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Darna distribution: properties and application

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In this paper, Darna distribution (DD) is suggested as a new continuous probability density function (PDF). The statistical properties of the DD as the moments, shapes of the distribution, measures of skewness, kurtosis, coefficient of variation are presented as well as some calculations are provided. Also, the maximum likelihood estimators, the Bonferroni and Lorenz curves, and Gini Index are obtained. The Stress-Strength Reliability, Rényi entropy, mean and median deviations are derived and proved. The distribution of order statistics are presented. The reliability analysis including hazard, reliability, odds, and reverse hazard functions are presented. An application of Wheaton River data is considered.

MSC 2010: 60E05; 62F10

keywords: Mixture distribution; Variance; Bonferroni curve; Stress-Strength reliability; Lorenz curve; Gini index; Rényi entropy; Stochastic ordering; Reliability analysis.

1 Introduction

The distribution $f(x)$ is a mixture of k components distributions $\varphi_1(x), \varphi_2(x), ..., \varphi_k(x)$ if $f(x) = \sum_{i=1}^{k} \eta_i \varphi_i(x)$ where η_i is the mixing weights, such that $0 \leqslant \eta_i \leqslant 1$, and $\sum_{i=1}^{k} \eta_i = 1$. A random variable X is said to have a mixture of two distributions $\varphi_1(x)$ and $\varphi_2(x)$ if its probability density function (PDF) is given by

$$f(x) = \eta_1 \varphi_1(x) + \eta_2 \varphi_2(x), \tag{1}$$

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where η_1, η_2 are positive and $\eta_1+\eta_2=1$. Shanker (2017) suggested Akshaya distribution with PDF given by

$$f(x;\theta) = \frac{\theta^4}{\theta^3+2\theta^2+6\theta+6}(1+x)^3 e^{-\theta x} \;,\; x>0\,,\theta>0,$$

which is a mixture of four components, $Exp(\theta), Gamma(2,\theta), Gamma(3,\theta)$, and $Gamma(4,\theta)$ distributions. Shanker and Ghebretsadik (2013) suggested new Quasi Lindley distribution with PDF given by

$$f(x;\theta) = \frac{\theta^2}{\theta+1}\,(1+x)\,e^{-\theta x}\,, x>0\,,\theta>0,$$

that is a mixture of $Exp(\theta)$ and $Gamma(2,\theta)$ distributions with mixture factor $\eta = \frac{\alpha}{\theta^2+\alpha}$.

Daniyal and Aleem (2014) proposed a new distribution as a mixture of Burr and Weibull distributions. Hall and Zhou (2003) have introduced nonparametric estimation of component distribution in a multivariate mixture. Cruz-Medina and Hettmansperger (2004) have proposed nonparametric estimation in semiparametric univariate mixture models. Roy et al. (1992) introduced a class of passion mixture distribution and in (1995) introduced some negative binomial mixture distribution. Sultan (2007) introduced mixture of two inverse Weibull distributions. Al-Omari et al. (2019c) suggested power length-biased Suja distribution. Jiang et al. (1999) proposed Weibull and inverse Weibull mixture. Kamaruzzaman et al. (2012) introduced mixtures of normal distributions. Balakrishnan and Mohanty (1972) proposed finite mixture of Laguerre distributions. Nareeat et al. (2015) proposed new mixture Pareto distribution. Roy and Sinha (1995) proposed negative binomial mixture of normal moment distribution. Adnan (2009) (2009) suggested Laplace mixture distribution. Al-Omari et al. (2019a) suggested exponentiated new Weibull-Pareto distribution. Al-Omari et al. (2019b) proposed size-biased Ishita distribution. Al-Omari and Alsmairan (2019) introduced length-biased Suja distribution. Al-Omari and Gharaibeh (2018) suggested Topp-Leone Mukherjee-Islam distribution. Tamandi et al. (2019) suggested a generalized Birnbaum-Sauders distribution with application to the air pollution data. Mdlongwa et al. (2017) proposed the Burr XII modified Weibull distribution.

The rest of this paper is organized as follows: In Section 2, we introduce the PDF and CDF of the Darna distribution and illustrate the shapes of the distribution for various values of the distribution parameters. Section 3, provides the moments includes the rth moment, the coefficients of skewness, kurtosis, and coefficient of variation, as well as some simulations for the distribution moments. Also, we present the moment generating function and the mode of the distribution Section 4, includes the Bonferroni and Lorenz curves and Gini index of the Darna distribution and some simulations for the Gini index for different distribution parameters. Median and mean deviations and the stress-strength reliability are given in Section 5. The distributions of order statistics from Darna distribution are provided in Section 6. In Section 7, the maximum likelihood estimation of the distribution parameters and the Rényi entropy with some simulation are presented. Stochastic ordering and reliability analysis are given in Section 8. An application of real data is given in Section 9. Finally, the paper is concluded in Section 10.

2 The Darna Distribution

In this section, we will define the PDF and cumulative distribution function (CDF) of the Darna distribution and illustrate the shapes of the distribution. However, no any work has been conducted on the Darna distribution to our best knowledge.

Definition: A random variable X is said to have a Darna distribution with two parameters α and θ, if its probability density function is given by

$$f_{DD}(x;\theta,\alpha)=\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta x}{\alpha}};\, x>0,\alpha>0,\theta>0,\theta>\alpha,\theta\neq\alpha. \quad (2)$$

Based on Equation (1), the proposed Darna distribution is a mixture of two distributions, namely, exponential $Exp\left(\frac{\theta}{\alpha}\right)$ and gamma $G\left(3,\frac{\theta}{\alpha}\right)$, distributions where $\varphi_1=Exp\left(\frac{\theta}{\alpha}\right)=\frac{\theta}{\alpha}e^{-\frac{\theta}{\alpha}x}, x>0$, and $\varphi_2=G\left(3,\frac{\theta}{\alpha}\right)=\frac{\theta^3}{2\alpha^3}e^{-\frac{\theta}{\alpha}x}x^2, x>0$, with mixing factors $\eta=\frac{2\alpha^2}{2\alpha^2+\theta^2}$ and $1-\eta=\frac{\theta^2}{2\alpha^2+\theta^2}$.

Theorem 1: The corresponding cumulative distribution function is given by

$$F_{DD}(x;\alpha,\theta)=1-\frac{\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)}{2\alpha^2\left(2\alpha^2+\theta^2\right)}e^{-\frac{\theta x}{\alpha}};x>0,\alpha>0,\theta>0. \quad (3)$$

Proof: The proof is direct using integration by parts as

$$\begin{aligned}F_{DD}\left(x,\alpha,\theta\right)=P(X\leqslant x)\ &=\int_0^x f_{DD}(t;\alpha,\theta)\,dt\\ &=\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha\int_0^x e^{-\frac{\theta t}{\alpha}}\,dt+\frac{\theta^4}{2\alpha^3}\int_0^x t^2e^{-\frac{\theta t}{\alpha}}\,dt\right).\end{aligned}$$

Now, let $I_1=2\alpha\int_0^x e^{-\frac{\theta t}{\alpha}}\,dt\ =\ -2\frac{\alpha^2}{\theta}e^{-\frac{\theta t}{\alpha}}\Big|_0^x=-2\frac{\alpha^2}{\theta}\left(e^{-\frac{\theta x}{\alpha}}-1\right)$ and

$$I_2 = \frac{\theta^4}{2\alpha^3}\int_0^x t^2 e^{-\frac{\theta t}{\alpha}} dt$$
$$= \frac{\theta^4}{2\alpha^3}\left(-\frac{\alpha}{\theta}t^2 e^{-\frac{\theta t}{\alpha}}\Big|_0^x + 2\frac{\alpha}{\theta}\int_0^x t e^{-\frac{\theta t}{\alpha}} dt\right)$$
$$= \frac{\theta^4}{2\alpha^3}\left[-\frac{\alpha}{\theta}t^2 e^{-\frac{\theta t}{\alpha}}\Big|_0^x + 2\frac{\alpha}{\theta}\left(-\frac{\alpha}{\theta}t e^{-\frac{\theta t}{\alpha}}\Big|_0^x + \frac{\alpha}{\theta}\int_0^x e^{-\frac{\theta t}{\alpha}} dt\right)\right]$$
$$= -\frac{\theta^3}{2\alpha^3}x^2 e^{-\frac{\theta x}{\alpha}} - \frac{\theta^2}{\alpha}x e^{-\frac{\theta x}{\alpha}} - \theta\left(e^{-\frac{\theta x}{\alpha}} - 1\right).$$

Therefore,

$$F_{DD}(x,\alpha,\theta) = \frac{\theta}{2\alpha^2+\theta^2}(I_1+I_2) = 1 - \frac{e^{-\frac{\theta x}{\alpha}}\left(4\alpha^4+\theta^4x^2+2x\alpha\theta^3+2\alpha^2\theta^2\right)}{(2\alpha^2+\theta^2)\,2\alpha^2}.$$

It is easy to show that
$f_{DD}(x;\theta,\alpha) = \frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta x}{\alpha}} \geqslant 0$ and $\int_0^\infty f_{DD}(x;\theta,\alpha)dx = 1$.

The shapes of the PDF and CDF of the Darna distribution are given in Figure (1) for different values of the parameter α when $\theta = 3$. It is clear from Figure (1) that the

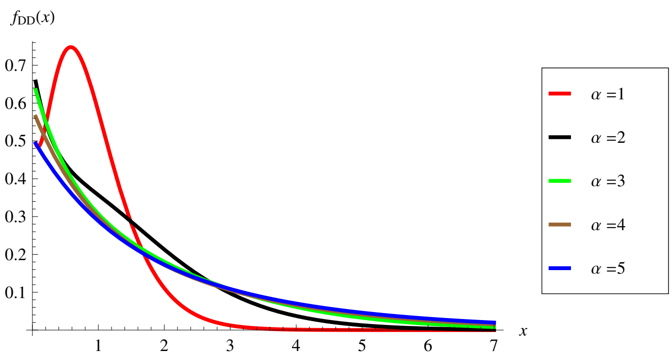

Figure 1: The PDF of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

Darna distribution is skewed to the right.

3 The Moments of the Darna Distribution

In this section, we will provide some moments of the Darna distribution and present some tables of the mean, standard deviation, coefficient of skewness, coefficient of variation, and coefficient of kurtosis for some selected parameters.

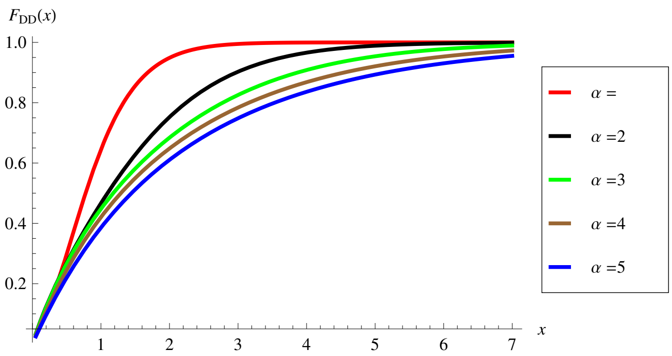

Figure 2: The CDF of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

Theorem 1: Let $X \sim f_{DD}(x;\alpha,\theta)$. Then, the rth moment of X is

$$E\left(X^r\right) = r!\left(\frac{\alpha}{\theta}\right)^r \left(\frac{4\alpha^2 + (r+1)(r+2)\theta^2}{2\left(2\alpha^2+\theta^2\right)}\right), \frac{\theta}{\alpha} > 0, r > -1. \tag{4}$$

Proof: The rth moment of the Darna distribution can be derived as

$$\begin{aligned}
E\left(X^r\right) &= \int_0^\infty x^r \frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha + \frac{\theta^4}{2\alpha^3}x^2\right) e^{-\frac{\theta x}{\alpha}}\, dx \\
&= \frac{\theta}{2\alpha^2+\theta^2} \int_0^\infty \left(2\alpha x^r + \frac{\theta^4}{2\alpha^3}x^{r+2}\right) e^{-\frac{\theta x}{\alpha}}\, dx \\
&= \frac{\theta}{2\alpha^2+\theta^2} \left(\int_0^\infty 2\alpha x^r e^{-\frac{\theta x}{\alpha}}\, dx + \int_0^\infty \frac{\theta^4}{2\alpha^3} x^{r+2} e^{-\frac{\theta x}{\alpha}}\, dx\right) \\
&= \frac{\theta}{2\alpha^2+\theta^2} \left(2\alpha \frac{r!}{\left(\frac{\theta}{\alpha}\right)^{r+1}} + \frac{\theta^4}{2\alpha^3} \frac{(r+2)!}{\left(\frac{\theta}{\alpha}\right)^{r+3}}\right) \\
&= \frac{\theta}{2\alpha^2+\theta^2} r!\left(\frac{\theta}{\alpha}\right)^{-r} \left(2\alpha \frac{1}{\left(\frac{\theta}{\alpha}\right)} + \frac{\theta^4}{2\alpha^3} \frac{(r+2)(r+1)}{\left(\frac{\theta}{\alpha}\right)^3}\right) \\
&= \frac{\theta}{2\alpha^2+\theta^2} r!\left(\frac{\theta}{\alpha}\right)^{-r} \left(\frac{2\alpha^2}{\theta} + \frac{\theta(r+2)(r+1)}{2}\right) \\
&= r!\left(\frac{\alpha}{\theta}\right)^r \left(\frac{4\alpha^2 + (r+1)(r+2)\theta^2}{2\left(2\alpha^2+\theta^2\right)}\right).
\end{aligned}$$

Hence, the first four moments of the $DD(\theta,\alpha)$ distributed random variable can be found by substituting $r = 1, 2, 3, 4$, respectively, in Equation (4) as

$$E\left(X\right)=\frac{\alpha}{2\theta}\left(\frac{4\alpha^{2}+6\theta^{2}}{2\alpha^{2}+\theta^{2}}\right),\theta>0,E\left(X^{2}\right)=\frac{\alpha^{2}}{\theta^{2}}\left(\frac{4\alpha^{2}+12\theta^{2}}{2\alpha^{2}+\theta^{2}}\right),\theta>0,$$

$$E\left(X^{3}\right)=\frac{3\alpha^{3}}{\theta^{3}}\left(\frac{4\alpha^{2}+20\theta^{2}}{2\alpha^{2}+\theta^{2}}\right),\theta>0,E\left(X^{4}\right)=\frac{12\alpha^{2}}{\theta^{4}}\left(\frac{4\alpha^{2}+30\theta^{2}}{2\alpha^{2}+\theta^{2}}\right),\theta>0.$$

Then, the variance of the Darna distribution is given by

$$V\left(X\right)=E\left(X^{2}\right)-\left[E(X)\right]^{2}=\frac{4\alpha^{6}+16\alpha^{4}\theta^{2}+3\alpha^{2}\theta^{4}}{\left(2\theta^{2}+\theta^{3}\right)^{2}}.$$

The coefficient of skewness, coefficient of kurtosis, and coefficient of variation of the Darna distribution, respectively, are given by

$$\begin{aligned}Sk_{DD}&=\frac{E\left(X^{3}\right)-3\mu\sigma^{2}-\mu^{3}}{\sigma^{3}}\\&=\frac{2\alpha^{11}+17\alpha^{9}\theta^{2}+20\alpha^{7}\theta^{4}+9\alpha^{5}\theta^{6}+1.875\alpha^{3}\theta^{8}+0.1875\,\alpha\theta^{10}}{\left(\alpha^{2}+0.5\,\theta^{2}\right)^{3}\sqrt{\frac{4\alpha^{6}+16\alpha^{4}\theta^{2}+3\alpha^{2}\theta^{4}}{\left(2\alpha^{2}\theta+\theta^{3}\right)^{2}}}\left(\alpha^{4}\theta+4\alpha^{2}\theta^{3}+0.75\theta^{5}\right)},\end{aligned}\tag{5}$$

$$\begin{aligned}Ku_{DD}&=\frac{E\left(X^{4}\right)-4\mu E\left(X^{3}\right)+6E\left(X^{2}\right)\sigma^{2}+3E\left(X^{4}\right)}{\sigma^{8}}\\&=\frac{0.56\,\theta^{4}\left(2\,\alpha^{2}+\theta^{2}\right)^{2}\begin{pmatrix}12.8\,\alpha^{12}+149.333\,\alpha^{10}\theta^{2}+248.533\,\alpha^{8}\theta^{4}\\+181.333\,\alpha^{6}\theta^{6}+69.6\alpha^{4}\theta^{8}+13.6\,\alpha^{2}\theta^{10}+\theta^{12}\end{pmatrix}}{\left(1.33\,\alpha^{5}+5.33\,\alpha^{3}\theta^{2}+\alpha\theta^{4}\right)^{4}},\end{aligned}\tag{6}$$

and

$$Cv_{DD}=\frac{\sigma_{DD}}{\mu_{DD}}=\frac{\theta\left(2\,\alpha^{2}+\theta^{2}\right)\sqrt{\frac{4\,\alpha^{6}+16\,\alpha^{4}\theta^{2}+3\,\alpha^{2}\theta^{4}}{\left(2\,\alpha^{2}\theta+\theta^{3}\right)^{2}}}}{2\alpha^{3}+3\,\alpha\theta^{2}}.\tag{7}$$

Some values of the mean, standard deviation, coefficient of variation, coefficient skewness and coefficient kurtosis for the Darna distribution are obtained for various values of the parameters and the results are presented in Table (1).

Table (1) indicates that the mean, standard deviation, coefficients of variation are increasing as the values of α increasing. The values of skewness are decreasing from $\alpha=0.1$ to $\alpha=1$, then start increasing for $\alpha>1$ up to $\alpha=4$.

Theorem 3: The moment generating function (MGF) of the Darna distribution is

$$M_{X}(t)=\frac{\theta}{\alpha^{2}\left(2\alpha^{2}+\theta^{2}\right)}\left[\frac{-2\alpha^{4}(\alpha t-\theta)^{2}-\alpha^{2}\theta^{4}}{\left(\alpha t-\theta\right)^{3}}\right].\tag{8}$$

Table 1: The mean, standard deviation, coefficients of variation, skewness and kurtosis for the $DD(\theta, \alpha)$ with different values of the parameter α when $\theta = 3$

α	μ_{DD}	σ_{DD}	Cv_{DD}	Sk_{DD}	Ku_{DD}
0.1	0.09985	0.05778	0.578630	1.15217	4.99412
0.2	0.19883	0.11580	0.582435	1.14495	4.97702
0.3	0.29608	0.17429	0.588659	1.13413	4.95027
0.4	0.39084	0.23339	0.597140	1.12125	4.91627
0.5	0.48246	0.29317	0.607665	1.10806	4.87801
0.6	0.57037	0.35362	0.619985	1.09623	4.83874
0.7	0.65418	0.41464	0.633836	1.08715	4.80170
0.8	0.73359	0.47606	0.648942	1.08184	4.76983
0.9	0.80848	0.53766	0.665033	1.08093	4.74562
1	0.87879	0.59920	0.681852	1.08465	4.73103
1.1	0.94460	0.66043	0.699161	1.09297	4.72745
1.2	1.00606	0.72109	0.716741	1.10564	4.73572
1.3	1.06338	0.78095	0.734400	1.12225	4.75621
1.4	1.11682	0.83982	0.751971	1.14234	4.78893
1.5	1.16667	0.89753	0.769309	1.16540	4.83353
1.6	1.21322	0.95395	0.786295	1.19093	4.88948
1.7	1.25679	1.00898	0.802828	1.21846	4.95604
1.8	1.29767	1.06257	0.818830	1.24754	5.03240
1.9	1.33617	1.11468	0.834239	1.27778	5.11765
2	1.37255	1.16531	0.849009	1.30883	5.21087
2.1	1.40707	1.21445	0.863108	1.34039	5.31114
2.2	1.43997	1.26216	0.876516	1.37217	5.41752
2.3	1.47147	1.30846	0.889223	1.40396	5.52913
2.4	1.50175	1.35342	0.901227	1.43556	5.64512
2.5	1.53101	1.39710	0.912534	1.46680	5.76469
2.6	1.55938	1.43956	0.923156	1.49756	5.88706
2.7	1.58702	1.48087	0.933109	1.52771	6.01155
2.8	1.61405	1.52110	0.942412	1.55717	6.13747
2.9	1.64056	1.56032	0.951090	1.58586	6.26424
3	1.66667	1.59861	0.959166	1.61372	6.39130
3.1	1.69244	1.63603	0.966668	1.64071	6.51815
3.2	1.71796	1.67264	0.973622	1.66680	6.64432
3.3	1.74327	1.70851	0.980057	1.69197	6.76942
3.4	1.76845	1.74369	0.986001	1.71619	6.89306
3.5	1.79353	1.77825	0.991480	1.73947	7.01494
3.6	1.81856	1.81223	0.996523	1.76181	7.13475
3.7	1.84356	1.84569	1.001160	1.78320	7.25227
3.8	1.86857	1.87867	1.005400	1.80367	7.36726
3.9	1.89361	1.91120	1.009290	1.82322	7.47956
4	1.91870	1.94334	1.012840	1.84187	7.58900

Proof: The mgf of the Darna distribution can be proved as

$$E\left(e^{tx}\right) = \int_0^\infty e^{tx}\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta x}{\alpha}}dx$$
$$= \frac{\theta}{2\alpha^2+\theta^2}\left(\int_0^\infty 2\alpha e^{\left(t-\frac{\theta}{\alpha}\right)x}dx + \frac{\theta^4}{2\alpha^3}\int_0^\infty x^2e^{\left(t-\frac{\theta}{\alpha}\right)x}dx\right).$$

Let

$$I_1 = \int_0^\infty 2\alpha e^{\left(t-\frac{\theta}{\alpha}\right)x}dx = \frac{-2\alpha}{t-\frac{\theta}{\alpha}},$$

and

$$I_2 = \frac{\theta^4}{2\alpha^3}\int_0^\infty x^2e^{\left(t-\frac{\theta}{\alpha}\right)x}dx$$
$$= \frac{\theta^4}{2\alpha^3}\left(\frac{x^2}{t-\frac{\theta}{\alpha}}e^{\left(t-\frac{\theta}{\alpha}\right)x}\Big|_0^\infty - \frac{2}{t-\frac{\theta}{\alpha}}\int_0^\infty xe^{\left(t-\frac{\theta}{\alpha}\right)x}dx\right)$$
$$= \frac{-2\theta^4}{2\alpha^3\left(t-\frac{\theta}{\alpha}\right)^3}.$$

Therefore,

$$E\left(e^{tx}\right) = \frac{-4\alpha^4\left(t-\frac{\theta}{\alpha}\right)^2-2\theta^4}{2\alpha^3\left(t-\frac{\theta}{\alpha}\right)^3}\frac{\theta}{2\alpha^2+\theta^2}$$
$$= \frac{-2\alpha^4\left(t-\frac{\theta}{\alpha}\right)^2-\theta^4}{\alpha^3\left(t-\frac{\theta}{\alpha}\right)^3}\frac{\theta}{2\alpha^2+\theta^2}$$
$$= \frac{\theta}{\alpha^2\left(2\alpha^2+\theta^2\right)}\left[\frac{-2\alpha^4(\alpha t-\theta)^2-\alpha^2\theta^4}{(\alpha t-\theta)^3}\right].$$

Theorem 4: The mode of the Darna distribution is the solution of the equation

$$x\left(1-\frac{\theta}{2\alpha}x\right) = 2\left(\frac{\alpha}{\theta}\right)^3. \tag{9}$$

Proof:

$$\log\left[f\left(x;\alpha,\theta\right)\right] = \log\left[\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta x}{\alpha}}\right]$$
$$= \log\left[\frac{\theta}{2\alpha^2+\theta^2}\right]+\log\left[2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right]+\log\left[e^{-\frac{\theta x}{\alpha}}\right].$$

with the first derivative given by

$$\frac{d}{dx}\log\left[f\left(x;\alpha,\theta\right)\right]=\frac{2x\frac{\theta^4}{2\alpha^3}}{2\alpha+\frac{\theta^4x^2}{2\alpha^3}}-\frac{\theta}{\alpha}.$$

Setting it to zero and with simple algebra we get the proof.

4 Bonferroni and Lorenz Curves and Gini Index

Assume that the random variable X is non-negative with continuous and twice differentiable cumulative distribution function $F(x)$. The Bonferroni Curve of the random variable X is defined as

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{1}{p\mu}\left(\int_0^\infty xf(x)dx-\int_q^\infty xf(x)dx\right)=\frac{1}{p\mu}\left(\mu-\int_q^\infty xf(x)dx\right),$$

where $q=\ F^{-1}(p)$ and $p\in(0,1]$. The Lorenz curve is defined as

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{1}{\mu}\left(\int_0^\infty xf(x)dx-\int_q^\infty xf(x)dx\right)=\frac{1}{\mu}\left(\mu-\int_q^\infty xf(x)dx\right).$$

The Gini index is given by

$$G=1-\frac{1}{\mu}\int_0^\infty(1-F(x))^2dx=\frac{1}{\mu}\int_0^\infty F(x)(1-F(x))dx.$$

Now, for the Darna distribution, the Bonferroni curves, Lorenz curves and Gini index are in the following theorem.

Theorem 5: The Bonferroni curve, Lorenz curve and Gini index for the Darna distribution, respectively, are

$$B(p)=\frac{1}{2p}\left(2-\frac{e^{-\frac{\theta q}{\alpha}}\left(4\alpha^5+6\alpha^3\theta^2+\theta^5q^3+3\alpha\theta^4q^2+4\alpha^4\theta q+6\alpha^2\theta^3q\right)}{\alpha^3\left(2\alpha^2+3\theta^2\right)}\right),\qquad(10)$$

$$L(p)=1-\frac{\mathrm{e}^{-\frac{q\theta}{\alpha}}\left(4\alpha^5+4q\alpha^4\theta+6\alpha^3\theta^2+6q\alpha^2\theta^3+3q^2\alpha\theta^4+q^3\theta^5\right)}{2\alpha^3\left(2\alpha^2+3\theta^2\right)},\qquad(11)$$

and the Gini index is

$$G_{DD}(\theta,\alpha)=\frac{32\alpha^4+72\alpha^2\theta^2+15\theta^4}{64\alpha^4+128\alpha^2\theta^2+48\theta^4}.\qquad(12)$$

Table (2) shows that the Gini index values for the Darna distribution for $\theta=2$ are increasing as α increasing up to $\alpha=3.6$, then it is decreasing. But for $\theta=5$ the values of the Gini index are increasing for all considered values of α. However, for fixed α, as θ increasing the Gini index values are decreasing.

Table 2: Gini index values for the Darna distribution for $\theta = 2, 5$ and selected values of α

α	$G_{DD}(2,\alpha)$	α	$G_{DD}(2,\alpha)$	α	$G_{DD}(2,\alpha)$	α	$G_{DD}(5,\alpha)$	α	$G_{DD}(5,\alpha)$	α	$G_{DD}(5,\alpha)$
0.1	0.314157	1.6	0.475150	3.1	0.514098	0.1	0.312766	1.6	0.367578	3.1	0.444452
0.2	0.319017	1.7	0.481482	3.2	0.514438	0.2	0.313563	1.7	0.373135	3.2	0.448517
0.3	0.326761	1.8	0.486988	3.3	0.514675	0.3	0.314880	1.8	0.378728	3.3	0.452414
0.4	0.336911	1.9	0.491746	3.4	0.514825	0.4	0.316705	1.9	0.384325	3.4	0.456145
0.5	0.348889	2	0.495833	3.5	0.514900	0.5	0.319017	2	0.389901	3.5	0.459710
0.6	0.362088	2.1	0.499326	3.6	0.514913	0.6	0.321792	2.1	0.395429	3.6	0.463112
0.7	0.375929	2.2	0.502293	3.7	0.514874	0.7	0.325003	2.2	0.400888	3.7	0.466353
0.8	0.389901	2.3	0.504799	3.8	0.514790	0.8	0.328616	2.3	0.406258	3.8	0.469437
0.9	0.403585	2.4	0.506904	3.9	0.514670	0.9	0.332598	2.4	0.411523	3.9	0.472368
1	0.416667	2.5	0.508658	4	0.514520	1	0.336911	2.5	0.416667	4	0.475150
1.1	0.428924	2.6	0.510109	4.1	0.514345	1.1	0.341517	2.6	0.421678	4.1	0.477787
1.2	0.440219	2.7	0.511299	4.2	0.514151	1.2	0.346376	2.7	0.426546	4.2	0.480284
1.3	0.450487	2.8	0.512263	4.3	0.513940	1.3	0.351450	2.8	0.431263	4.3	0.482647
1.4	0.459710	2.9	0.513033	4.4	0.513716	1.4	0.356700	2.9	0.435822	4.4	0.484880
1.5	0.467914	3	0.513636	4.5	0.513483	1.5	0.362088	3	0.440219	4.5	0.486988

5 Mean and Median Deviations

To measure the scatter in the population, the mean deviation about the mean $\mathbb{Z}_1(x)$, and the mean deviation about the median $\mathbb{Z}_2(x)$, can be used, where

$$\mathbb{Z}_1(x) = \int_0^\infty |x-\mu| f(x)dx = \int_0^\mu (\mu - x)f(x)dx + \int_\mu^\infty (x-\mu)f(x)dx = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx,$$

and

$$\mathbb{Z}_2(x) = \int_0^\infty |x-M| f(x)dx = \int_0^M (M - x)f(x)dx + \int_M^\infty (x-M)f(x)dx = \mu - 2\int_0^M xf(x)dx,$$

where μ and M are the population mean and median, respectively. The mean and median deviations about the mean and median for the Darna distribution are defined in the following theorem.

Theorem 6: Let X has $f_{DD}(x;\theta,\alpha)$, the mean and median deviations about the mean and median, respectively, are

$$\mathbb{Z}_1(\theta,\alpha) = \frac{\alpha\left(2\alpha^2+3\theta^2\right)^2\left(4\alpha^2+3\theta^2\right)}{\theta(2\alpha^2+\theta^2)^3} e^{\frac{4\alpha^2}{2\alpha^2+\theta^2}-3}, \tag{13}$$

and

$$\mathbb{Z}_2(\theta,\alpha) = \frac{e^{-\frac{\theta M}{\alpha}}\left[\theta^5 M^3 + 3\alpha\theta^4 M^2 - 2\alpha^5\left(e^{\frac{\theta M}{\alpha}}-2\right) + 4\alpha^4\theta M - 3\alpha^3\theta^2\left(e^{\frac{\theta M}{\alpha}}-2\right) + 6\alpha^2\theta^3 M\right]}{\alpha^2\left(2\alpha^2\theta+\theta^3\right)}. \tag{14}$$

Proof: For the Darna distribution we have

$$F_{DD}(\mu;\alpha,\theta) = 1 - \frac{e^{-\theta\left(\frac{2\theta}{2\alpha^2+\theta^2}+\frac{1}{\theta}\right)}\left[4\alpha^4 + 2\alpha^2\theta^2 + \alpha^2\theta^4\left(\frac{2\theta}{2\alpha^2+\theta^2}+\frac{1}{\theta}\right)^2 + 2\alpha^2\theta^3\left(\frac{2\theta}{2\alpha^2+\theta^2}+\frac{1}{\theta}\right)\right]}{2\alpha^2\left(2\alpha^2+\theta^2\right)},$$

and

$$\int_0^\mu xf(x)dx = \frac{\alpha\left(2\alpha^2+3\theta^2\right)\left[e^3\left(2\alpha^2+\theta^2\right)^3 - e^{\frac{4\alpha^2}{2\alpha^2+\theta^2}}\left(16\alpha^6+40\alpha^4\theta^2+38\alpha^2\theta^4+13\theta^6\right)\right]}{e^3\theta(2\alpha^2+\theta^2)^4}.$$

Hence, the mean deviation about mean for the Darna distribution is given by

$$\mathbb{Z}_1(x) = 2\mu F(\mu) - 2\int_0^\mu xf(x)dx = \frac{\alpha e^{\frac{4\alpha^2}{2\alpha^2+\theta^2}-3}\left(2\alpha^2+3\theta^2\right)^2\left(4\alpha^2+3\theta^2\right)}{\theta(2\alpha^2+\theta^2)^3}.$$

Also, $\int_0^M xf(x)dx = \frac{\theta\left(6\alpha^3 - e^{-\frac{\theta M}{\alpha}}\left(6\alpha^3+\theta^3M^3+3\alpha\theta^2M^2+6\alpha^2\theta M\right)\right)}{2\alpha^2(2\alpha^2+\theta^2)} + \frac{2\alpha^2\left(\alpha - e^{-\frac{\theta M}{\alpha}}(\alpha+\theta M)\right)}{2\alpha^2\theta+\theta^3}$. Therefore,

$$\begin{aligned}\mathbb{Z}_2(x) &= \mu - 2\int_0^M xf(x)dx \\ &= \frac{e^{-\frac{\theta M}{\alpha}}\left[\theta^5M^3 + 3\alpha\theta^4M^2 - 2\alpha^5\left(e^{\frac{\theta M}{\alpha}}-2\right) + 4\alpha^4\theta M - 3\alpha^3\theta^2\left(e^{\frac{\theta M}{\alpha}}-2\right) + 6\alpha^2\theta^3 M\right]}{\alpha^2\left(2\alpha^2\theta+\theta^3\right)}.\end{aligned}$$

6 The Stress-Strength Reliability and Order Statistics

6.1 The stress-strength reliability

The stress-strength reliability explain the life of a component that has a random strength Y that is subjected to a random stress X. For independent random variables X and Y the stress-strength reliability is defined as

$$R = P(Y < X) = \int_0^\infty P(Y < X|X = x)\, f(x)dx = \int_0^\infty f(x;\theta,\alpha)F(x;,\lambda)dx.$$

Let the random variables X and Y are independent and observed from the Darna distribution.

$$R_{DD} = P\left(Y_{DD} < X_{DD}\right)$$
$$= \frac{\alpha\Upsilon}{\left(2\alpha^2+\theta^2\right)\left(2\lambda^2+\Upsilon^2\right)\left(\alpha\Upsilon+\theta\lambda\right)^5}\begin{pmatrix} 2\alpha^6\left(\Upsilon^6+2\lambda^2\Upsilon^4\right)+18\alpha\theta^5\lambda^5\Upsilon \\ +6\theta^6\lambda^6+4\alpha^5\theta\lambda\Upsilon^3\left(4\lambda^2+\Upsilon^2\right) \\ +\alpha^4\theta^2\Upsilon^2\left(24\lambda^4+\Upsilon^4+4\lambda^2\Upsilon^2\right) \\ +\alpha^3\theta^3\lambda\Upsilon\left(16\lambda^4+5\Upsilon^4+10\lambda^2\Upsilon^2\right) \\ +2\alpha^2\theta^4\lambda^2\left(2\lambda^4+5\Upsilon^4+10\lambda^2\Upsilon^2\right) \end{pmatrix}. \quad (15)$$

Remark: It can be seen that R_{DD} is a function of the stress parameters (θ, α) and strength parameters (Υ, λ). Also, $R_{DD} = 0.5$ if $\Upsilon = \theta = 1$, or $\Upsilon = \theta = \alpha = \lambda = 1$, and for $\alpha = \lambda = 1$, we have .

$$R_{DD} = \frac{\Upsilon}{\left(\theta^2+2\right)\left(\Upsilon^2+2\right)\left(\theta+\Upsilon\right)^5}\begin{pmatrix} 6\theta^6+2\left(\Upsilon^6+2\Upsilon^4\right)+2\theta^4\left(5\Upsilon^4+10\Upsilon^2+2\right) \\ +\theta^3\Upsilon\left(5\Upsilon^4+10\Upsilon^2+16\right)+4\theta\Upsilon^3\left(\Upsilon^2+4\right) \\ +\theta^2\Upsilon^2\left(\Upsilon^4+4\Upsilon^2+24\right)+18\theta^5\Upsilon \end{pmatrix}.$$

6.2 Order statistics

Let $X_{(1:m)}, X_{(2:m)}, ..., X_{(m:m)}$ be the order statistics of the random sample $X_1, X_2, ..., X_m$ selected from a PDF and CDF $f(x)$ and $F(x)$, respectively. The PDF of the ith order statistics $X_{(i:m)}$ as defined by David and Nagaraja (2003) as

$$f_{(i:m)}(x) = \frac{m!}{(i-1)!(m-i)!}[F(x)]^{i-1}[1-F(x)]^{m-i}f(x), i = 1, 2, ..., m. \quad (16)$$

From Equation (17), the PDF of smallest order statistic, $X_{(1:m)}$, and largest order statistic, $X_{(m:m)}$, are respectively, given by

$$f_{(1:m)}(x;\theta,\alpha) = \frac{\theta m\left(4\alpha^4+\theta^4x^2\right)}{2^m\alpha^{2m+1}(2\alpha^2+\theta^2)^m}\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)^{m-1}e^{-m\frac{\theta x}{\alpha}}, \quad (17)$$

with CDF given by

$$F_{(1:m)}(x;\theta,\alpha) = \frac{1}{2}m\delta_{1-m}\left(1-\frac{e^{-\frac{\theta x}{\alpha}}\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)}{2\alpha^2\left(2\alpha^2+\theta^2\right)}\right)^2, \quad (18)$$

where $\delta_{m_1,m_2,...}$ equal to 1 if all m_i are equal, and 0 otherwise. The PDF of largest order statistic, $X_{(m:m)}$, is given as

$$f_{(m:m)}(x;\theta,\alpha) = \frac{\theta m e^{-\frac{\theta x}{\alpha}}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)}{2\alpha^2+\theta^2}\left(1-\frac{e^{-\frac{\theta x}{\alpha}}\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)}{2\alpha^2\left(2\alpha^2+\theta^2\right)}\right)^{m-1}, \quad (19)$$

with CDF

$$F_{(m:m)}(x;\theta,\alpha)=\frac{m}{m+1}\left(1-\frac{e^{-\frac{\theta x}{\alpha}}\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)}{2\alpha^2\left(2\alpha^2+\theta^2\right)}\right)^{m+1}. \tag{20}$$

7 Maximum Likelihood Estimation and Rényi Entropy

7.1 Maximum likelihood estimation

For a random sample of size n, $X_1,X_2,...,X_n$ from the Darna distribution with parameters $\theta>0,\alpha>0$. The maximum likelihood estimator of the Darna distribution parameter can be obtained as follows

$$L(x;\theta,\alpha)=\prod_{i=1}^{n}f(x_i,\theta,\alpha)=\left(\frac{\theta}{2\alpha^2+\theta^2}\right)^n\prod_{i=1}^{n}\left(2\alpha+\frac{\theta^4x_i^2}{2\alpha^3}\right)e^{-\frac{\theta x_i}{\alpha}}.$$

The logarithm of the last equation is

$$\ln L(x;\theta,\alpha)=n\ln\left(\frac{\theta}{2\alpha^2+\theta^2}\right)+\sum_{i=1}^{n}\ln\left(2\alpha+\frac{\theta^4x_i^2}{2\alpha^3}\right)-\sum_{i=1}^{n}\frac{\theta x_i}{\alpha}$$

and its derivative with respect θ and α, respectively, are

$$\frac{d\ln}{d\theta}=\frac{n}{\theta}-\frac{2n\theta}{2\alpha^2+\theta^2}+\sum_{i=1}^{n}\left(\frac{\frac{4\theta^3x_i^2}{2\alpha^3}}{2\alpha+\frac{\theta^4x_i^2}{2\alpha^3}}\right)-\sum_{i=1}^{n}\frac{x_i}{\alpha},$$

and

$$\frac{d\ln}{d\alpha}=\frac{-4n\alpha}{2\alpha^2+\theta^2}+\sum_{i=1}^{n}\left(\frac{2+\left(-3\alpha^{-4}\right)x_i^2\frac{\theta^4}{2}}{2\alpha+\frac{\theta^4x_i^2}{2\alpha^3}}\right)+\sum_{i=1}^{n}\frac{\theta x_i}{\alpha^2}.$$

Since there is no closed from for these equations, then the MLE $\hat{\theta}$ and $\hat{\alpha}$ of θ and α, respectively can be solved numerically.

7.2 The Rényi entropy

The Rényi entropy of a random variable X is a measure of variation of the uncertainty and it defined as

$$RE(\beta)=\frac{1}{1-\beta}\log\left(\int_{0}^{\infty}f(x)^{\beta}dx\right),$$

where $\beta>0$ and $\beta\neq0$. For more about entropy see Zamanzade and Al-Omari (2016), Zamanzade (2015), and Al-Omari and Haq (2019b).

Theorem 8: If $X\sim f_{DD}\left(x;\theta,\alpha\right)$, the Rényi entropy is defined as

$$RE(\beta,\theta,\alpha)=\frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^{\beta}\sum_{j=0}^{\beta}\binom{\beta}{j}\left(\frac{\theta}{\alpha}\right)^{2j-1}\frac{(2j)!}{4^j\beta^{2j+1}}\right]. \tag{21}$$

Proof: The Rényi entropy of the Darna distribution can be derived as

$$\begin{aligned}
RE(\beta,\theta,\alpha) &= \frac{1}{1-\beta}\log\left(\int_0^\infty [f_{DD}(x;\theta,\alpha)]^\beta dx\right)\\
&= \frac{1}{1-\beta}\log\left[\int_0^\infty\left(\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta x}{\alpha}}\right)^\beta dx\right]\\
&= \frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^\beta\int_0^\infty\left(1+\frac{\theta^4x^2}{4\alpha^4}\right)^\beta e^{-\frac{\theta\beta}{\alpha}x}dx\right]\\
&= \frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^\beta\int_0^\infty\sum_{j=0}^{\beta}\binom{\beta}{j}\left(\frac{\theta^4x^2}{4\alpha^4}\right)^j e^{-\frac{\theta\beta}{\alpha}x}dx\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^\beta\sum_{j=0}^{\beta}\binom{\beta}{j}\left(\frac{\theta^4}{4\alpha^4}\right)^j\frac{(2j)!}{\left(\frac{\beta\theta}{\alpha}\right)^{2j+1}}\right]\\
&= \frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^\beta\sum_{j=0}^{\beta}\binom{\beta}{j}\left(\frac{\theta^4}{4\alpha^4}\right)^j\frac{\alpha^{2j+1}(2j)!}{(\beta\theta)^{2j+1}}\right]\\
&= \frac{1}{1-\beta}\log\left[\left(\frac{2\,\alpha\,\theta}{2\alpha^2+\theta^2}\right)^\beta\sum_{j=0}^{\beta}\binom{\beta}{j}\left(\frac{\theta}{\alpha}\right)^{2j-1}\frac{(2j)!}{4^j\beta^{2j+1}}\right].
\end{aligned}$$

Based on Table 3 we can conclude the following:

- For fixed values of θ and β, the Rényi entropy is increasing as α is increasing. As an example, for $\theta = 2$ and $\beta = 3$, the Rényi entropy for $\alpha = 1, 2$ and 15, are 0.87156, 1.05089, and 2.576, respectively.

- For fixed θ and α the Rényi entropy of Darna Distribution is decreasing as is increasing. As an example for =6, =2, the Rényi entropy values are 1.71950, 1.51996 for $\beta = 3, 6$, respectively.

Table 3: Rényi entropy values for selected values of the Darna distribution parameters

α	$RE(3,2,\alpha)$	α	$RE(3,2,\alpha)$	α	$RE(3,2,\alpha)$	α	$RE(6,2,\alpha)$	α	$RE(6,2,\alpha)$	α	$RE(6,2,\alpha)$
1	0.87156	16	2.63911	31	3.29292	1	0.77338	16	2.44687	31	3.10162
2	1.05089	17	2.69856	32	3.32450	2	0.82452	17	2.50646	32	3.13322
3	1.21434	18	2.75473	33	3.35511	3	0.99656	18	2.56275	33	3.16385
4	1.39694	19	2.80796	34	3.38482	4	1.18849	19	2.61609	34	3.19358
5	1.56716	20	2.85854	35	3.41368	5	1.36426	20	2.66676	35	3.22246
6	1.71950	21	2.90671	36	3.44173	6	1.51996	21	2.71501	36	3.25052
7	1.85513	22	2.95270	37	3.46902	7	1.65774	22	2.76106	37	3.27783
8	1.97647	23	2.99668	38	3.49559	8	1.78052	23	2.80510	38	3.30441
9	2.08580	24	3.03883	39	3.52147	9	1.89087	24	2.84730	39	3.33030
10	2.18508	25	3.07929	40	3.54670	10	1.99088	25	2.88781	40	3.35554
11	2.27587	26	3.11819	41	3.57132	11	2.08222	26	2.92675	41	3.38016
12	2.35942	27	3.15565	42	3.59534	12	2.16620	27	2.96424	42	3.40420
13	2.43677	28	3.19176	43	3.61880	13	2.24388	28	3.00038	43	3.42767
14	2.50874	29	3.22662	44	3.64173	14	2.31610	29	3.03527	44	3.45060
15	2.57600	30	3.26032	45	3.66414	15	2.38358	30	3.06899	45	3.47302

8 Stochastic Ordering and Reliability Analysis

8.1 Stochastic ordering

The stochastic ordering can be used to compare two positive continuous distributions. A random variable X is smaller than random variable Y in

1. Mean residual life order denoted by $X \underset{MRLO}{\leq} Y$, if $m_X(x) \leq m_Y(x)$ for all x.
2. Hazard rate order denoted by $X \underset{HRO}{\leq} Y$, if $h_X(x) \geq h_Y(x)$ for all x.
3. Stochastic order denoted by $X \underset{SO}{\leq} Y$, if $F_X(x) \geq F_Y(x)$ for all x.
4. Likelihood ratio order denoted by $X \underset{LRO}{\leq} Y$, if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

It is shown by Shaked and Shanthikumar (1994) that

$$X \underset{LRO}{\leq} Y \Rightarrow X \underset{HRO}{\leq} Y \Rightarrow X \underset{MRLO}{\leq} Y.$$

$$\Downarrow$$

$$X \underset{SO}{\leq} Y$$

Theorem 9: Let $X \sim f_X(x;\alpha,\theta)$, $Y \sim f_Y(x;\beta,\eta)$, and if ($\alpha < \beta$, and $\theta \geqslant \eta$), then $X \underset{LRO}{\leqslant} Y$, $X \underset{HRO}{\leqslant} Y$, $X \underset{MRLO}{\leqslant} Y$ and $X \underset{SO}{\leqslant} Y$.

Proof: Let $X \sim f_X(x;\alpha,\theta)$, $Y \sim f_Y(x;\beta,\eta)$. To prove the theorem, it is sufficient to show that $\frac{f_X(x;\alpha,\theta)}{f_Y(x;\beta,\eta)}$ is a deceasing function of x, where its log is

$$\begin{aligned}
\log \frac{f_X(x;\alpha,\theta)}{f_Y(x;\beta,\eta)} &= \log\left[\frac{\frac{\theta}{2\alpha^2+\theta^2}\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right)e^{-\frac{\theta}{\alpha}x}}{\frac{\eta}{2\beta^2+\eta^2}\left(2\alpha+\frac{\eta^4x^2}{2\beta^3}\right)e^{-\frac{\eta}{\beta}x}}\right] \\
&= \log\left[\frac{\theta\left(2\beta^2+\eta^2\right)}{\eta\left(2\alpha^2+\theta^2\right)}\right] + \log\left(\frac{2\alpha+\frac{\theta^4x^2}{2\alpha^3}}{2\alpha+\frac{\eta^4x^2}{2\beta^3}}\right) + \log e^{-\left(\frac{\theta}{\alpha}-\frac{\eta}{\beta}\right)x}
\end{aligned}$$

$$= \log\left[\frac{\theta\left(2\beta^2+\eta^2\right)}{\eta\left(2\alpha^2+\theta^2\right)}\right] + \log\left(2\alpha+\frac{\theta^4x^2}{2\alpha^3}\right) - \log\left(2\beta+\frac{\eta^4x^2}{2\beta^3}\right) - \left(\frac{\theta}{\alpha}-\frac{\eta}{\beta}\right)x.$$

Taking the derivative of the last equation with respect to x yields

$$\begin{aligned}\frac{d}{dx}log\left(\frac{f_X\left(x;\alpha,\theta\right)}{f_Y\left(x;\beta,\eta\right)}\right) &= \frac{\frac{2x\theta^4}{2\alpha^3}}{2\alpha+\frac{\theta^4x^2}{2\alpha^3}} - \frac{\frac{2x\eta^4}{2\beta^3}}{2\beta+\frac{\eta^4x^2}{2\beta^3}} - \left(\frac{\theta}{\alpha}-\frac{\eta}{\beta}\right)\\ &= \frac{2\theta^4x}{4\alpha^4+\theta^4x^2} - \frac{2\eta^4x}{4\beta^4+\eta^4x^2} - \left(\frac{\theta}{\alpha}-\frac{\eta}{\beta}\right).\end{aligned}$$

Hence, if $(\alpha \leqslant \beta, \theta \geqslant \eta)$, then $\frac{d}{dx}log\left(\frac{f_X(x;\alpha,\theta)}{f_Y(x;\beta,\eta)}\right) < 0$. Therefore, the theorem is proved.

8.2 Reliability analysis

The reliability and hazard function of function of the Darna distribution are given by

$$\begin{aligned}R_{DD}(x;\theta,\alpha) &= 1 - F_{DD}(x;\theta,\alpha)\\ &= \frac{\left(4\alpha^4+2\alpha^2\theta^2+\theta^4x^2+2\alpha\theta^3x\right)}{2\alpha^2\left(2\alpha^2+\theta^2\right)}e^{-\frac{\theta x}{\alpha}};\ x>0,\ \alpha>0,\ \theta>0.\end{aligned} \quad (22)$$

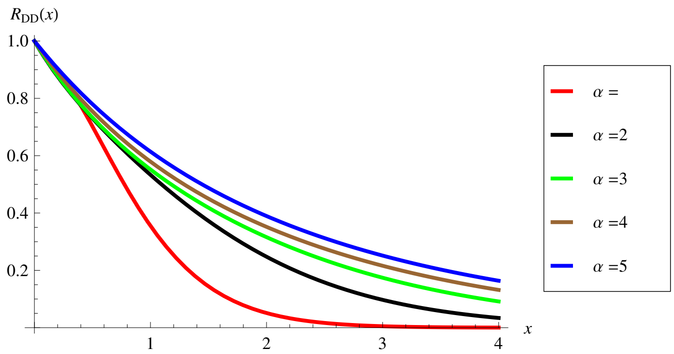

Figure 3: The reliability function of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

and

$$\begin{aligned}H_{DD}(x;\theta,\alpha) &= \frac{f_{DD}(x;\theta,\alpha)}{1-F_{DD}(x;\theta,\alpha)}\\ &= \frac{4\alpha^4\theta+\theta^5x^2}{4\alpha^5+2\alpha^3\theta^2+\alpha\theta^4x^2+2\alpha^2\theta^3x}.\end{aligned} \quad (23)$$

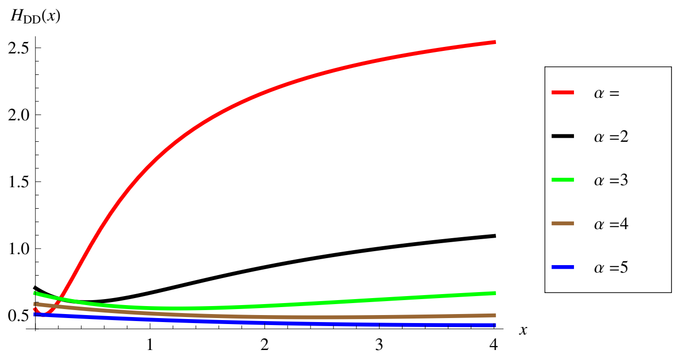

Figure 4: The hazard function of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

The reversed hazard rate and odds functions for the Darna distribution, respectively, are defined as

$$\begin{aligned} RH_{DD}(x;\theta,\alpha) &= \frac{f_{DD}(x;\theta,\alpha)}{F_{DD}(x;\theta,\alpha)} \\ &= \frac{4\alpha^4\theta + \theta^5 x^2}{2\alpha^3\left(2\alpha^2+\theta^2\right)e^{\frac{\theta x}{\alpha}} - \alpha\left(4\alpha^4 + 2\alpha^2\theta^2 + \theta^4 x^2 + 2\alpha\theta^3 x\right)}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} O_{DD}(x;\theta,\alpha) &= \frac{F_{DD}(x;\theta,\alpha)}{1 - F_{DD}(x;\theta,\alpha)} \\ &= \frac{2\alpha^2\left(2\alpha^2+\theta^2\right)e^{\frac{\theta x}{\alpha}}}{4\alpha^4 + 2\alpha^2\theta^2 + \theta^4 x^2 + 2\alpha\theta^3 x} - 1. \end{aligned} \tag{25}$$

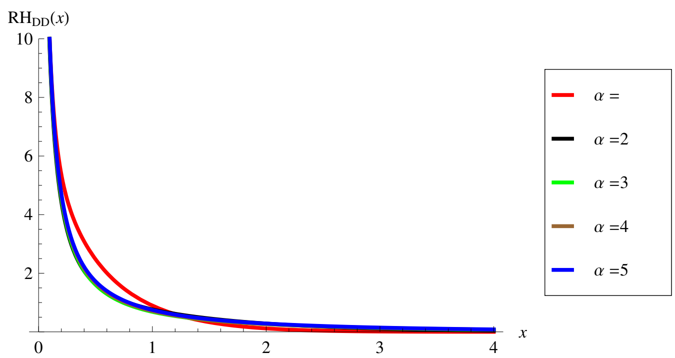

Figure 5: The reversed hazard rate functions of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

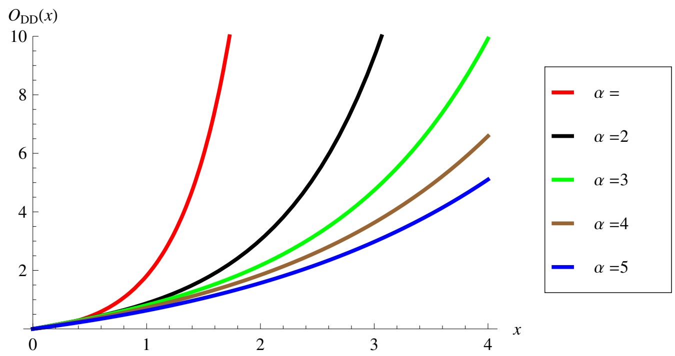

Figure 6: The odds functions of DD for $\alpha = 1, 2, 3, 4, 5$ and $\theta = 3$

9 An Application of Real Data

In this section, to justify the suitability of the Darna distribution in a practical application we used a real data set from the excesses of flood peaks (in m^3/s) Wheaton River near Car cross in the Yukon Territory, Canada. 72 exceedances of the years 1958 to 1984 are recorded, rounded to one decimal place. These data is considered by Choulakian and Stephens (2001), and Bodhisuwan et al. (2016) are

1.7, 2.2, 14.4, 1.1, 0.4, 20.6, 5.3, 0.7, 1.9, 13.0, 12.0, 9.3, 1.4, 18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 2.5, 14.4, 1.7, 37.6, 0.6, 2.2, 39.0, 0.3, 15.0, 11.0, 7.3, 22.9, 1.7, 0.1, 1.1, 0.6, 9.0, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.0

The Darna distribution is fitted to the real data and compared with the following distributions

1. Size biased Ishita distribution, $SBID(\alpha)$: $f(x;\alpha) = \frac{\alpha^4}{\alpha^3+6} x \left(\alpha + x^2\right) e^{-\alpha x}; x > 0, \alpha > 0.$

2. Janardan distribution, $JD(\alpha,\theta)$: $f(x;\alpha,\theta) = \frac{\theta^2}{\alpha(\theta+\alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha} x}; x > 0, \theta > 0, \alpha > 0.$

3. Ishita distribution, $ID(\tau)$: $f(x;\tau) = \frac{\tau^3}{\tau^3+2} \left(\tau + x^2\right) e^{-\tau x}$; $x > 0$, $\tau > 0$.

4. Sushila distribution, $SD(\delta,\eta)$: $f(x;\eta,\delta) = \frac{\delta^2}{\eta(\delta+1)} \left(1 + \frac{x}{\eta}\right) e^{-\frac{\delta}{\eta} x}; x > 0, \delta > 0, \eta > 0.$

The distributions parameters are estimated based on maximum likelihood method, and the negative maximized log-likelihood values (MLL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion

Table 4: The MLEs of the unknown parameter with the corresponding standard errors and the confidence intervals for the 72 exceedances data.

Model	MLE	Std. Dev.	0.95%CI
$ID(\tau)$	$\hat{\tau} = 0.250504$	0.016805	(0.217566, 0.283442)
$SBID(\alpha)$	$\hat{\alpha} = 0.330328$	0.019469	(0.292170, 0.368486)
$JD(\alpha, \theta)$	$\hat{\alpha} = 0.031261$	0.005674	(0.020139, 0.042382)
	$\hat{\theta} = 0.003172$	0.000565	(0.002064, 0.004279)
$SD(\delta, \eta)$	$\hat{\eta} = 40.63191$	46.152651	(-49.825630, 131.0894)
	$\hat{\delta} = 3.988327$	4.008477	(-3.868143, 11.84480)
$DD(x; \theta, \alpha)$	$\hat{\alpha} = 0.897484$	5.404177	(-9.694508, 11.489475)
	$\hat{\theta} = 0.074041$	0.445767	(-0.799646,0.947728)

Table 5: The AIC, CAIC, BIC, HQIC, -2LL, KS, P-value for 72 exceedances data

Model	$ID(\tau)$	$SBID(\alpha)$	$SD(\delta, \eta)$	$JD(\alpha, \theta)$	$DD(x; \theta, \alpha)$
AIC	603.7872	698.8290	508.5015	508.6630	508.2554
$CAIC$	603.8444	698.8862	508.6755	508.8369	508.4293
BIC	606.0639	701.1057	513.0549	513.2163	512.8087
$HQIC$	604.6936	699.7354	510.3142	510.4757	510.0681
$-2LL$	300.8936	348.4145	252.2508	252.3315	252.1277
$K.S$	0.159313	0.340367	0.148772	0.150300	0.141658
$P-Value$	0.051733	1.14e-07	0.082572	0.077313	0.111173

(BIC), Hannan-Quinn Information Criterion (HQIC), and Kolmogorov-Smirnov (KS) test statistics where these measures are defined as

- AIC $= -2MLL + 2\kappa$,
- CAIC $= -2MLL + \frac{2\kappa n}{n-\kappa-1}$,
- HQIC $= 2Log\{Log(n)[\kappa - 2MLL]\}$,
- BIC $= -2MLL + \kappa\, Log(n)$,
- $K.S = Sup_n |F_n(x) - F(x)|$, $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$,

where κ is the number of parameters and n is the sample size. $F_n(x)$ is empirical distribution function and $F(x)$ is cumulative distribution. The results are summarized in Tables (4) and (5).

It is clear that the Darna distribution have the smallest values of the criteria AIC, CAIC, BIC, and HQIC. Also, the Darna distribution have the smallest value of the KS among the other distributions with largest P-value of 0.111173. Hence, we can say that the Darna distribution represents a good fit to

10 Conclusions

A new continuous two parameters lifetime distribution is suggested in this paper and called as Darna distribution. The main statistical properties of the distribution are provided. The distributions of order statistics from the Darna distribution are presented. The reliability, hazard, reversed hazard and odds functions are given. Also, the maximum likelihood estimation of the distribution parameters are presented. The Fisher's information, and Rényi entropy are proved for the Darna distribution. The stochastic ordering, stress-strength, mean and median deviations about the mean and the median of the Darna distribution are established. A real data set of 72 exceedances is considered for illustration. Also, for future research the authors will estimate the parameters of the newly developed distributions using ranked set sampling. See for example Al-Omari and Haq (2019a), Zamanzade and Mahdizadeh (2017), Haq et al. (2016), Haq et al. (2015) and Al-Omari and Zamanzade (2019) proposed new ranked set sapling for estimating the population mean.

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