.039(0.279)	0.682(0.569)	0.962(0.271)	1.169(0.664)	0.990(0.204)	1.026(0.329)
0.25	0.75	α	1.213(0.288)	0.708(0.353)	0.869(0.337)	0.705(0.061)	0.788(0.300)	0.552(0.359)	0.498(0.299)	1.227(0.409)	0.900(0.030)	0.958(0.040)
		β	1.141(2.689)	1.032(3.065)	1.052(2.999)	0.838(0.478)	1.164(2.802)	0.561(1.956)	1.117(2.646)	1.114(3.619)	0.984(0.140)	1.028(0.229)
		λ	1.079(0.111)	0.967(0.123)	1.001(0.114)	0.888(0.148)	1.112(0.130)	0.469(0.962)	0.971(0.145)	1.082(0.166)	0.938(0.094)	0.951(0.107)
		γ_1	1.122(0.292)	1.122(0.292)	1.030(0.327)	0.964(0.262)	1.014(0.223)	0.701(0.480)	0.942(0.239)	1.153(0.596)	1.002(0.144)	1.037(0.232)
		γ_2	1.151(0.476)	1.151(0.476)	1.040(0.460)	0.990(0.716)	1.044(0.361)	0.674(0.773)	0.967(0.369)	1.177(0.878)	1.020(0.349)	1.075(0.677)
0.5	0.25	α	1.153(0.305)	0.856(0.364)	0.951(0.347)	0.674(0.149)	0.950(0.307)	0.505(0.479)	0.808(0.271)	1.151(0.428)	0.862(0.069)	0.898(0.097)
		β	1.080(1.732)	0.958(1.092)	1.002(1.035)	0.907(0.003)	1.110(1.134)	0.887(1.219)	0.961(1.196)	1.025(1.262)	0.911(0.003)	0.905(0.003)
		λ	1.067(0.097)	0.957(0.120)	0.994(0.110)	0.875(0.143)	1.099(0.115)	0.458(0.958)	0.965(0.133)	1.069(0.155)	0.940(0.092)	0.950(0.108)
		γ_1	1.133(0.442)	1.133(0.442)	1.035(0.515)	0.902(0.380)	1.026(0.323)	0.657(0.637)	0.954(0.320)	1.161(0.861)	0.962(0.225)	0.998(0.407)
		γ_2	1.141(0.531)	1.141(0.531)	1.037(0.555)	0.900(0.434)	1.036(0.415)	0.661(0.760)	0.960(0.441)	1.165(0.979)	0.959(0.258)	0.995(0.468)
0.5	0.5	α	1.113(0.304)	0.841(0.378)	0.924(0.353)	0.778(0.130)	0.905(0.314)	0.581(1.450)	0.741(0.309)	1.129(0.441)	0.926(0.058)	0.979(0.084)
		β	1.276(2.819)	1.128(1.143)	1.168(1.096)	0.919(0.025)	1.294(1.921)	0.711(1.463)	1.244(1.588)	1.211(1.711)	0.962(0.008)	0.968(0.015)
		λ	1.069(0.097)	0.960(0.118)	0.994(0.104)	0.887(0.136)	1.098(0.113)	0.473(0.929)	0.963(0.129)	1.070(0.152)	0.940(0.091)	0.956(0.108)
		γ_1	1.139(0.572)	1.139(0.572)	1.040(0.601)	0.949(0.432)	1.030(0.426)	0.669(0.737)	0.949(0.376)	1.179(1.166)	0.992(0.241)	1.039(0.465)
		γ_2	1.175(0.769)	1.175(0.769)	1.059(0.722)	0.949(0.591)	1.064(0.566)	0.659(0.951)	0.980(0.567)	1.201(1.348)	0.993(0.330)	1.043(0.684)
0.5	0.75	α	1.111(0.304)	0.843(0.376)	0.924(0.352)	0.822(0.111)	0.899(0.313)	0.518(0.448)	0.745(0.325)	1.126(0.439)	0.951(0.055)	1.009(0.081)
		β	1.198(2.829)	1.076(3.142)	1.110(3.101)	0.957(0.077)	1.215(2.929)	0.671(1.683)	1.156(1.156)	1.156(3.715)	0.987(0.024)	1.008(0.041)
		λ	1.068(0.093)	0.961(0.114)	0.994(0.101)	0.889(0.131)	1.095(0.109)	0.481(0.907)	0.963(0.123)	1.069(0.147)	0.936(0.091)	0.954(0.107)
		γ_1	1.141(0.644)	1.141(0.644)	1.041(0.681)	0.971(0.498)	1.030(0.478)	0.670(0.842)	0.952(0.476)	1.182(1.324)	1.009(0.292)	1.062(0.588)
		γ_2	1.181(0.993)	1.181(0.993)	1.062(0.938)	0.982(0.884)	1.069(0.731)	0.650(1.231)	0.476(0.728)	1.209(1.776)	1.014(0.491)	1.078(1.126)
0.75	0.75	α	1.097(0.321)	0.908(0.385)	0.967(0.373)	0.855(0.131)	0.951(0.324)	0.536(0.537)	0.863(0.337)	1.114(0.453)	0.953(0.071)	0.996(0.107)
		β	1.129(2.777)	1.005(2.169)	1.033(2.080)	0.955(0.021)	1.147(2.847)	0.745(2.635)	1.005(2.765)	1.084(2.757)	0.980(0.009)	0.989(0.014)
		λ	1.063(0.085)	0.962(0.097)	0.993(0.091)	0.889(0.121)	1.084(0.094)	0.513(0.800)	0.962(0.106)	1.066(0.124)	0.937(0.081)	0.953(0.096)
		γ_1	1.154(1.114)	1.154(1.114)	1.049(1.170)	0.953(0.800)	1.038(0.810)	0.649(1.581)	0.958(0.815)	1.198(2.101)	1.002(0.550)	1.060(1.018)
		γ_2	1.179(1.677)	1.179(1.677)	1.054(1.580)	0.954(1.220)	1.064(1.236)	0.632(2.171)	0.962(1.115)	1.218(2.178)	1.004(0.832)	1.066(1.566)

Table 2: Average values of MREs and MSEs (in parentheses)for $n = 100$.

Initial values			Method									
α	β	Par	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	α	1.063(0.105)	0.888(0.137)	0.965(0.124)	0.760(0.060)	0.867(0.107)	0.605(0.199)	0.731(0.103)	1.100(0.146)	0.995(0.021)	1.043(0.028)
		β	1.267(0.960)	1.143(1.221)	1.202(1.133)	0.923(0.009)	1.286(0.978)	0.748(0.640)	1.155(0.873)	1.174(1.303)	0.977(0.003)	0.988(0.004)
		λ	1.031(0.039)	0.987(0.051)	1.008(0.044)	0.920(0.069)	1.048(0.043)	0.634(0.620)	0.972(0.058)	1.033(0.058)	0.980(0.040)	0.988(0.044)
		γ_1	1.045(0.065)	1.045(0.065)	1.020(0.074)	0.942(0.100)	0.996(0.055)	0.783(0.185)	0.958(0.062)	1.058(0.104)	1.009(0.046)	1.027(0.067)
		γ_2	1.059(0.081)	1.059(0.081)	1.032(0.093)	0.942(0.124)	1.011(0.068)	0.781(0.230)	0.968(0.080)	1.068(0.128)	1.010(0.056)	1.028(0.084)
0.25	0.5	α	1.061(0.105)	0.890(0.137)	0.965(0.123)	0.777(0.042)	0.861(0.107)	0.711(0.189)	0.697(0.120)	1.098(0.145)	1.046(0.018)	1.091(0.024)
		β	1.142(0.963)	1.067(1.225)	1.103(1.137)	0.926(0.076)	1.155(0.980)	0.724(0.848)	1.132(1.067)	1.097(1.306)	1.055(0.020)	1.095(0.032)
		λ	1.031(0.037)	0.987(0.049)	1.007(0.043)	0.922(0.066)	1.046(0.041)	0.645(0.597)	0.973(0.055)	1.032(0.056)	0.976(0.039)	0.986(0.043)
		γ_1	1.045(0.073)	1.045(0.073)	1.020(0.083)	0.966(0.110)	0.996(0.062)	0.790(0.200)	0.956(0.071)	1.058(0.116)	1.030(0.056)	1.050(0.084)
		γ_2	1.061(0.103)	1.061(0.103)	1.033(0.118)	0.971(0.199)	1.012(0.086)	0.781(0.302)	0.972(0.104)	1.071(0.164)	1.040(0.095)	1.066(0.146)
0.25	0.75	α	1.059(0.104)	0.892(0.136)	0.965(0.123)	0.885(0.030)	0.915(0.106)	0.615(0.184)	0.703(0.122)	1.096(0.145)	1.066(0.015)	1.118(0.020)
		β	1.100(0.966)	1.041(1.228)	1.070(1.142)	0.941(0.144)	1.111(0.982)	0.767(0.944)	1.081(1.110)	1.071(1.312)	1.111(0.069)	1.153(0.092)
		λ	1.030(0.036)	0.987(0.048)	1.007(0.041)	0.924(0.063)	1.044(0.039)	0.655(0.569)	0.973(0.053)	1.032(0.054)	0.970(0.039)	0.982(0.042)
		γ_1	1.046(0.081)	1.046(0.081)	1.021(0.093)	0.991(0.125)	0.995(0.069)	0.789(0.221)	0.956(0.078)	1.059(0.130)	1.047(0.070)	1.072(0.106)
		γ_2	1.063(0.131)	1.063(0.131)	1.034(0.151)	1.002(0.291)	1.014(0.109)	0.777(0.382)	0.972(0.134)	1.073(0.212)	1.074(0.173)	1.110(0.276)
0.5	0.25	α	1.083(0.114)	0.963(0.146)	1.019(0.133)	0.825(0.067)	0.982(0.113)	0.646(0.266)	0.892(0.114)	1.078(0.156)	0.976(0.026)	0.998(0.037)
		β	0.896(1.012)	0.902(1.275)	0.909(1.189)	0.954(0.001)	0.896(1.026)	0.712(0.631)	0.878(0.931)	0.930(1.360)	0.971(0.001)	0.953(0.001)
		λ	1.024(0.034)	0.980(0.045)	1.002(0.040)	0.923(0.063)	1.034(0.036)	0.618(0.618)	0.964(0.052)	1.024(0.050)	0.972(0.036)	0.978(0.042)
		γ_1	1.055(0.129)	1.055(0.129)	1.026(0.144)	0.943(0.178)	1.003(0.108)	0.755(0.394)	0.961(0.122)	1.062(0.197)	1.000(0.086)	1.017(0.137)
		γ_2	1.050(0.145)	1.050(0.145)	1.024(0.173)	0.941(0.202)	0.998(0.124)	0.748(0.461)	0.956(0.138)	1.061(0.233)	0.999(0.098)	1.016(0.157)
0.5	0.5	α	1.082(0.113)	0.981(0.147)	1.033(0.136)	0.861(0.057)	0.977(0.112)	0.374(0.274)	0.879(0.129)	1.095(0.158)	1.018(0.024)	1.031(0.035)
		β	0.969(1.014)	0.905(1.272)	0.927(1.199)	0.965(0.006)	0.979(1.027)	0.702(0.907)	0.996(1.123)	0.931(1.354)	1.008(0.003)	0.996(0.005)
		λ	1.030(0.033)	0.990(0.041)	1.010(0.037)	0.926(0.059)	1.040(0.035)	0.630(0.614)	0.971(0.045)	1.033(0.046)	0.978(0.034)	0.981(0.039)
		γ_1	1.056(0.137)	1.056(0.137)	1.031(0.158)	0.957(0.185)	1.003(0.114)	0.758(0.454)	0.960(0.135)	1.068(0.218)	1.024(0.101)	1.036(0.152)
		γ_2	1.054(0.182)	1.054(0.182)	1.024(0.202)	0.956(0.247)	1.002(0.155)	0.745(0.641)	0.962(0.182)	1.061(0.272)	1.026(0.139)	1.038(0.208)
0.5	0.75	α	1.043(0.095)	0.959(0.123)	1.005(0.112)	0.906(0.042)	0.953(0.096)	0.680(0.227)	0.880(0.117)	1.052(0.130)	1.027(0.018)	1.048(0.027)
		β	0.986(0.841)	0.942(1.067)	0.956(0.987)	0.989(0.018)	0.992(0.850)	0.773(0.946)	0.970(1.037)	0.965(1.124)	1.032(0.011)	1.035(0.015)
		λ	1.013(0.024)	0.979(0.033)	0.997(0.028)	0.923(0.050)	1.020(0.024)	0.698(0.464)	0.967(0.036)	1.014(0.035)	0.964(0.028)	0.970(0.033)
		γ_1	1.034(0.115)	1.034(0.115)	1.013(0.135)	0.976(0.169)	0.989(0.103)	0.788(0.409)	0.954(0.125)	1.042(0.180)	1.030(0.093)	1.047(0.148)
		γ_2	1.032(0.161)	1.032(0.161)	1.007(0.185)	0.979(0.279)	0.988(0.143)	0.771(0.642)	0.951(0.189)	1.038(0.251)	1.037(0.158)	1.057(0.256)
0.75	0.75	α	1.056(0.115)	0.986(0.145)	1.021(0.132)	0.935(0.056)	0.983(0.114)	0.692(0.302)	0.914(0.145)	1.066(0.156)	1.031(0.030)	1.057(0.045)
		β	0.985(1.002)	0.937(1.242)	0.952(1.168)	0.989(0.009)	0.990(1.012)	0.813(0.966)	0.988(1.200)	0.965(1.321)	1.007(0.002)	1.015(0.003)
		λ	1.025(0.030)	0.986(0.039)	1.005(0.034)	0.929(0.053)	1.031(0.031)	0.697(0.459)	0.971(0.040)	1.026(0.043)	0.971(0.032)	0.982(0.038)
		γ_1	1.060(0.279)	1.060(0.279)	1.032(0.321)	0.980(0.376)	1.004(0.230)	0.770(0.835)	0.957(0.277)	1.074(0.449)	1.042(0.241)	1.072(0.392)
		γ_2	1.059(0.406)	1.059(0.406)	1.027(0.494)	0.983(0.573)	1.004(0.337)	0.755(1.200)	0.956(0.388)	1.072(0.658)	1.045(0.359)	1.077(0.583)

Table 3: Boot-P and Boot-BCP (in parentheses) confidence interval length for $n = 40$.

Initial values			Method									
α	β	Par	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	α	2.146(2.185)	2.279(2.374)	2.320(2.332)	1.405(1.349)	2.174(2.229)	1.313(2.347)	2.052(2.203)	2.508(2.451)	0.923(0.951)	1.264(1.310)
		β	1.527(1.521)	1.709(1.696)	1.557(1.557)	0.744(1.525)	1.789(1.789)	1.388(1.849)	1.406(1.171)	2.322(2.402)	0.322(0.327)	0.893(0.917)
		λ	1.326(1.137)	1.289(1.482)	1.318(1.536)	1.325(1.669)	1.460(1.160)	1.406(1.681)	1.387(1.655)	1.565(1.446)	1.150(1.376)	1.190(1.439)
		γ_1	2.119(1.690)	1.723(2.072)	1.906(2.241)	1.671(2.678)	1.712(1.194)	1.861(1.867)	0.861(0.953)	2.529(2.215)	1.267(1.588)	1.481(1.380)
		γ_2	2.113(1.697)	1.901(2.590)	1.974(2.689)	1.856(3.121)	1.780(1.210)	1.949(2.920)	0.949(1.057)	2.908(2.208)	1.385(1.753)	1.680(1.513)
0.25	0.5	α	2.126(2.085)	2.376(2.442)	2.387(2.367)	1.101(1.053)	2.147(2.144)	0.916(2.149)	2.181(2.222)	2.567(2.479)	0.809(0.823)	0.950(0.991)
		β	1.739(1.879)	1.835(1.972)	1.826(1.914)	1.959(1.912)	1.797(1.801)	2.413(2.155)	1.921(1.924)	1.934(1.961)	0.915(1.102)	1.263(1.263)
		λ	1.321(1.202)	1.360(1.675)	1.362(1.478)	1.296(1.528)	1.431(1.247)	1.526(1.470)	1.485(1.910)	1.621(1.488)	1.110(1.423)	1.283(1.409)
		γ_1	1.979(1.775)	1.784(2.484)	1.905(2.234)	1.510(2.622)	1.575(1.260)	1.877(1.974)	0.877(1.129)	2.667(2.343)	1.286(1.504)	1.777(2.217)
		γ_2	2.547(2.137)	2.078(2.891)	2.066(2.570)	2.045(3.488)	2.035(1.512)	1.995(3.349)	1.095(1.555)	3.105(2.735)	1.687(1.958)	2.429(3.278)
0.25	0.75	α	2.122(2.092)	2.367(2.433)	2.387(2.361)	0.907(0.854)	2.143(2.149)	2.293(2.143)	2.235(2.269)	2.554(2.478)	0.678(0.724)	0.803(0.851)
		β	1.735(1.759)	1.260(1.197)	1.386(2.036)	1.199(2.076)	1.789(1.836)	1.706(1.823)	1.998(1.892)	1.945(2.037)	1.654(1.629)	2.256(2.216)
		λ	1.300(1.185)	1.339(1.626)	1.339(1.455)	1.269(1.466)	1.403(1.222)	1.012(1.458)	1.446(1.870)	1.593(1.461)	1.092(1.336)	1.252(1.338)
		γ_1	2.100(1.878)	1.895(2.627)	2.003(2.401)	1.647(2.934)	1.663(1.326)	0.877(2.095)	0.877(1.041)	2.837(2.469)	1.457(1.639)	1.992(2.649)
		γ_2	2.885(2.403)	2.337(3.285)	2.386(2.949)	2.759(2.113)	2.320(1.711)	2.298(2.914)	1.298(1.642)	2.572(3.124)	2.245(2.301)	2.450(2.727)
0.5	0.25	α	2.145(2.164)	2.324(2.445)	2.313(2.357)	1.394(1.237)	2.151(2.152)	2.206(1.652)	2.064(2.362)	2.531(2.490)	0.980(0.964)	1.167(1.245)
		β	1.547(1.553)	1.872(1.836)	1.852(1.845)	0.225(7.070)	1.515(1.515)	0.456(3.956)	1.364(1.697)	1.477(1.370)	0.162(0.192)	0.186(0.220)
		λ	1.170(1.103)	1.245(1.507)	1.250(1.333)	1.143(1.333)	1.304(1.114)	1.277(1.392)	1.380(1.570)	1.477(1.329)	1.084(1.241)	1.206(1.281)
		γ_1	2.641(2.141)	2.222(2.901)	2.406(2.827)	1.895(2.937)	2.113(1.541)	2.037(2.391)	1.437(1.331)	1.336(2.770)	1.140(1.975)	2.082(2.914)
		γ_2	2.834(2.274)	2.318(2.344)	2.502(2.821)	2.031(2.135)	2.370(1.634)	2.088(2.853)	1.588(1.423)	1.574(2.947)	1.249(2.121)	2.214(2.111)
0.5	0.5	α	2.126(2.116)	2.376(2.444)	2.298(2.291)	1.302(1.351)	2.088(2.255)	1.130(2.015)	2.328(2.339)	2.605(2.549)	0.940(0.956)	1.115(1.241)
		β	2.535(2.565)	2.939(2.983)	2.811(2.799)	2.681(2.265)	2.715(2.679)	1.510(2.046)	2.061(2.836)	2.599(2.609)	0.384(0.383)	0.533(0.500)
		λ	1.186(1.074)	1.244(1.356)	1.229(1.287)	1.204(1.294)	1.256(1.125)	0.940(1.265)	1.322(1.588)	1.520(1.373)	1.015(1.149)	1.174(1.317)
		γ_1	2.782(2.145)	2.586(2.579)	2.463(3.225)	2.042(2.092)	2.263(1.579)	1.606(1.796)	1.106(1.441)	3.884(2.807)	1.108(1.329)	2.363(1.794)
		γ_2	3.367(2.708)	2.791(3.885)	2.889(3.347)	2.437(2.049)	2.715(1.978)	2.302(2.308)	1.302(1.640)	3.300(3.316)	1.124(1.291)	2.815(2.548)
0.5	0.75	α	2.127(2.120)	2.370(2.436)	2.304(2.298)	1.196(1.273)	2.089(2.250)	1.121(1.998)	2.378(2.372)	2.596(2.554)	0.908(0.928)	1.074(1.234)
		β	2.502(2.574)	2.951(2.983)	2.822(2.836)	1.313(2.339)	2.713(2.683)	2.725(2.975)	2.962(6.546)	2.610(2.626)	0.750(0.798)	0.972(1.004)
		λ	1.162(1.051)	1.225(1.330)	1.206(1.237)	1.184(1.257)	1.222(1.104)	0.965(1.238)	1.291(1.492)	1.497(1.356)	0.976(1.107)	1.134(1.258)
		γ_1	2.979(2.263)	2.755(3.823)	2.609(3.316)	2.214(3.402)	2.384(1.672)	1.102(2.026)	1.102(1.602)	4.128(2.976)	2.006(1.265)	2.639(2.375)
		γ_2	3.834(3.081)	3.151(3.402)	3.244(3.798)	2.979(3.171)	3.107(2.217)	1.441(3.803)	1.441(1.792)	3.922(3.808)	2.711(1.261)	3.680(3.785)
0.75	0.75	α	2.299(2.218)	2.350(2.562)	2.349(2.425)	1.264(1.325)	2.331(2.394)	2.362(1.904)	2.308(2.462)	2.599(2.516)	1.015(1.092)	1.298(1.392)
		β	2.722(2.971)	2.314(2.061)	2.238(2.126)	0.619(2.884)	2.972(2.025)	2.395(2.359)	2.143(2.025)	2.074(2.016)	0.397(0.403)	0.487(0.485)
		λ	1.099(0.996)	1.230(1.359)	1.225(1.263)	1.005(1.045)	1.189(1.053)	1.040(0.999)	1.277(1.442)	1.430(1.259)	0.945(1.228)	1.109(1.230)
		γ_1	4.174(3.252)	3.497(3.453)	3.890(3.798)	2.975(2.949)	3.394(2.375)	3.511(3.781)	3.511(3.293)	3.582(4.575)	2.704(2.541)	3.096(3.444)
		γ_2	4.965(3.968)	4.428(4.884)	4.862(4.890)	3.706(3.106)	4.159(2.941)	4.850(4.718)	4.850(4.660)	4.574(4.862)	3.306(3.592)	4.051(3.462)

Table 4: Boot-P and Boot-BCP (in parentheses) confidence interval length for $n = 100$.

Initial values			Method									
α	β	Par	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	α	1.248(1.239)	1.447(1.448)	1.384(1.384)	0.927(0.853)	1.247(1.255)	0.971(1.310)	1.349(1.434)	1.490(1.495)	0.584(0.570)	0.709(0.707)
		β	1.317(1.768)	1.371(1.362)	1.226(1.243)	0.481(1.142)	1.427(1.542)	1.178(1.212)	1.257(1.341)	1.495(1.505)	0.243(0.238)	0.310(0.310)
		λ	0.749(0.700)	0.855(0.871)	0.818(0.810)	0.857(1.037)	0.781(0.706)	1.177(0.698)	0.863(0.889)	0.924(0.881)	0.741(0.778)	0.810(0.912)
		γ_1	1.036(0.927)	1.108(1.212)	1.084(1.086)	1.126(1.360)	0.903(0.779)	0.817(1.064)	0.817(0.925)	1.280(1.192)	0.848(0.920)	1.084(1.238)
		γ_2	1.103(1.001)	1.149(1.214)	1.183(1.177)	1.284(1.643)	1.007(0.882)	0.930(1.131)	0.930(1.076)	1.327(1.246)	0.937(1.017)	1.209(1.370)
0.25	0.5	α	1.245(1.239)	1.445(1.453)	1.384(1.369)	0.753(0.759)	1.245(1.246)	0.873(1.438)	1.395(1.535)	1.489(1.484)	0.522(0.515)	0.638(0.637)
		β	1.311(1.423)	1.359(1.338)	1.231(1.245)	1.268(1.287)	1.368(1.425)	1.702(1.850)	1.475(1.530)	1.488(1.493)	0.672(0.579)	0.642(0.696)
		λ	0.734(0.682)	0.839(0.858)	0.803(0.804)	0.825(1.003)	0.763(0.693)	1.192(0.702)	0.854(0.873)	0.905(0.862)	0.733(0.777)	0.800(0.888)
		γ_1	1.096(0.984)	1.176(1.282)	1.159(1.147)	1.268(1.512)	0.950(0.827)	0.817(1.141)	0.817(0.949)	1.367(1.285)	0.937(1.069)	1.204(1.355)
		γ_2	1.232(1.133)	1.287(1.378)	1.337(1.330)	1.682(1.770)	1.134(0.989)	1.061(1.312)	1.061(1.267)	1.502(1.402)	1.292(1.428)	1.639(1.844)
0.25	0.75	α	1.242(1.241)	1.438(1.454)	1.382(1.382)	0.640(0.652)	1.242(1.246)	0.874(1.264)	1.365(1.515)	1.487(1.484)	0.455(0.449)	0.565(0.547)
		β	1.521(1.415)	1.137(1.163)	1.226(1.226)	1.184(1.186)	1.564(1.529)	1.627(1.371)	1.347(1.354)	1.475(1.481)	1.225(1.050)	1.270(1.131)
		λ	0.720(0.677)	0.825(0.847)	0.788(0.794)	0.801(0.971)	0.747(0.682)	1.203(0.689)	0.839(0.846)	0.891(0.846)	0.711(0.759)	0.802(0.871)
		γ_1	1.161(1.046)	1.243(1.367)	1.225(1.211)	1.379(1.683)	1.004(0.884)	0.860(1.207)	0.860(1.029)	1.442(1.354)	1.073(1.236)	1.350(1.603)
		γ_2	1.381(1.294)	1.456(1.533)	1.504(1.501)	2.115(2.086)	1.269(1.112)	1.224(1.496)	1.224(1.558)	1.713(1.589)	1.790(1.512)	2.209(2.233)
0.5	0.25	α	1.383(1.334)	1.507(1.511)	1.422(1.395)	0.939(0.883)	1.385(1.390)	1.083(1.343)	1.413(1.511)	1.549(1.533)	0.610(0.620)	0.724(0.719)
		β	1.177(1.164)	1.167(1.178)	1.181(1.127)	0.166(1.191)	1.177(1.123)	0.723(0.999)	1.140(1.227)	1.198(1.222)	0.116(0.167)	0.125(0.140)
		λ	0.713(0.682)	0.773(0.841)	0.742(0.742)	0.821(0.896)	0.743(0.659)	1.113(0.806)	0.828(0.818)	0.832(0.816)	0.700(0.723)	0.741(0.787)
		γ_1	1.461(1.310)	1.517(1.681)	1.454(1.454)	1.456(1.537)	1.321(1.080)	1.064(1.652)	1.064(1.219)	1.751(1.597)	1.089(1.263)	1.329(1.585)
		γ_2	1.433(1.293)	1.456(1.634)	1.471(1.471)	1.568(1.640)	1.299(1.112)	1.152(1.699)	1.152(1.319)	1.713(1.645)	1.176(1.348)	1.420(1.699)
0.5	0.5	α	1.304(1.284)	1.464(1.489)	1.387(1.387)	0.876(0.852)	1.301(1.323)	1.106(0.987)	1.359(1.371)	1.511(1.495)	0.604(0.613)	0.725(0.715)
		β	1.384(1.372)	1.364(1.487)	1.304(1.138)	0.439(1.237)	1.269(1.229)	1.209(1.211)	1.240(1.189)	1.577(4.586)	0.232(0.254)	0.260(0.283)
		λ	0.692(0.688)	0.771(0.842)	0.726(0.757)	0.784(0.811)	0.693(0.686)	1.135(0.652)	0.798(0.919)	0.828(0.800)	0.677(0.736)	0.730(0.764)
		γ_1	1.404(1.397)	1.527(1.693)	1.512(1.579)	1.506(1.896)	1.255(1.225)	1.067(1.417)	1.067(1.225)	1.783(1.683)	1.247(1.348)	1.501(1.581)
		γ_2	1.595(1.528)	1.726(2.022)	1.740(1.875)	1.739(2.348)	1.420(1.313)	1.279(1.845)	1.279(1.484)	2.069(1.858)	1.467(1.560)	1.763(1.832)
0.5	0.75	α	1.193(1.207)	1.377(1.384)	1.281(1.277)	0.789(0.759)	1.209(1.207)	1.142(1.031)	1.379(1.397)	1.416(1.414)	0.558(0.523)	0.624(0.626)
		β	1.567(1.518)	1.026(1.047)	1.173(1.208)	0.544(1.051)	1.562(1.588)	1.570(1.606)	1.344(1.427)	1.147(1.171)	0.524(0.459)	0.443(0.410)
		λ	0.631(0.603)	0.703(0.742)	0.646(0.646)	0.721(0.782)	0.652(0.586)	1.187(0.651)	0.760(0.784)	0.751(0.719)	0.605(0.674)	0.668(0.675)
		γ_1	1.319(1.242)	1.437(1.727)	1.379(1.373)	1.555(1.596)	1.181(1.098)	1.218(1.300)	1.218(1.323)	1.641(1.538)	1.268(1.080)	1.481(1.736)
		γ_2	1.640(1.525)	1.774(1.943)	1.684(1.662)	2.010(2.259)	1.467(1.324)	1.535(1.471)	1.535(1.754)	2.044(1.876)	1.626(1.013)	1.979(2.269)
0.75	0.75	α	1.250(1.262)	1.444(1.450)	1.325(1.340)	0.915(0.936)	1.257(1.251)	1.246(1.202)	1.418(1.470)	1.498(1.471)	0.645(0.642)	0.812(0.812)
		β	1.965(1.974)	1.579(1.583)	1.429(1.400)	0.419(1.333)	1.978(1.903)	1.853(1.913)	1.546(1.648)	1.732(1.744)	0.187(0.208)	0.205(0.198)
		λ	0.681(0.632)	0.759(0.771)	0.734(0.732)	0.752(0.747)	0.679(0.655)	1.199(0.702)	0.757(0.834)	0.806(0.799)	0.653(0.687)	0.728(0.772)
		γ_1	2.070(1.872)	2.090(2.395)	2.105(2.117)	2.214(2.310)	1.822(1.630)	1.757(2.249)	1.757(2.253)	2.519(2.376)	1.837(2.150)	2.408(2.187)
		γ_2	2.634(2.377)	2.744(3.048)	2.675(2.758)	2.692(2.582)	2.347(2.057)	2.171(3.218)	2.171(2.743)	3.340(2.966)	2.255(2.619)	2.962(2.369)

Table 5: The predicted shape parameter under $S_u = 0.05$.

Initial values		Method									
α	β	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
		$N = 40$									
0.25	0.25	1.190	1.073	1.108	0.942	1.077	0.709	0.972	1.261	0.974	0.960
		0.705	0.711	0.758	0.202	0.583	0.371	0.336	1.334	0.102	0.185
0.25	0.5	1.156	1.040	1.079	0.938	1.047	0.729	0.974	1.217	0.984	1.010
		0.511	0.649	0.649	0.165	0.409	0.808	0.428	1.222	0.091	0.140
0.25	0.75	1.156	1.040	1.078	0.958	1.045	0.744	0.969	1.216	0.996	1.022
		0.538	0.683	0.678	0.155	0.428	0.765	0.403	0.669	0.089	0.135
0.5	0.25	1.178	1.039	1.088	0.903	1.067	0.669	0.985	1.231	0.963	0.999
		0.883	0.928	1.097	0.356	0.670	0.651	0.527	1.784	0.210	0.379
0.5	0.5	1.175	1.052	1.087	0.949	1.063	0.694	0.973	1.248	0.992	1.037
		1.202	1.331	1.272	0.369	0.937	0.948	0.653	2.548	0.206	0.385
0.5	0.75	1.175	1.052	1.088	0.967	1.061	0.703	0.981	1.248	1.006	1.055
		1.265	1.398	1.346	0.376	0.981	0.996	0.907	2.679	0.225	0.433
0.75	0.75	1.194	1.055	1.102	0.952	1.073	0.679	1.000	1.266	1.002	1.057
		1.942	2.008	2.221	0.648	1.460	1.987	1.380	1.937	0.447	0.821
		$N = 100$									
0.25	0.25	1.053	1.015	1.032	0.942	1.003	0.792	0.966	1.074	1.009	1.026
		0.123	0.158	0.145	0.090	0.108	0.188	0.101	0.198	0.041	0.060
0.25	0.5	1.053	1.015	1.032	0.965	1.002	0.805	0.963	1.074	1.025	1.043
		0.130	0.166	0.152	0.082	0.113	0.190	0.115	0.207	0.043	0.063
0.25	0.75	1.053	1.016	1.032	0.988	1.001	0.806	0.965	1.074	1.035	1.056
		0.136	0.175	0.160	0.079	0.118	0.196	0.120	0.218	0.044	0.065
0.5	0.25	1.073	1.022	1.047	0.943	1.021	0.766	0.977	1.086	1.001	1.018
		0.246	0.278	0.266	0.167	0.212	0.406	0.212	0.354	0.081	0.128
0.5	0.5	1.073	1.029	1.053	0.957	1.019	0.776	0.976	1.094	1.023	1.035
		0.241	0.299	0.288	0.159	0.206	0.449	0.228	0.383	0.086	0.129
0.5	0.75	1.048	1.011	1.032	0.974	1.002	0.808	0.970	1.063	1.026	1.043
		0.198	0.251	0.236	0.132	0.176	0.389	0.204	0.305	0.071	0.113
0.75	0.75	1.077	1.029	1.053	0.980	1.020	0.790	0.976	1.098	1.041	1.070
		0.450	0.534	0.507	0.304	0.380	0.813	0.439	0.692	0.197	0.321

6 Real Data Analysis

In this section, we illustrate the applicability of the different estimators by analyzing real life data set from Nelson (2004). The data describes the times to breakdown of an insulating fluid subjected to various constant elevated test voltages. Recently, these data analyzed by many authors, see for example Nassar and Dey (2018) and Dey and Nassar (2020). Table 8 presents the set of the data under the different three stress levels, i.e. 30, 32 and 34 kilovolt (kv). We use the maximum likelihood method to estimate

Table 6: Average values of MREs and MSEs (in parentheses) of the Predicted reliability function for $n = 40$.

Initial values			Method									
α	β	x_0	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	0.2	0.983(0.010)	0.963(0.014)	0.968(0.013)	0.970(0.005)	0.940(0.015)	1.014(0.016)	0.951(0.011)	0.981(0.014)	0.983(0.003)	0.943(0.025)
		0.5	0.991(0.017)	0.989(0.019)	0.988(0.019)	0.991(0.004)	0.909(0.020)	1.209(0.030)	0.956(0.015)	0.998(0.022)	0.994(0.004)	0.933(0.029)
		1	0.988(0.010)	1.052(0.012)	1.031(0.012)	1.060(0.003)	0.863(0.010)	1.812(0.184)	1.001(0.010)	1.009(0.014)	1.037(0.003)	0.929(0.015)
		2	0.977(0.001)	1.236(0.002)	1.153(0.002)	1.326(0.002)	0.802(0.001)	1.208(0.103)	1.194(0.002)	1.033(0.002)	1.206(0.001)	0.902(0.002)
0.25	0.5	0.2	0.971(0.011)	0.948(0.014)	0.955(0.013)	0.969(0.004)	0.929(0.016)	0.999(0.007)	0.936(0.014)	0.965(0.013)	0.981(0.002)	0.984(0.002)
		0.5	0.966(0.018)	0.959(0.019)	0.961(0.019)	0.985(0.004)	0.887(0.022)	1.172(0.029)	0.933(0.019)	0.967(0.022)	0.987(0.003)	0.991(0.003)
		1	0.953(0.010)	1.007(0.012)	0.991(0.012)	1.054(0.003)	0.836(0.011)	1.729(0.177)	0.971(0.011)	0.965(0.013)	1.027(0.003)	1.026(0.003)
		2	0.945(0.001)	1.176(0.002)	1.102(0.002)	1.333(0.002)	0.779(0.001)	1.617(0.101)	1.156(0.003)	0.985(0.002)	1.194(0.001)	1.182(0.001)
0.25	0.75	0.2	0.963(0.011)	0.941(0.014)	0.948(0.013)	0.971(0.003)	0.923(0.016)	0.991(0.007)	0.930(0.015)	0.958(0.013)	0.980(0.002)	0.982(0.002)
		0.5	0.952(0.018)	0.944(0.020)	0.946(0.019)	0.984(0.003)	0.876(0.023)	1.152(0.028)	0.918(0.019)	0.952(0.022)	0.984(0.003)	0.987(0.003)
		1	0.935(0.011)	0.985(0.012)	0.970(0.012)	1.051(0.003)	0.822(0.012)	1.284(0.174)	0.948(0.012)	0.946(0.014)	1.023(0.003)	1.021(0.003)
		2	0.922(0.002)	1.142(0.002)	1.071(0.002)	1.326(0.002)	0.764(0.001)	1.247(0.102)	1.122(0.003)	0.961(0.002)	1.194(0.001)	1.178(0.001)
0.5	0.25	0.2	0.984(0.007)	0.961(0.010)	0.968(0.009)	0.966(0.005)	0.952(0.010)	0.983(0.014)	0.959(0.008)	0.977(0.009)	0.982(0.002)	0.983(0.002)
		0.5	0.990(0.016)	0.973(0.018)	0.977(0.017)	0.975(0.005)	0.922(0.019)	1.116(0.018)	0.958(0.015)	0.987(0.020)	0.986(0.003)	0.990(0.004)
		1	0.990(0.011)	1.025(0.013)	1.012(0.013)	1.040(0.003)	0.878(0.012)	1.189(0.167)	0.995(0.011)	0.999(0.015)	1.022(0.003)	1.024(0.003)
		2	0.982(0.002)	1.194(0.003)	1.119(0.002)	1.301(0.002)	0.819(0.001)	1.195(0.106)	1.161(0.003)	1.023(0.002)	1.177(0.002)	1.168(0.002)
0.5	0.5	0.2	0.975(0.007)	0.955(0.009)	0.961(0.008)	0.972(0.004)	0.945(0.011)	0.979(0.004)	0.947(0.010)	0.970(0.008)	0.985(0.001)	0.987(0.002)
		0.5	0.970(0.016)	0.956(0.018)	0.959(0.017)	0.982(0.005)	0.904(0.021)	1.097(0.017)	0.936(0.018)	0.971(0.019)	0.990(0.003)	0.995(0.004)
		1	0.962(0.012)	0.998(0.013)	0.986(0.013)	1.042(0.003)	0.856(0.014)	1.538(0.161)	0.964(0.013)	0.974(0.015)	1.026(0.003)	1.027(0.004)
		2	0.949(0.002)	1.148(0.003)	1.083(0.002)	1.283(0.002)	0.797(0.002)	3.890(0.103)	1.124(0.003)	0.986(0.002)	1.180(0.002)	1.164(0.002)
0.5	0.75	0.2	0.970(0.006)	0.950(0.009)	0.956(0.008)	0.973(0.003)	0.941(0.011)	0.973(0.005)	0.943(0.011)	0.965(0.008)	0.984(0.001)	0.987(0.002)
		0.5	0.958(0.016)	0.943(0.018)	0.947(0.018)	0.980(0.004)	0.894(0.022)	1.077(0.015)	0.924(0.019)	0.958(0.019)	0.987(0.003)	0.992(0.003)
		1	0.944(0.013)	0.977(0.014)	0.966(0.013)	1.036(0.003)	0.842(0.015)	1.490(0.155)	0.946(0.014)	0.955(0.016)	1.022(0.003)	1.023(0.003)
		2	0.927(0.002)	1.116(0.003)	1.054(0.003)	1.269(0.002)	0.782(0.002)	1.203(0.101)	1.095(0.003)	0.962(0.003)	1.180(0.002)	1.160(0.002)
0.75	0.75	0.2	0.978(0.003)	0.961(0.006)	0.966(0.005)	0.976(0.002)	0.958(0.006)	0.963(0.004)	0.959(0.006)	0.973(0.005)	0.986(0.001)	0.987(0.001)
		0.5	0.964(0.014)	0.945(0.017)	0.950(0.016)	0.972(0.005)	0.912(0.020)	1.015(0.010)	0.933(0.017)	0.962(0.017)	0.982(0.003)	0.987(0.004)
		1	0.953(0.015)	0.971(0.016)	0.965(0.015)	1.015(0.003)	0.860(0.017)	1.314(0.126)	0.948(0.016)	0.963(0.018)	1.008(0.004)	1.012(0.004)
		2	0.930(0.003)	1.090(0.004)	1.037(0.003)	1.224(0.003)	0.796(0.003)	1.991(0.091)	1.072(0.004)	0.961(0.003)	1.145(0.002)	1.130(0.002)

Table 7: Average values of MREs and MSEs (in parentheses) of the Predicted reliability function for $n = 100$.

Initial values			Method									
α	β	x_0	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
0.25	0.25	0.2	0.986(0.004)	0.978(0.005)	0.982(0.005)	0.977(0.003)	0.965(0.005)	1.003(0.004)	0.970(0.005)	0.987(0.005)	0.995(0.001)	0.997(0.001)
		0.5	0.982(0.006)	0.983(0.008)	0.982(0.007)	0.988(0.002)	0.942(0.007)	1.125(0.020)	0.969(0.007)	0.987(0.008)	0.999(0.001)	1.002(0.002)
		1	0.973(0.003)	1.002(0.004)	0.988(0.004)	1.032(0.001)	0.913(0.004)	1.108(0.145)	0.990(0.004)	0.983(0.004)	1.016(0.001)	1.015(0.001)
		2	0.965(0.000)	1.068(0.001)	1.020(0.001)	1.190(0.001)	0.880(0.000)	1.160(0.094)	1.087(0.001)	0.988(0.001)	1.082(0.001)	1.070(0.001)
0.25	0.5	0.2	0.978(0.004)	0.970(0.006)	0.973(0.005)	0.979(0.002)	0.958(0.006)	0.994(0.004)	0.959(0.006)	0.978(0.005)	0.993(0.001)	0.995(0.001)
		0.5	0.967(0.007)	0.968(0.008)	0.967(0.008)	0.988(0.002)	0.929(0.008)	1.102(0.019)	0.950(0.008)	0.972(0.008)	0.996(0.001)	0.998(0.001)
		1	0.954(0.004)	0.980(0.004)	0.968(0.004)	1.030(0.001)	0.896(0.004)	1.159(0.142)	0.965(0.005)	0.963(0.005)	1.013(0.001)	1.011(0.001)
		2	0.943(0.001)	1.040(0.001)	0.994(0.001)	1.185(0.001)	0.862(0.001)	1.286(0.063)	1.052(0.001)	0.964(0.001)	1.082(0.001)	1.067(0.001)
0.25	0.75	0.2	0.971(0.004)	0.963(0.006)	0.966(0.005)	0.980(0.002)	0.951(0.006)	0.987(0.004)	0.958(0.006)	0.972(0.006)	0.988(0.001)	0.989(0.001)
		0.5	0.953(0.007)	0.953(0.008)	0.952(0.008)	0.987(0.002)	0.916(0.009)	1.087(0.018)	0.946(0.008)	0.963(0.009)	0.990(0.001)	0.991(0.001)
		1	0.935(0.004)	0.960(0.005)	0.948(0.004)	1.028(0.001)	0.880(0.005)	1.130(0.140)	0.961(0.005)	0.955(0.005)	1.010(0.001)	1.008(0.001)
		2	0.920(0.001)	1.013(0.001)	0.970(0.001)	1.179(0.001)	0.844(0.001)	1.195(0.065)	1.049(0.001)	0.960(0.001)	1.093(0.001)	1.082(0.001)
0.5	0.25	0.2	0.992(0.002)	0.982(0.004)	0.987(0.003)	0.980(0.002)	0.978(0.003)	0.990(0.002)	0.978(0.003)	0.989(0.003)	0.994(0.001)	0.995(0.001)
		0.5	0.994(0.006)	0.987(0.008)	0.990(0.007)	0.985(0.002)	0.961(0.007)	1.081(0.012)	0.977(0.006)	0.993(0.008)	0.997(0.001)	0.999(0.002)
		1	0.993(0.004)	1.008(0.005)	1.000(0.005)	1.023(0.001)	0.941(0.005)	1.106(0.146)	1.001(0.005)	0.997(0.006)	1.014(0.001)	1.015(0.001)
		2	0.993(0.001)	1.077(0.001)	1.034(0.001)	1.166(0.001)	0.918(0.001)	1.177(0.068)	1.098(0.001)	1.007(0.001)	1.083(0.001)	1.076(0.001)
0.5	0.5	0.2	0.985(0.003)	0.977(0.004)	0.980(0.003)	0.979(0.002)	0.971(0.004)	0.982(0.003)	0.969(0.004)	0.983(0.003)	0.992(0.001)	0.993(0.001)
		0.5	0.977(0.007)	0.972(0.008)	0.974(0.008)	0.980(0.002)	0.945(0.008)	1.060(0.012)	0.958(0.008)	0.978(0.009)	0.992(0.001)	0.994(0.002)
		1	0.966(0.005)	0.981(0.006)	0.974(0.005)	1.012(0.001)	0.915(0.005)	1.364(0.141)	0.970(0.006)	0.971(0.006)	1.003(0.001)	1.004(0.001)
		2	0.951(0.001)	1.028(0.001)	0.989(0.001)	1.145(0.001)	0.881(0.001)	2.965(0.069)	1.047(0.001)	0.962(0.001)	1.059(0.001)	1.057(0.001)
0.5	0.75	0.2	0.980(0.002)	0.972(0.003)	0.976(0.003)	0.981(0.001)	0.968(0.003)	0.973(0.003)	0.966(0.004)	0.978(0.003)	0.991(0.001)	0.992(0.001)
		0.5	0.964(0.006)	0.960(0.007)	0.962(0.007)	0.981(0.002)	0.938(0.007)	1.021(0.010)	0.947(0.008)	0.965(0.007)	0.990(0.001)	0.993(0.001)
		1	0.952(0.004)	0.965(0.005)	0.959(0.005)	1.013(0.001)	0.910(0.005)	1.143(0.123)	0.952(0.005)	0.957(0.005)	1.007(0.001)	1.007(0.001)
		2	0.948(0.001)	1.013(0.001)	0.980(0.001)	1.144(0.001)	0.890(0.001)	1.123(0.052)	1.016(0.001)	0.961(0.001)	1.078(0.001)	1.071(0.001)
0.7532	0.75	0.2	0.986(0.001)	0.980(0.002)	0.983(0.002)	0.986(0.001)	0.977(0.002)	0.973(0.002)	0.973(0.002)	0.984(0.002)	0.994(0.001)	0.995(0.001)
		0.5	0.971(0.006)	0.963(0.008)	0.967(0.007)	0.980(0.002)	0.945(0.008)	0.995(0.007)	0.949(0.009)	0.971(0.008)	0.991(0.001)	0.995(0.002)
		1	0.956(0.006)	0.964(0.007)	0.961(0.007)	1.003(0.001)	0.912(0.007)	1.170(0.113)	0.948(0.008)	0.961(0.008)	1.002(0.001)	1.004(0.002)
		2	0.943(0.001)	1.008(0.002)	0.978(0.001)	1.121(0.001)	0.881(0.001)	1.361(0.052)	1.008(0.002)	0.956(0.002)	1.061(0.001)	1.050(0.001)

the unknown parameters and use these estimates to see whether the EEx distribution provides a good fit to the data set. We obtain the Kolmogorov-Smirnov(K-S) distance and the corresponding p-value (PV) for each stress level. These values are displayed in Table 8. Due to all PVs in Table 8 being greater than 0.05, we can conclude that the EEx distribution provides good fit to mentioned data for the given three stress levels.

Table 8: Times to breakdown of an insulating fluid, K-S distance and the corresponding PV.

Stress	Data									KS (PV)
30 kV	7.74	17.05	20.46	21.02	22.66	43.4	47.3	139.07	144.12	0.2739(0.3813)
	175.88	194.9								
32 kV	0.27	0.40	0.69	0.79	2.75	3.91	9.88	13.95	15.93	0.2138(0.4995)
	27.80	53.24	82.85	89.29	100.58	215.10				
34 kV	0.19	0.78	0.96	1.31	2.78	3.16	4.15	4.67	4.85	0.2068(0.3910)
	6.50	7.35	8.01	8.27	12.06	31.75	32.52	33.91	36.71	
	72.89									

Now, we use the different methods of estimation to estimate the EEx distribution parameters under the different stress levels. The various estimates are presented in Table 9. We use 1000 bootstrap samples to estimate the standard error of the different estimates. These values are also displayed in Table 9. From Table 9, we can observe that the estimates of the shape parameter $\gamma_j, j = 1, ..., 3$ decrease as the stress level increases for all methods of estimation. Also, it is noted that the ADEs perform better than other estimates in terms of minimum estimated standard error of estimate. Table 10 presents the two bootstrap confidence intervals of the different estimates. The results in Table 10 shows that the two bootstrap confidence intervals based on the ADEs have the shortest length in compare with other estimates. We use the different estimates in Table 9 to predict the reliability function and the shape parameter under the usual stress level, i.e. 20 kv. The predicted values are displayed in Table 11. From Table 11 we noted that the predicted shape parameter values based on the different estimation methods are greater than those values based on the higher stress levels which obtained in Table 9. Also, it is noted that the estimated reliability value decreases as the remission time increases in all the cases. Generally, we can conclude that the ADEs are more efficient than the other estimates based on the minimum estimated standard error of the estimates and the shortest confidence interval length to estimate the EEx distribution parameters under CSALT for Nelson (2004) data.

7 Conclusions

In this paper, we have investigated the estimation of the EEx parameters under CSALT by considering ten classical estimation methods. These estimation methods are the

Table 9: The estimates and the corresponding estimated standard errors (in parentheses) using various methods for the real data set.

	Estimates					
Method	α	β	λ	γ_1	γ_2	γ_3
MLEs	7.791(3.181)	-0.258(0.097)	0.018(0.005)	1.040(0.396)	0.620(0.149)	0.370(0.116)
LSEs	6.959(3.363)	-0.243(0.102)	0.012(0.004)	0.726(0.291)	0.447(0.095)	0.275(0.066)
WLSEs	7.879(3.677)	-0.262(0.111)	0.019(0.006)	1.023(0.567)	0.606(0.160)	0.359(0.092)
PCEs	14.41(6.991)	-0.482(0.221)	0.011(0.004)	0.939(0.493)	0.358(0.153)	0.136(0.082)
MPSEs	7.179(3.308)	-0.242(0.101)	0.020(0.006)	0.910(0.345)	0.560(0.116)	0.345(0.084)
MSADEs	4.264(3.153)	-0.166(0.097)	0.001(0.004)	0.488(0.282)	0.350(0.097)	0.251(0.078)
MSALDEs	9.947(3.553)	-0.329(0.109)	0.014(0.005)	1.088(0.417)	0.564(0.119)	0.292(0.073)
CMEs	8.828(3.165)	-0.303(0.096)	0.008(0.002)	0.776(0.348)	0.424(0.096)	0.231(0.049)
ADEs	7.722(3.098)	-0.264(0.093)	0.013(0.001)	0.811(0.228)	0.478(0.090)	0.282(0.047)
RTADEs	8.889(3.891)	-0.302(0.119)	0.012(0.004)	0.856(0.484)	0.468(0.118)	0.256(0.074)

Table 10: Boot-P (first row) and Boot-BCP (second row) for the real data set.

	Bootstrap confidence intervals					
Method	α	β	λ	γ_1	γ_2	γ_3
MLEs	(-0.087,12.568)	(-0.398,-0.013)	(0.012,0.030)	(0.566,2.004)	(0.474,1.021)	(0.293,0.730)
	(3.2480,16.850)	(-0.561,-0.122)	(0.011,0.026)	(0.486,1.658)	(0.414,0.908)	(0.222,0.502)
LSEs	(-0.307,12.400)	(-0.414,-0.024)	(0.004,0.018)	(0.336,1.284)	(0.266,0.644)	(0.159,0.414)
	(1.6630,14.540)	(-0.478,-0.072)	(0.008,0.039)	(0.550,2.082)	(0.362,0.801)	(0.189,0.551)
WLSEs	(0.7040,14.414)	(-0.463,-0.044)	(0.010,0.032)	(0.500,2.194)	(0.382,0.956)	(0.220,0.564)
	(1.3930,15.025)	(-0.478,-0.061)	(0.013,0.049)	(0.639,4.463)	(0.450,1.473)	(0.244,0.658)
PCEs	(-1.209,25.476)	(-0.847,-0.013)	(0.002,0.016)	(0.128,1.900)	(0.072,0.647)	(0.020,0.334)
	(4.7940,39.135)	(-0.618,-0.119)	(0.006,0.023)	(0.583,2.866)	(0.256,0.809)	(0.026,0.350)
MPSEs	(0.4860,13.898)	(-0.448,-0.040)	(0.014,0.038)	(0.467,1.785)	(0.368,0.813)	(0.210,0.526)
	(0.7370,14.136)	(-0.036,0.0440)	(0.012,1.020)	(0.560,2.152)	(0.414,0.944)	(0.230,0.615)
MSADEs	(0.8790,15.584)	(-0.505,-0.071)	(0.000,0.025)	(0.217,1.662)	(0.171,1.131)	(0.125,0.853)
	(2.2470,11.271)	(-0.392,-0.114)	(0.001,0.042)	(0.355,1.377)	(0.284,0.846)	(0.201,0.721)
MSALDEs	(2.3460,16.564)	(-0.536,-0.096)	(0.007,0.025)	(0.498,1.877)	(0.326,0.767)	(0.161,0.436)
	(5.5650,18.943)	(-0.590,-0.185)	(0.009,0.032)	(0.708,3.490)	(0.419,0.995)	(0.187,0.528)
CMEs	(3.2620,16.813)	(-0.615,-0.166)	(0.002,0.019)	(0.433,1.744)	(0.255,0.727)	(0.122,0.416)
	(1.5940,16.734)	(-0.645,-0.112)	(0.006,0.019)	(0.457,1.829)	(0.210,0.886)	(0.111,0.425)
ADEs	(0.8700,13.329)	(-0.439,-0.054)	(0.006,0.020)	(0.416,1.473)	(0.311,0.685)	(0.174,0.440)
	(2.0530,14.502)	(-0.466,-0.091)	(0.009,0.025)	(0.501,1.961)	(0.361,0.833)	(0.190,0.478)
RTADEs	(1.2660,16.363)	(-0.535,-0.068)	(0.006,0.020)	(0.402,1.850)	(0.289,0.745)	(0.150,0.431)
	(2.4060,17.028)	(-0.558,17.028)	(0.008,0.027)	(0.469,2.549)	(0.324,0.846)	(0.162,0.490)

Table 11: The predicted reliability function and shape parameter ($\hat{\gamma}^*$) under $S_u = 20kv$.

x_0	MLEs	LSEs	WLSEs	PCEs	MPSEs	MSADEs	MSALDEs	CMEs	ADEs	RTADEs
5	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
50	0.999	0.999	0.999	1.000	0.992	0.999	1.000	1.000	1.000	1.000
125	0.802	0.882	0.728	1.000	0.601	0.992	0.997	0.999	0.925	0.984
200	0.339	0.558	0.254	1.000	0.184	0.976	0.839	0.976	0.603	0.781
$\hat{\gamma}^*$	13.776	8.216	14.035	116.88	10.277	2.550	31.518	16.016	11.409	17.444

MLEs, PCEs, LSEs, WLSEs, MPSEs, MSADEs, MSALDEs, CMEs, ADEs and RTADEs. We also used the different estimation methods to obtain the predicted shape parameter and the reliability function of the EEX distribution under normal use conditions. In addition, we have considered two bootstrap confidence intervals, namely the percentile bootstrap and bias corrected percentile bootstrap confidence intervals for the unknown parameters. Since it is not easy to compare the performance of the different estimators theoretically, we have conducted a simulation study to compare the various estimators by considering different combinations of the unknown parameters and different sample sizes. The numerical results showed that the ADE method gives the best results compared with the other estimation methods in terms of MRE, MSE and confidence interval length. Furthermore, the empirical analysis of a real-life data set supports the numerical results to show that the ADE method performs better than the other studied methods. The work in this paper can be extended by considering the estimation of the EEx parameters under CSALT using Bayesian approach. Further, it is of interest to investigate the same estimation procedures in the presence of censoring samples.

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