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Bivariate modified Weibull distribution derived from Farlie-Gumbel-Morgenstern copula: a simulation study

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In recent years, the use of copulas has grown rapidly, especially in survival analysis. In this paper, we introduce a bivariate modified Weibull distribution derived from the Farlie–Gumbel–Morgenstern (FGM), a copula function commonly used to model very weak linear dependences. Considering the presence of non censored data and censored data, an extensive simulation study was developed to check the performance of the maximum likelihood method in estimating the parameters of the proposed model. Maximum likelihood and Bayesian approaches for the estimation of the model parameters are presented. In the Bayesian analysis, the posterior distributions of the parameters are estimated using Markov chain Monte Carlo (MCMC) methodology. An example, considering a real data set, is introduced to illustrate the proposed methodology.

keywords: Bayesian estimates, bivariate data, copula function, simulation study, survival analysis.

1 Introduction

In the lifetime data analysis, researchers commonly use standard non-parametric techniques, as for example, the Kaplan–Meier estimators for the survival function, the log-rank test or semi-parametrical Cox proportional hazard models (Kleinbaum and Klein,

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2012). However, in some situations more complex models are needed. A very common example in the survival analysis is given when for each patient we observe two lifetimes, that is, the time to the events can be modeled by a bivariate distribution function. In the statistical literature, many authors introduced different solutions to modeling bivariate time-to-event data. As examples, we can consider Hougaard (1987), Liang et al. (1995), Parner (2001) and Hougaard (2012).

Commonly, a bivariate survival dataset presents dependence between the times to events and the study of this dependence structure has been the goal of many researchers. For the study of this dependence, a popular technique is the use of frailty models proposed by Vaupel et al. (1979). In these models one or more random effects are included to model the dependence between the observations. In this case the marginal times are conditionally independent given the frailty variable. An other frequently used approach involves bivariate parametric distributions that introduce specific parameters to capture the dependence between the lifetimes, for example: Gumbel bivariate exponential (Gumbel, 1960), Marshall-Olkin bivariate exponential (Marshall and Olkin, 1967), Block and Basu bivariate exponential (Block and Basu, 1974) and Basu-Dhar Bivariate geometric distributions (Basu and Dhar, 1995; De Oliveira and Achcar, 2018).

An alternative method to model dependence between the survival times is the use of copula functions, described for example by Nelsen (1999), Balakrishnan and Lai (2009), Jaworski et al. (2010) and Joe (2014). Copulas are basically functions that "join "or couple univariate distributions creating multivariate distributions. The copula functions allow us to define different distributions for the margins, with a dependence structure of the copula, creating a multivariate distribution with the selected margins. In this way copulas are multivariate distributions modeling the dependence structure between variables, irrespective of their marginal distributions. Copula functions have been applied in many fields, including: hydrology and climate (Favre et al., 2004; Zhang and Singh, 2006), management science (Abbas, 2006), finance and economics (Roch and Alegre, 2006; Patton, 2006; Rivieccio, 2015) and medical science (Viswanathan and Manatunga, 2001; Achcar et al., 2016). It is important to note that different copula functions introduce different structures of dependence among the variables.

The focus of this article is related to bivariate survival data in different kinds of situations including censored data where two lifetimes are observed for the same patient. For example, the interest could be in studying the lifetimes of paired human organs such as kidneys and eyes or the times between a first and a second hospitalization for a particular disease, among others. We consider the modified Weibull distribution, since this distribution is more flexible in relation to the risk function when compared, for example, with the standard exponential and Weibull distributions (Rinne, 2008; Bhattacharjee and Misra, 2016). We also present a simulation study of the bivariate modified Weibull distribution derived from a Farlie-Gumbel-Morgenstern copula, in the presence of different sample sizes, percentage of censored data and different correlations between the times, with a brief comparison among classic and Bayesian estimates. Applications to real data are also presented, considering both the frequentist and Bayesian approaches.

2 Methods

2.1 Modified Weibull Distribution (MW)

Let T be a random variable representing the time to some event of interest. The probability density function of the modified Weibull distribution (MW) with three parameters, introduced by Lai et al. (2003), is given by

$$f(t) = \alpha t^{\beta-1}(\beta + \lambda t)\exp(\lambda t - \alpha t^{\beta} e^{\lambda t}), \tag{1}$$

where $t \geq 0$, $\alpha > 0$, $\beta > 0$ and $\lambda > 0$. The corresponding survival function is given by

$$S(t) = 1 - P(T < t) = \exp(-\alpha t^{\beta} e^{\lambda t}), \tag{2}$$

and the hazard function takes the following form:

$$h(t) = \frac{f(t)}{S(t)} = \alpha t^{\beta-1}(\beta + \lambda t)\exp(\lambda t). \tag{3}$$

The MW distribution contains as special sub-models three well-known distributions:

(1) For $\lambda = 0$, the expression (1) is the probability density function of a two-parameter Weibull distribution, with parameters α and β.

(2) When $\lambda = 0$ and $\beta = 1$, the expression (1) is reduced to the probability density function of an exponential distribution with only one parameter α.

(3) If $\lambda = 0$ and $\beta = 2$, the expression (1) is reduced to the probability density function of a Rayleigh distribution with parameter α.

An important characteristic of the MW distribution is the flexibility of its hazard function. As observed by Lai et al. (2003), if $\beta > 1$ in the expression (3) we have an increasing form for the hazard function. However, if $0 < \beta < 1$ we observed a bathtub shape for expression (3). Figure 1 illustrates the probability density function, the survival function and the hazard function of the MW distribution, considering different values for the parameters α, β and λ.

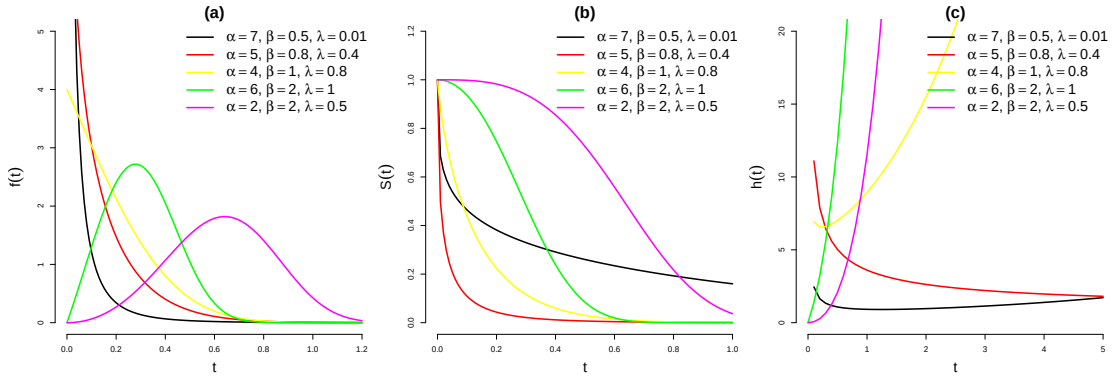

Figure 1: The probability density function (a), survival function (b) and hazard function (c) of the modified Weibull distributions for some values of α, β and λ.

2.2 Farlie-Gumbel-Morgenstern Copula (FGM)

Copula functions are used to represent the joint distribution function of two marginal univariate distributions. If $S(t_k)$ is the univariate survival function for T_k, $k = 1, 2$, the joint survival function $S(t_1, t_2)$ is defined by a copula function given by

$$S(t_1, t_2) = C_\phi(S(t_1), S(t_2)), \tag{4}$$

for $t_1 > 0$ and $t_2 > 0$, where ϕ is a measure of the dependence between T_1 and T_2.

Farlie–Gumbel–Morgenstern copula was originally proposed by Morgenstern (1956) and further studied by Gumbel (1960) and Farlie (1960). The joint survival function considering the FGM copula for T_1 and T_2 is given by

$$S(t_1, t_2) = S(t_1)S(t_2)\left\{1 + \phi[1 - S(t_1)][1 - S(t_2)]\right\}, \tag{5}$$

where $-1 \leq \phi \leq 1$. When $\phi = 0$ the joint survival function (5) takes the form $S(t_1, t_2) = S(t_1)S(t_2)$, that is, in this case T_1 and T_2 are independent. The parameter ϕ is related to the Kendall rank correlation coefficient by the expression

$$\tau(\phi) = \frac{2\phi}{9}. \tag{6}$$

We observe that $-2/9 \leq \tau(\phi) \leq 2/9$, or to say, the FGM copula is only appropriate to model weak dependences. In other situations, more appropriate copula functions are given by Gumbel copula (Gumbel, 1960), Clayton copula (Clayton, 1978) and Ali-Mikhail-Haq copula (Ali et al., 1978).

2.3 Bivariate Modified Weibull Distribution Derived From FGM (BMW)

Considering the bivariate lifetime distributions with a dependence structure given by FGM copula functions, we assume as a special model, the modified Weibull distribution. In this way, the marginal MW distributions for the lifetimes T_1 and T_2 have density functions given by

$$f_1(t_1) = \alpha_1 t_1^{\beta_1 - 1}(\beta_1 + \lambda_1 t)\exp(\lambda_1 t_1 - \alpha_1 t_1^{\beta_1} e^{\lambda_1 t_1}) \tag{7}$$

and

$$f_2(t_2) = \alpha_2 t_2^{\beta_2 - 1}(\beta_2 + \lambda_2 t)\exp(\lambda_2 t_2 - \alpha_2 t_2^{\beta_2} e^{\lambda_2 t_2}). \tag{8}$$

The survival functions are given by

$$S_1(t_1) = \exp(-\alpha_1 t_1^{\beta_1} e^{\lambda_1 t_1}) \tag{9}$$

and

$$S_2(t_2) = \exp(-\alpha_2 t_2^{\beta_2} e^{\lambda_2 t_2}), \tag{10}$$

with distribution functions given, respectively, by $F_1(t_1) = 1 - S_1(t_1)$ and $F_2(t_2) = 1 - S_2(t_2)$.

From the expressions (9) and (10), the joint survival function based on the FGM copula expressed in (5) is given by

$$\begin{aligned} S(t_1,t_2) &= \exp(-\alpha_1 t_1^{\beta_1} e^{\lambda_1 t_1} - \alpha_2 t_2^{\beta_2} e^{\lambda_2 t_2}) \\ &\times \left\{1+\phi[1-\exp(-\alpha_1 t_1^{\beta_1} e^{\lambda_1 t_1})][1-\exp(-\alpha_2 t_2^{\beta_2} e^{\lambda_2 t_2})]\right\}, \end{aligned} \tag{11}$$

The joint probability density function for T_1 and T_2 is given by the second derivate of $S(t_1,t_2)$ with respect to t_1 and t_2, that is,

$$f(t_1,t_2) = \frac{\partial^2 S(t_1,t_2)}{\partial t_1 \partial t_2}. \tag{12}$$

2.4 Maximum Likelihood Estimation

Let us consider that either T_1 and T_2 can be censored and that censoring is independent of the time to the events of interest in the study. Assuming a random sample of size n, each i^{th} observation $(i = 1, ..., n)$ can be classified into one of four groups following:

(1) $C1 : t_{1i}$ and t_{2i} are complete observations, they are uncensored lifetimes;

(2) $C2 : t_{1i}$ is a complete observation and t_{2i} is a censored lifetime;

(3) $C3 : t_{2i}$ is a complete observation and t_{1i} is a censored lifetime;

(4) $C4 : t_{1i}$ and t_{2i} are censored lifetimes.

Thus, the likelihood function is given by

$$L = \prod_{i\in C_1}\left[\frac{\partial^2 S(t_{1i},t_{2i})}{\partial t_{1i}\partial t_{2i}}\right]\prod_{i\in C_2}\left[-\frac{\partial S(t_{1i},t_{2i})}{\partial t_{1i}}\right]\prod_{i\in C_3}\left[-\frac{\partial S(t_{1i},t_{2i})}{\partial t_{2i}}\right]\prod_{i\in C_4}[S(t_{1i},t_{2i})] \tag{13}$$

Let us consider two indicator variables, denoted by δ_{1i} and δ_{2i}, where $\delta_{ji} = 1$ when t_{ji} is an observed lifetime and $\delta_{ji} = 0$ when t_{ji} a censored observation, $j = 1, 2$ and $i = 1, ..., n$. In this way, we can rewrite the likelihood function as

$$\begin{aligned} L &= \prod_{i=1}^{n}\left[\frac{\partial^2 S(t_{1i},t_{2i})}{\partial t_{1i}\partial t_{2i}}\right]^{\delta_{1i}\delta_{2i}}\left[-\frac{\partial S(t_{1i},t_{2i})}{\partial t_{1i}}\right]^{\delta_{1i}(1-\delta_{2i})} \\ &\times \left[-\frac{\partial S(t_{1i},t_{2i})}{\partial t_{2i}}\right]^{\delta_{2i}(1-\delta_{1i})}[S(t_{1i},t_{2i})]^{(1-\delta_{1i})(1-\delta_{2i})}. \end{aligned} \tag{14}$$

We observe that if there is no censored observations, the expression above is reduced to the form,

$$L = \prod_{i=1}^{n}\frac{\partial^2 S(t_{1i},t_{2i})}{\partial t_{1i}\partial t_{2i}} = \prod_{i=1}^{n} f(t_1,t_2), \tag{15}$$

where $S(t_{1i}, t_{2i})$ is given by equation (5) considering the FGM copula.

The first partial derivatives of $S(t_{1i}, t_{2i})$ with respect to t_{1i} and t_{2i} are obtained from the relations,

$$-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{1i}} = f_1(t_1)S_2(t_2)\left[1 + \phi(1 - S_2(t_2))(1 - 2S_1(t_1))\right] \tag{16}$$

and

$$-\frac{\partial S(t_{1i}, t_{2i})}{\partial t_{2i}} = f_2(t_2)S_1(t_1)\left[1 + \phi(1 - S_1(t_1))(1 - 2S_2(t_2))\right]. \tag{17}$$

In addition, we have

$$\frac{\partial^2 S(t_{1i}, t_{2i})}{\partial t_{1i}\partial t_{2i}} = f_1(t_1)f_2(t_2)\left[1 + \phi(1 - 2S_1(t_1))(1 - 2S_2(t_2))\right], \tag{18}$$

the joint probability density function obtained from a FGM copula function. Replacing (7), (8) , (9) and (10) in the above expression, we obtain the joint probability density function of the bivariate modified Weibull distribution based on a FGM copula.

2.5 Bayesian Analysis

Assuming the proposed model, let $\boldsymbol{\theta} = (\alpha_1, \beta_1, \lambda_1, \alpha_2, \beta_2, \lambda_2, \phi)$ be the vector of unknown parameters. Under a Bayesian framework, the joint posterior distribution for the model parameters is obtained by combining the joint prior distribution of the parameters and the likelihood function given by equation (14). To simulate samples from the joint posterior distribution, we could consider the use of MCMC (Markov Chain Monte Carlo) algorithms implemented in the OpenBUGS software, where we just need to specify the data distribution and the prior distribution for the parameters.

In this proposed model under a Bayesian approach, we assume independent gamma prior distributions for the parameters α_1, β_1, λ_1, α_2, β_2 and λ_2, since these parameters are positive. That is, we assume $\alpha_1 \sim Gamma(a_1, b_1)$, $\alpha_2 \sim Gamma(a_2, b_2)$, $\beta_1 \sim Gamma(a_3, b_3)$, $\beta_2 \sim Gamma(a_4, b_4)$, $\lambda_1 \sim Gamma(a_5, b_5)$ and $\lambda_2 \sim Gamma(a_6, b_6)$, where a_k and b_k,$k = 1, ..., 6$, are known hyperparameters, and $Gamma(a, b)$ denotes a gamma distribution with mean a/b and variance a/b^2. We also assume that the dependence parameter ϕ follows a uniform prior distribution $(1 - \phi)/2 \sim Beta(c, d)$. This choice assures that $\phi \in (-1, 1)$.

2.6 A simulation method to generate a random variable with a BMW distribution

Adapting an algorithm suggested by Balakrishnan and Lai (2009), we simulate a sample of size n from the bivariate modified Weibull distributions based on the FGM copula with right-censored data following the steps:

Step 1. Fix values for the parameters: α_1, β_1, λ_1, α_2, β_2, λ_2 and ϕ.

Step 2. Generate n random samples from $u_{i1} \sim U(0,1)$.

Step 3. Get values from T_{1i} considering $F_1(t_{1i}) = 1 - S_1(t_{1i}) = u_{i1}$ and isolate t_{1i}, from which, we get

$$t'_{1i} = \exp\left\{-\frac{1}{\beta_1}\left[\beta_1 W\left(\frac{\lambda_1}{\beta_1} e^{\frac{1}{\beta_1}\ln\left(-\frac{\ln(1-u_{i1})}{\alpha_1}\right)}\right) - \ln\left(-\frac{\ln(1-u_{i1})}{\alpha_1}\right)\right]\right\}, \quad (19)$$

where $W(\cdot)$ is the Lambert W function (Corless et al., 1996).

Step 4. Generate n values for c_{1i} from an exponential distribution with parameter θ_1, where c_{1i} denotes the censored time to event data and θ_1 controls the proportion of censored observations.

Step 5. Calculate $t_{1i} = \min(t'_{1i}, c_{1i})$.

Step 6. Pairs of values (t_{1i}, δ_{1i}) are thus obtained, where $\delta_{1i} = 1$ if $t_{1i} < c_{1i}$ and $\delta_{1i} = 0$ if $t_{1i} > c_{1i}$.

Step 7. Generate n random quantities from $u_{i2} \sim U(0,1)$.

Step 8. Get values from w_i, considering the expression given by,

$$w_i = \frac{u_{i2} - \phi\, {u_{i1}}^2 + \phi\, u_{i1} - u_{i1}}{2\,\phi\, u_{i1}(1-u_{i1})}. \quad (20)$$

This expression is the derivative of (5) with respect to u_{i1}, when $S(t_{1i}) = u_{i1}$ and $S(t_{2i}) = w_i$. When this derivative is equalized to u_{i2}, we find w_i from the obtained mathematical expression. Thus, $w_i \sim U(0,1)$.

Step 9. Get values of T_{2i} considering $F_2(t_2) = 1 - S_2(t_2) = w_i$ and isolate t_{2i}, from which it is obtained,

$$t'_{2i} = \exp\left\{-\frac{1}{\beta_2}\left[\beta_2 W\left(\frac{\lambda_2}{\beta_2} e^{\frac{1}{\beta_2}\ln\left(-\frac{\ln(1-w_i)}{\alpha_2}\right)}\right) - \ln\left(-\frac{\ln(1-w_i)}{\alpha_2}\right)\right]\right\}. \quad (21)$$

Step 10. Generate n values for c_{2i} from an exponential distribution with parameter θ_2, where c_{2i} denotes the censored time to event data and θ_2 controls the proportion of censored observations.

Step 11. Calculate $t_{2i} = \min(t'_{2i}, c_{2i})$.

Step 12. Pairs of values (t_{2i}, δ_{2i}) are obtained, where $\delta_{2i} = 1$ if $t_{2i} < c_{2i}$ and $\delta_{2i} = 0$ if $t_{2i} > c_{2i}$.

The *R* function code based on these steps is shown in an Appendix at the end of this paper.

2.7 Model Comparison Criteria

In the literature, there are many approaches to analyze the adequacy of a probability distribution to be fitted by a dataset and in the selection of the best fit among different specifications of the model. In this article, we consider for comparison between frequentist models the Akaike information criterion (AIC) introduced by Akaike (1974). This criterion is based on the log-likelihood value calculated on the estimates of the model parameters. The expression for the AIC is given by

$$\text{AIC} = \ell(\hat{\theta}) + 2n, \tag{22}$$

where $\ell(\hat{\theta})$ is the log-likelihood function, evaluated in the maximum likelihood estimates of the parameters and n is the number of parameters in the model.

Under a Bayesian approach, we consider for comparison between Bayesian models the deviation information criterion (DIC) proposed by Spiegelhalter et al. (2002). The DIC value is given by

$$\text{DIC} = D(\hat{\theta}) + 2n_p = 2\bar{D} - D(\hat{\theta}), \tag{23}$$

where $D(\hat{\theta})$ is the deviance evaluated in the posterior mean of the parameter of interest obtained using MCMC simulation methods and n_p is the effective number of parameters in the model, with $n_p = \bar{D} - D(\hat{\theta})$, where $\bar{D} = E[D(\theta)]$ is the posterior mean of the deviance. Lower values of AIC and DIC indicate better model fit.

3 Results

3.1 A simulation study

In order to examine the performance of the maximum likelihood estimation method, it was considered a simulation study to observe the coverage probability of the Wald confidence interval for the parameters α_1, β_1, λ_1, α_2, β_2, λ_2 and ϕ, with their corresponding bias and mean squared errors (MSE). The coverage probability is the observed percentage of times that the confidence interval includes the respective parameter. The bias and MSE in the estimation of a parameter θ are estimated, respectively, by,

$$\widehat{\text{Bias}}(\widehat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}^{(i)} - \theta\right) \quad \text{and} \quad \widehat{\text{MSE}}(\widehat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}^{(i)} - \theta\right)^2, \tag{24}$$

where $\widehat{\boldsymbol{\theta}} = (\widehat{\alpha_1}, \widehat{\beta_1}, \widehat{\lambda_1}, \widehat{\alpha_2}, \widehat{\beta_2}, \widehat{\lambda_2}, \widehat{\phi})$ is the vector of maximum likelihood estimates and N is the number of simulated samples of size n.

Considering that the nominal parameter values may or may not be contained in their respective confidence intervals, we define the observed coverage probability as the number of times that the nominal value falls within the confidence interval which is modeled by a binomial distribution $Binomial(N, p)$, where N is number of simulated samples and p is the nominal coverage probability. In this simulation study we have used $N = 1000$ and $p = 0.95$, and we reject the equality between the nominal coverage probability and the

observed coverage probability assuming a significance level of 5%, that is, if the observed coverage probability is outside the range interval $(0.9365, 0.9635)$.

We generated random samples each of size $n = 30, 40, ..., 500$, from a BWM distributions with arbitrary parameters values of $\alpha_1 = 0.4,\ \alpha_2 = 0.6,\ \beta_1 = 0.6,\ \beta_2 = 0.5,\ \lambda_1 = 0.2,$ and $\lambda_2 = 0.1$ and five different values of $\phi = (-0.8, -0.3, 0, 0.3, 0.8)$. For each value of ϕ we considered different possibilities for the percentage of censored data, given by $0\%, 30\%, 50\%, 70\%$. We considered these parameter values since the samples are generated with $t < 10$ and the hazard function is bathtub shaped. We used 0.95 nominal confidence coefficients for the intervals and computed the maximum likelihood estimates and corresponding standard errors for each simulated sample using the *maxLik* package in *R* (Henningsen and Toomet, 2011), applying Nelder-Mead maximization method.

In Figure 2 it is shown in the plots of the coverage probability, the biases and the MSE of ϕ versus the sample sizes for the simulated data from the BWM distribution, for different proportions of censored data. We can observe from Figure 2 that the coverage probability generally is near the nominal value, except for the simulated samples considering $\phi = 0.8$ with 70% censored data and the biases generally approach zero with increased samples size, except again $\phi = 0.8$ with 70% censored data and the respective MSE for ϕ decreases to zero with increased samples sizes. In Figure 3 and 4 the plots show the coverage probability, the biases and the MSE for $\alpha_1,\ \beta_1,\ \lambda_1,\ \alpha_2,\ \beta_2$ and λ_2 versus sample size for the simulated data from BWM distributions, for different proportions of censored data and assuming, respectively, $\phi = -0.8$ and $\phi = 0.8$. In these plots, we can observe that the coverage probability generally is properly near the nominal value, for all parameters in this scenario.

From this study, we observed highest bias and MSE in the samples with 70% censored data. We observe that the bias and MSE for the parameters generally are enough small $(\widehat{\text{Bias}}(\widehat{\theta}) < 0.01)$ when the sample size is higher than 100 with 50% of censored data and are enough small when the sample size is higher than 200, considering 70% of censored data..

During the simulation process, it was noted the presence of simulated samples resulting in monotone likelihood functions (error informed by *maxLik*), mainly when we considered low samples sizes and high proportion of censored data or values for ϕ near to -1 and 1. For example, when $n = 30$ and $\phi = 0.8$ considering 70% of censored data, we found 74 (7.4%) simulated samples of size $n = 30$ resulting in monotone likelihood functions. In the case of sample sizes greater than $n = 250$ or not considering the presence of censored data, we did not find situations of monotone likelihood functions.

3.2 A simulation study data

In this section to exemplify the application of the proposed bivariate model and to compare the obtained inference results under the frequentist and Bayesian methods, samples from the BWM distribution were simulated for different sizes and fixed values for the parameters, with samples in presence of 70% of censored observations. We simulated samples of size $n = 50, 100, 200$ and 300 from the BMW distribution, with model parameters arbitrarily fixed at $\alpha_1 = 0.4,\ \alpha_2 = 0.6,\ \beta_1 = 0.6,\ \beta_2 = 0.5,\ \lambda_1 =$

0.2, and $\lambda_2 = 0.1$, and the parameter ϕ was fixed at -0.8 and 0.8.

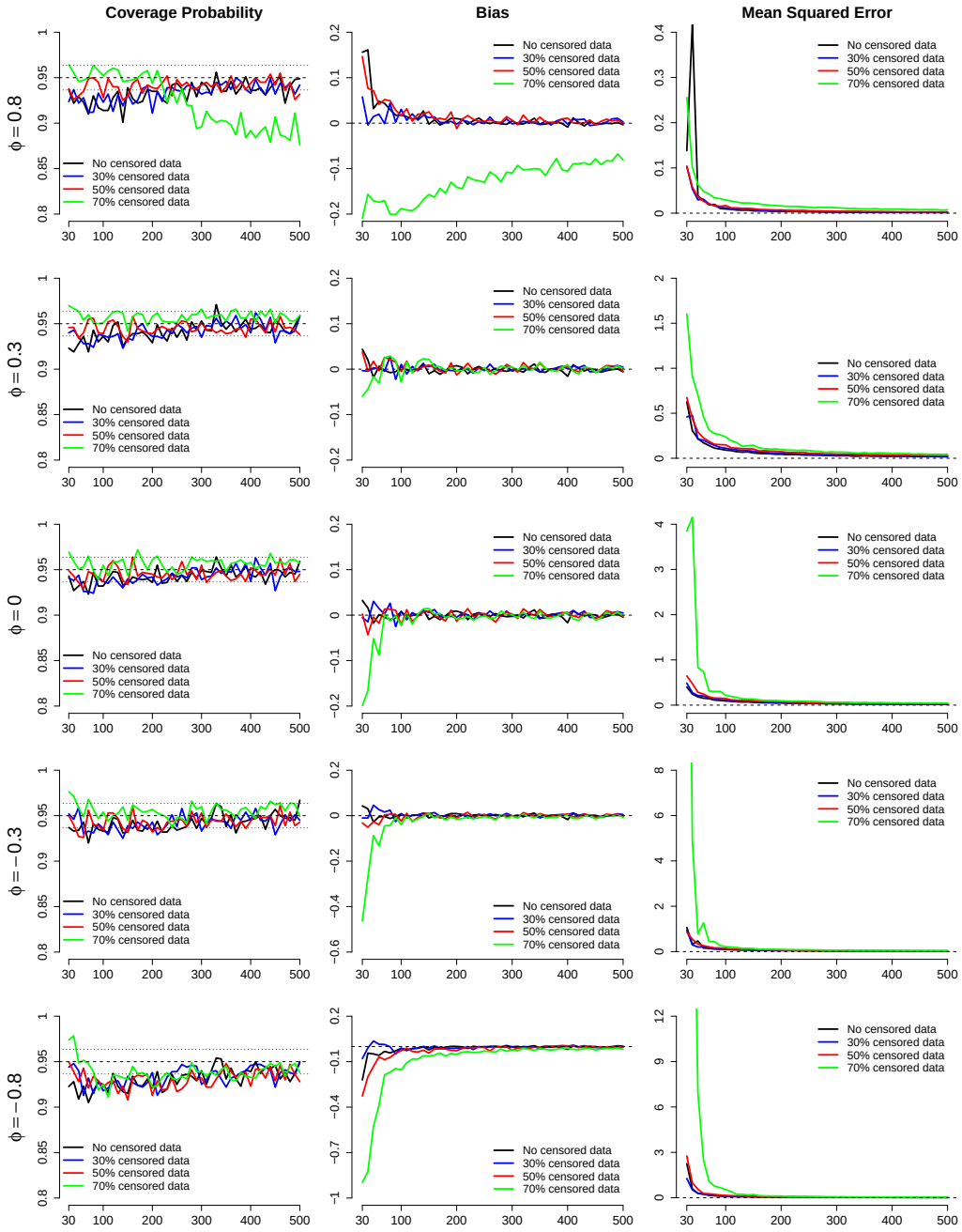

Figure 2: Plots of the coverage probability, biases and MSE of ϕ's versus sample sizes for simulated data with different proportions of censored data, considering the maximum likelihood estimation method.

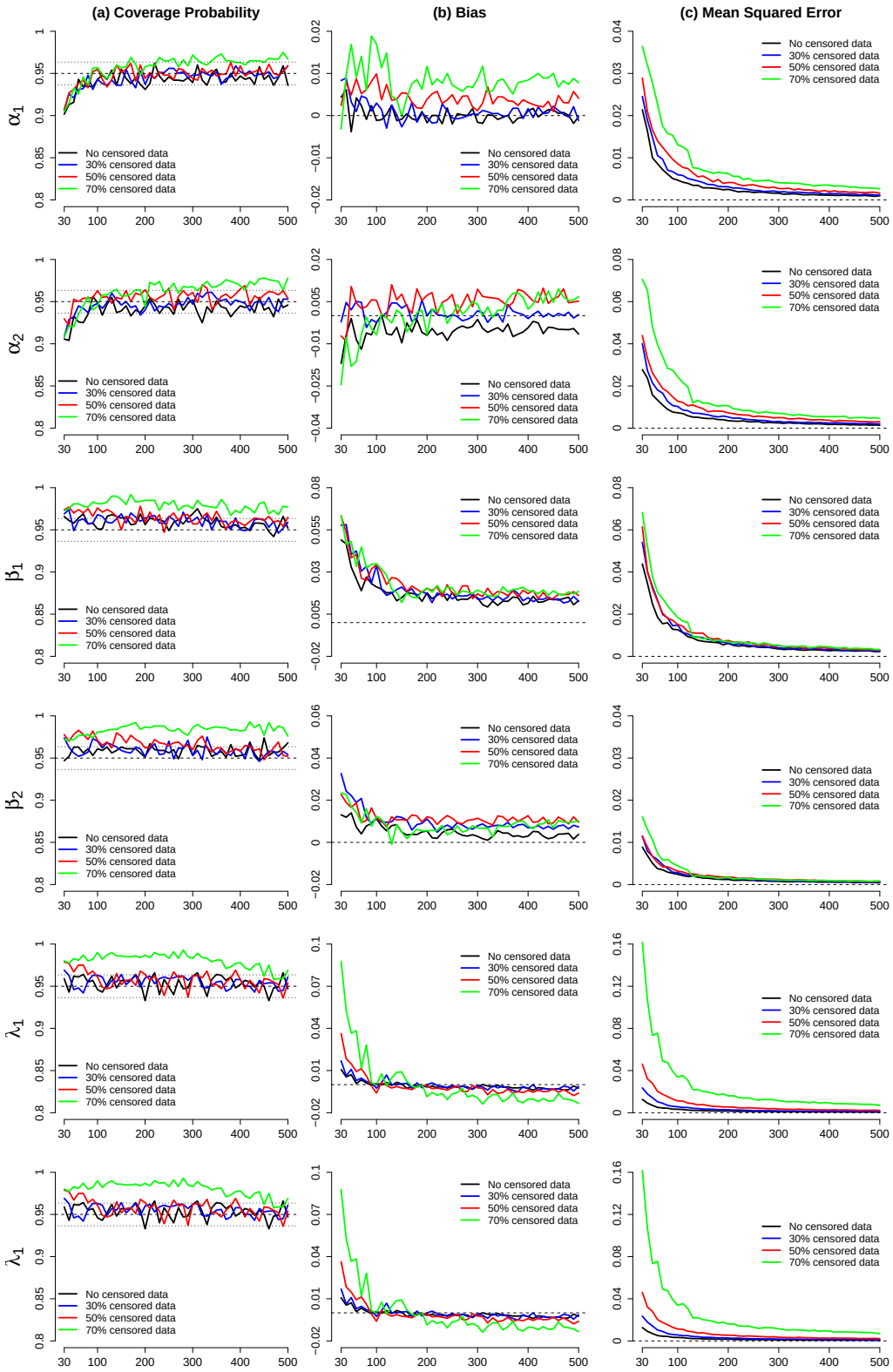

Figure 3: Plots of the coverage probability, biases and MSE for $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\beta_1 = 0.6$, $\beta_2 = 0.5$, $\lambda_1 = 0.2$, and $\lambda_2 = 0.1$ versus sample sizes for simulated data with different proportions of censored data from BWM distributions with $\phi = -0.8$, considering the maximum likelihood estimation method.

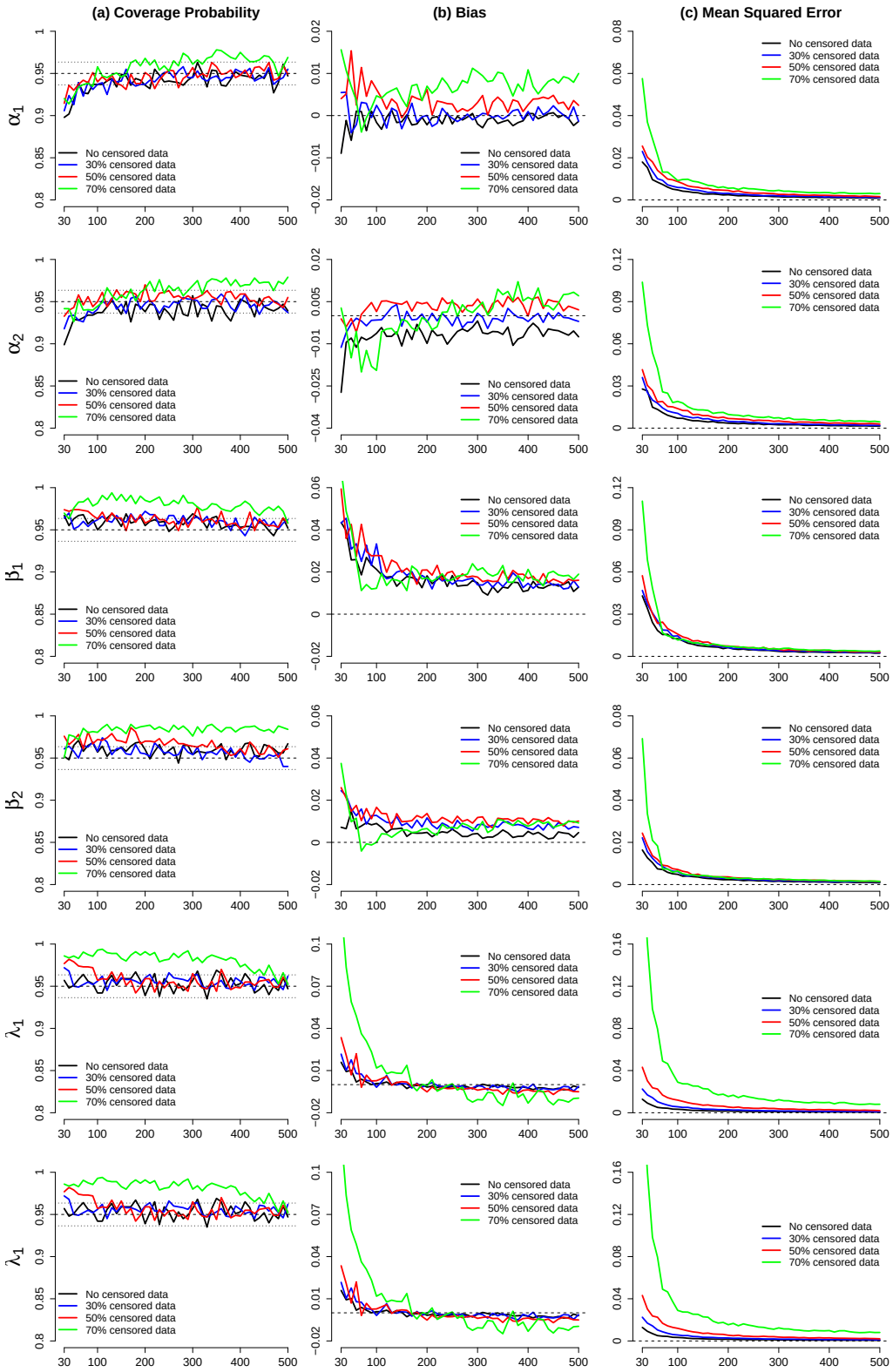

Figure 4: Plots of the coverage probability, biases and MSE for $\alpha_1 = 0.4$, $\alpha_2 = 0.6$, $\beta_1 = 0.6$, $\beta_2 = 0.5$, $\lambda_1 = 0.2$, and $\lambda_2 = 0.1$ versus sample sizes for simulated data with different proportions of censored data from BWM distributions with $\phi = 0.8$, considering the maximum likelihood estimation method.

Table 1 shows the maximum likelihood and Bayesian estimates for all parameters of the model with $\phi = 0.8$. The use of posterior medians instead of posterior means are considered due to the skewness of the distributions. We can note that MLE and Bayesian estimates are satisfactory close to the nominal values for each parameter. From these results, we observed that the Wald-type 95% confidence intervals can include negative values, extrapolating the set of possible values for the correspondent parameters, especially for λ_1, λ_2 and ϕ when using relatively small sample sizes. On the other hand, the bounds of the credible intervals are directly related to the posterior distributions of the parameters, avoiding values that are incompatible with the correspondent parametric space. Figure 5 shows the non-parametric survival functions estimated by the Kaplan-Meier method for each simulated data set and the corresponding parametric curves obtained from the fit of the model based on the BWM distribution with estimated parameters presented in Table 1.

Table 2 shows the maximum likelihood and Bayesian estimates for all parameters of the model with $\phi = -0.8$. In this case, we also can note that the MLE and Bayesian estimates are satisfatory close to the fixed values of the parameters, with exception to the dependence parameter ϕ. These results are also observed in Table 1. Figure 6 shows the estimated survival functions obtained by the Kaplan-Meier method for each simulated data set and the corresponding parametric curves obtained from the fit of the model based on the BWM distribution with parameter estimates presented in Table 2.

In a brief further statistical analysis, we also assumed a reparametrization for the parameters λ_1 and λ_2, considering $\gamma_k = \exp(\lambda_k)$ and also $\eta_k = \frac{1}{\lambda_k}$, k=1,2. However, no significant changes were observed with the obtained inference results when compared to the inference results presented in Table 1 and Table 2.

3.3 An application to a real data set

In this application, we consider the analysis of the dataset presented in the diabetic retinopathy study data by Group et al. (1976). This study consists of follow up times for 197 diabetic patients under 60 years of age. The main purpose of the study is to assess the efficacy of photocoagulation treatment for proliferative retinopathy. Each patient had one eye randomized to laser treatment and the other eye received no treatment. It was considered that T_1 is the time up to visual loss for the treatment eye, while T_2 is the time up to visual loss for the control eye. The censored observation was caused by death, abandonment or termination of the study, being that 73% of treated eyes and 43% of not treated eyes were censored.

Table 1: Maximum likelihood and Bayesian estimates for the model based on the BWM distribution simulated data with $\phi = 0.8$ and 70% censored data.

n		Nominal Values	Maximum Likelihood Estimate			Bayesian Estimate	
			Estimate	Std. Error	95%CI	Median	95%CrI
50	α_1	0.4	0.4260	0.2030	(0.0281,0.8239)	0.4434	(0.1951,0.7932)
	α_2	0.6	0.5498	0.1906	(0.1762,0.9234)	0.5265	(0.2835,0.8965)
	β_1	0.6	0.7564	0.2692	(0.2287,1.2840)	0.7678	(0.3709,1.2030)
	β_2	0.5	0.3198	0.0887	(0.1459,0.4937)	0.3360	(0.1952,0.5162)
	λ_1	0.2	0.3417	0.3348	(-0.3145,0.9979)	0.3052	(0.0165,0.861)
	λ_2	0.1	0.1714	0.2827	(-0.3826,0.7255)	0.2164	(0.0097,0.6699)
	ϕ	0.8	0.7400	0.4891	(-0.2186,1.6986)	0.4844	(-0.4649,0.9667)
			AIC = 72.1			DIC =68.6	
100	α_1	0.4	0.3401	0.1211	(0.1027, 0.5775)	0.3756	(0.1945,0.6516)
	α_2	0.6	0.5494	0.1718	(0.2126,0.8861)	0.5333	(0.3078,0.8337)
	β_1	0.6	0.6023	0.1596	(0.2895,0.9151)	0.6367	(0.3693,0.9541)
	β_2	0.5	0.4218	0.0930	(0.2395,0.6041)	0.4184	(0.2674,0.5829)
	λ_1	0.2	0.5491	0.2839	(-0.0073,1.1055)	0.4538	(0.0480, 0.9743)
	λ_2	0.1	0.2793	0.3505	(-0.4076,0.9662)	0.3041	(0.0175, 0.9107)
	ϕ	0.8	0.6009	0.4941	(-0.3675,1.5693)	0.4672	(-0.3340, 0.9674)
			AIC =303.00			DIC=300.1	
200	α_1	0.4	0.4143	0.0824	(0.2528,0.5759)	0.3906	(0.2724, 0.5276)
	α_2	0.6	0.5490	0.1304	(0.2934,0.8046)	0.5141	(0.3402, 0.7168)
	β_1	0.6	0.6347	0.0955	(0.4474,0.8220)	0.6120	(0.4558, 0.7793)
	β_2	0.5	0.52821	0.0837	(0.3640,0.6923)	0.5106	(0.3759, 0.6528)
	λ_1	0.2	0.0721	0.1138	(-0.1509,0.2952)	0.1012	(0.0053, 0.3005)
	λ_2	0.1	0.1089	0.2253	(-0.3327,0.5507)	0.1414	(0.0092, 0.5736)
	ϕ	0.8	0.8329	0.3855	(0.0772,1.5884)	0.6612	(0.0459, 0.9821)
			AIC= 271.0			DIC=278.4	
300	α_1	0.4	0.3605	0.0702	(0.2229,0.4981)	0.3601	(0.2452, 0.5073)
	α_2	0.6	0.5305	0.1061	(0.3225,0.7385)	0.5399	(0.3686, 0.7502)
	β_1	0.6	0.6153	0.0892	(0.4404,0.7901)	0.6165	(0.4599, 0.7945)
	β_2	0.5	0.5005	0.0668	(0.3696,0.6314)	0.5077	(0.3888, 0.6367)
	λ_1	0.2	0.2586	0.1365	(-0.0089,0.5261)	0.2521	(0.0342, 0.5099)
	λ_2	0.1	0.3878	0.2341	(-0.0710,0.8466)	0.3623	(0.0388, 0.8138)
	ϕ	0.8	0.9874	0.2999	(0.3995,1.5752)	0.7795	(0.3114, 0.9895)
			AIC=418.8			DIC=418.2	

Std. error: standard error; 95%CI: 95% confidence interval; 95%CrI: 95% credible interval; AIC: Akaike information criterion; DIC: Deviance information criterion.

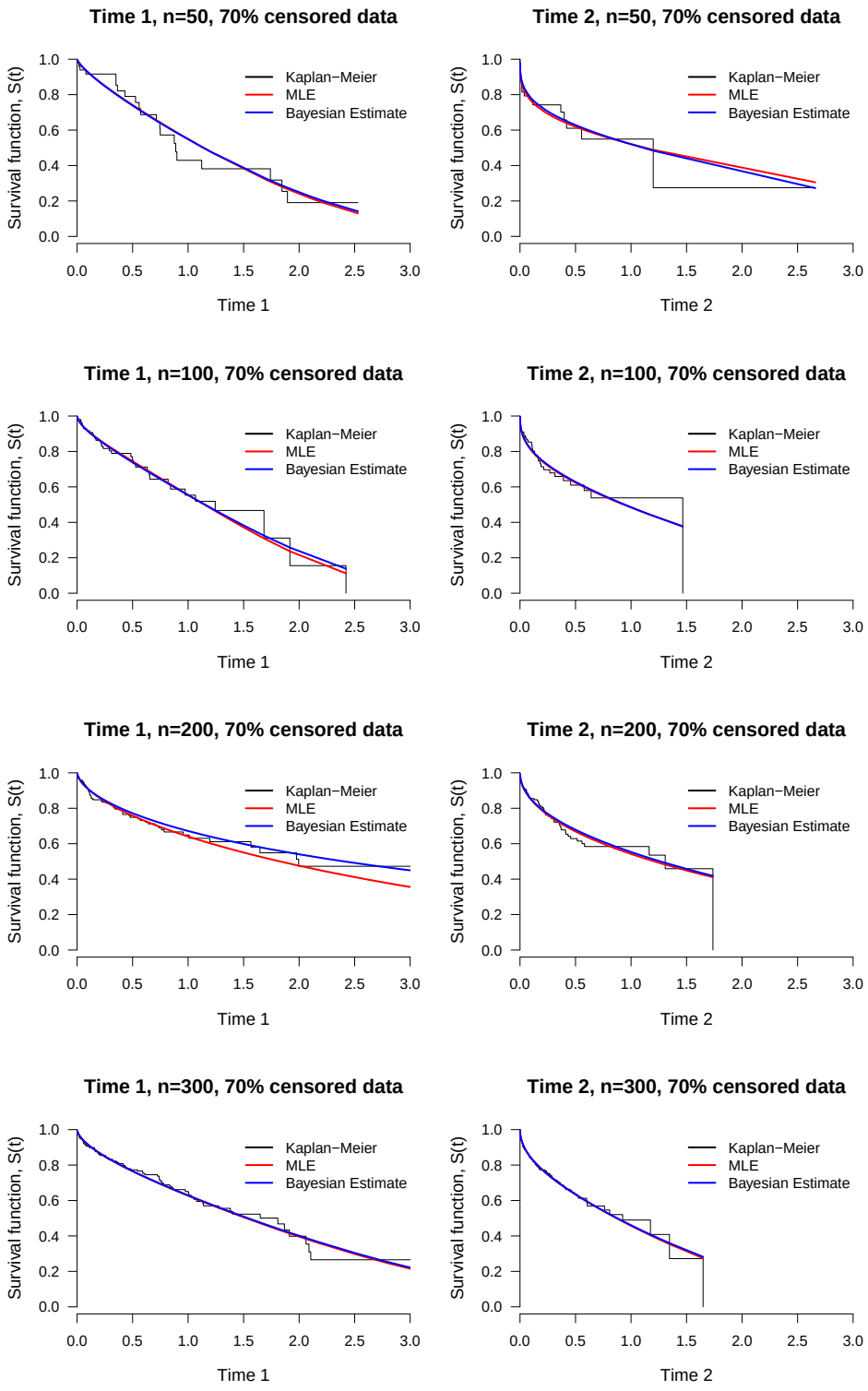

Figure 5: Plots of survival function estimated by the Kaplan-Meier and assuming the model based on the BMW distributions with $\phi = 0.8$ under maximum likelihood estimates and Bayesian estimates.

Table 2: Maximum likelihood and Bayesian estimates for the model based on the BWM distribution simulated data with $\phi = -0.8$ and 70% censored data.

n		Nominal Values	Maximum Likelihood Estimate			Bayesian Estimate	
			Estimate	Std. Error	95%CI	Median	95%CrI
50	α_1	0.4	0.3928	0.1662	(0.0671, 0.7184)	0.3813	(0.1823,0.6728)
	α_2	0.6	0.7849	0.3429	(0.1127, 1.4571)	0.6256	(0.3117,1.0790)
	β_1	0.6	0.4853	0.1607	(0.1704, 0.8002)	0.4779	(0.2456,0.7543)
	β_2	0.5	0.5639	0.1537	(0.2628, 0.8652)	0.5094	(0.2972,0.7641)
	λ_1	0.2	0.2385	0.2803	(-0.3110, 0.7880)	0.2233	(0.0125,0.6706)
	λ_2	0.1	0.0468	0.4119	(-0.7605, 0.8540)	0.2411	(0.0102,0.8862)
	ϕ	-0.8	-0.6616	0.5763	(-1.7912, 0.4681)	-0.4844	(-0.9628,0.7008)
				AIC = 73.3			DIC =69.8
100	α_1	0.4	0.5589	0.1389	(0.2866, 0.8312)	0.5020	(0.3217,0.7328)
	α_2	0.6	0.3974	0.1223	(0.1577, 0.6370)	0.4071	(0.2349,0.6503)
	β_1	0.6	0.5887	0.1086	(0.3757, 0.8016)	0.5499	(0.3802,0.7457)
	β_2	0.5	0.3564	0.0831	(0.1936, 0.5192)	0.3663	(0.2292,0.5318)
	λ_1	0.2	0.0241	0.1495	(-0.2690, 0.3172)	0.1072	(0.0052, 0.3657)
	λ_2	0;1	0.4013	0.3413	(-0.2677, 1.0703)	0.3446	(0.0222, 0.9504)
	ϕ	-0.8	-0.7589	0.3487	(-1.4423, -0.0756)	-0.5904	(-0.9728, 0.1677)
				AIC =112.1			DIC=109.4
200	α_1	0.4	0.3997	0.0868	(0.2295,0.5699)	0.4036	(0.2680, 0.5704)
	α_2	0.6	0.5024	0.1249	(0.2577,0.7472)	0.4800	(0.3106, 0.6938)
	β_1	0.6	0.5839	0.0961	(0.3955,0.7723)	0.5869	(0.4187, 0.7628)
	β_2	0.5	0.5549	0.0908	(0.3768,0.7329)	0.5455	(0.3976, 0.7030)
	λ_1	0.2	0.2089	0.1459	(-0.0771, 0.4948)	0.1992	(0.0163, 0.4661)
	λ_2	0.1	0.1778	0.2432	(-0.2990, 0.6546)	0.2091	(0.0111, 0.6288)
	ϕ	-0.8	-0.8337	0.2753	(-1.3735, -0.2940)	-0.7043	(-0.9833, -0.1306)
				AIC= 292.3			DIC=289.9
300	α_1	0.4	0.3542	0.0641	(0.2284,0.4800)	0.3657	(0.2587, 0.5103)
	α_2	0.6	0.4709	0.0869	(0.3004,0.6414)	0.4834	(0.3395, 0.6708)
	β_1	0.6	0.5163	0.0743	(0.3707,0.6619)	0.5274	(0.3936, 0.6799)
	β_2	0.5	0.4573	0.0606	(0.3386,0.5761)	0.4662	(0.3578, 0.5863)
	λ_1	0.2	0.3368	0.1277	(0.0864,0.5873)	0.3119	(0.0803, 0.5538)
	λ_2	0.1	0.4055	0.1887	(0.0357,0.7754)	0.3658	(0.0618, 0.7204)
	ϕ	-0.8	-0.7760	0.1952	(-1.1586 -0.3934)	-0.7208	(-0.97284, -0.2572)
				AIC=370.8			DIC=369.8

Std. error: standard error; 95%CI: 95% confidence interval; 95%CrI: 95% credible interval; AIC: Akaike information criterion; DIC: Deviance information criterion.

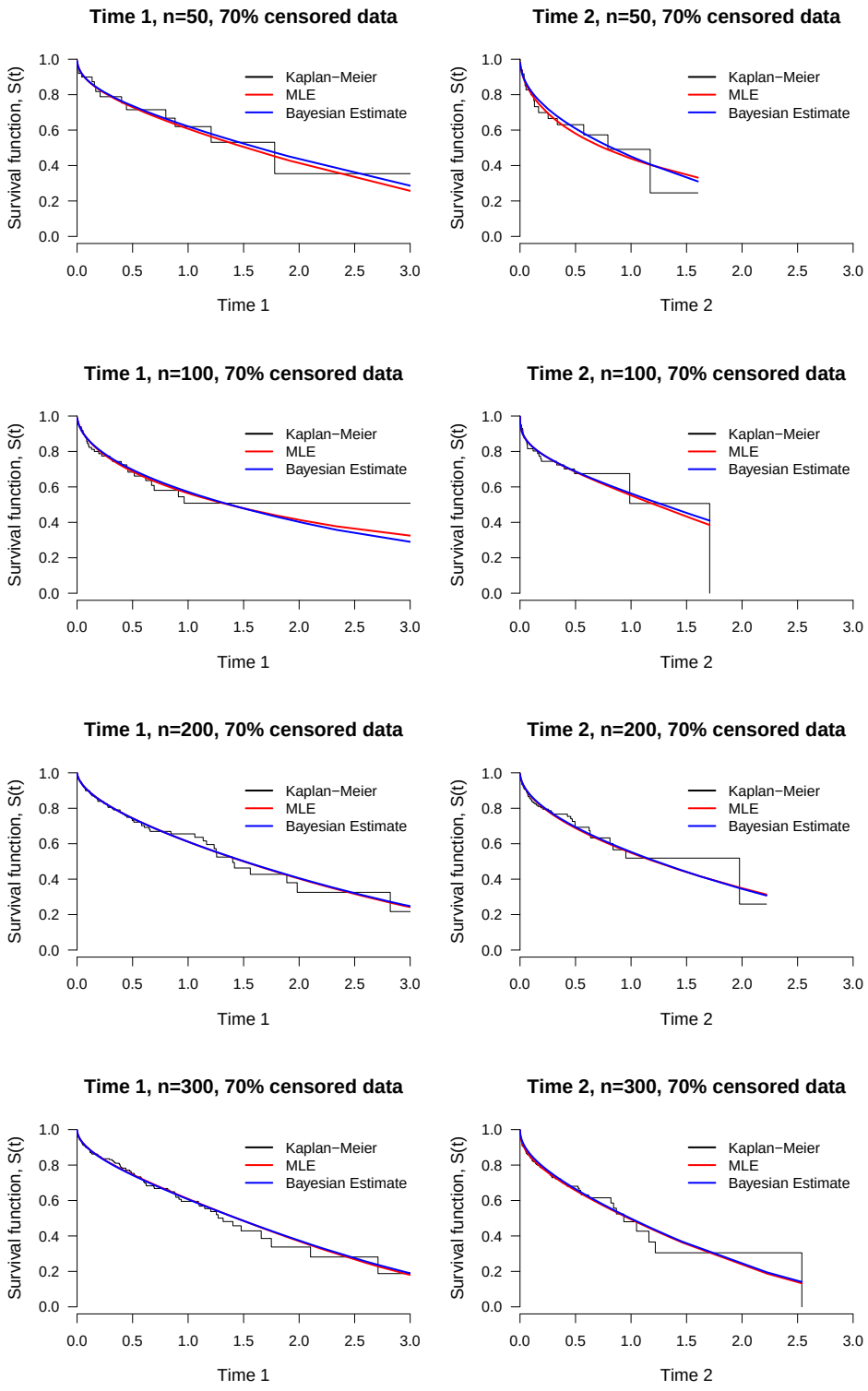

Figure 6: Plots of survival function estimated by the Kaplan-Meier and assuming the model based on the BMW distribution with $\phi = -0.8$ under maximum likelihood estimates and Bayesian estimates.

The procedure to fit the BMW distribution is similar to that presented in Section 3.2. That is, the likelihood maximization was performed by the *maxLik* package and Bayesian estimates were based on the simulated posterior samples recorded every 20th iteration from 220,000 Gibbs samples after a "burn-in" of 20,000 samples. We assumed the following independent prior distributions: $\alpha_k \sim Gamma(1,1)$, $\beta_k \sim Gamma(1,1)$, $\lambda_k \sim Gamma(1,1)$ and $(1-\phi)/2 \sim Beta(1,1)$, k=1,2. The convergence of the MCMC samples was checked by visual examination of traceplots of the simulated samples.

Table 3 shows the maximum likelihood and Bayesian estimates for the parameters of the models based on the BWM distributions and their special sub-models considering the retinopathy data. We observe that the Bayesian estimates are quite close to the maximum likelihood estimates. From these results, we observed that the Wald-type 95% confidence intervals can include negative values to λ_1 and λ_2 in the BWM distribution, including the value 0, suggesting that the model based on the bivariate standard Weibull distribution could be fitted by the data. The AIC and DIC values obtained from fitting the different models based on the BWM distributions and bivariate standard Weibull distribution are practically the same, suggesting that both models are suitable to be fitted by the dataset the dataset. The maximum likelihood estimators for ϕ are slightly lower than the obtained estimators using the Bayesian approach. Similar inference results for the parameter ϕ also were obtained by Louzada et al. (2013) and Martinez and Achcar (2014).

Figure 7 compares the survival curves S(t) estimated from the Kaplan-Meier method and from the fitted models based on the BWM distributions and their special sub-models, considering the frequentist and Bayesian approaches. Clearly, it is observed from these plots that the predicted values obtained from the model based on the BWM distribution are those closest to the empirical values. We also observe that the maximum likelihood and Bayesian methods produce similar survival curves.

4 Conclusion

Based on a modified Weibull distribution it was proposed in this study a new bivariate lifetime distribution constructed using the Farlie-Gumbel-Morgenstern copula function in presence of right-censored data. Under this new model,it was developed an extensive simulation study showing the performance of the obtained inference results under classical maximum likelihood and Bayeasian approaches. The simulation study showed that the maximum likelihood and Bayesian method are suitable approaches to estimate the parameters of the BWM distribution. However, in the situations where there is a high proportion of censored data (>70%) and small sample sizes we do not recommend the use of this distribution. We also observed that the estimates are more easily obtained if the lifetimes are smaller values than 10. The applications with simulated and real data showed that BWM distribution can be satisfactorily fitted by the data in almost all cases, under maximum likelihood and Bayesian approaches. Finally, it is important to point out that the computational algorithms for the proposed model can be easily implemented using *R* or OpenBUGS free softwares.

Table 3: Maximum likelihood and Bayesian estimates for the models based on the BWM distribution and yours special sub-models from retinopathy data.

Model		Maximum Likelihood Estimate			Bayesian Estimate	
		Estimate	Std. Error	95%CI	Median	95%CrI
BWM	$\alpha 1$	0.2362	0.0385	(0.1607, 0.3117)	0.2286	(0.1729,0.2960)
	$\alpha 2$	0.1133	0.0259	(0.0625, 0,1641)	0.1092	(0.0735,0.1543)
	$\beta 1$	0.8161	0.1216	(0.5777, 1,0544)	0.7803	(0.6248,0.9362)
	$\beta 2$	0.7658	0.1623	(0.4477, 1,0839)	0.7448	(0.5497,0.9536)
	$\lambda 1$	0.0021	0.0057	(-0.0091, 0.0133)	0.0190	(0.0007,0.0838)
	$\lambda 2$	0.0002	0.0078	(-0.0151, 0.0155)	0.0189	(0.0007,0.0914)
	ϕ	0.4348	0.2860	(-0,1257, 0.9954)	0.6217	(0.4143,0.9924)
		AIC = 907.5			DIC =901.5	
BW	$\alpha 1$	0.2379	0.0311	(0.1769, 0.2988)	0.2392	(0.1830, 0.3073)
	$\alpha 2$	0.1118	0.0208	(0.0710, 0.1526)	0.1148	(0.0787, 0.1611)
	$\beta 1$	0.8250	0.0725	(0.6829, 0,9671)	0.8187	(0.6842,0.9649)
	$\beta 2$	0.7980	0.0995	(0.6029, 0.9930)	0.7827	(0.6066,0.9831)
	ϕ	0.6352	0.280	(0.0864, 1.184)	0.8229	(0.3913, 0.9923)
		AIC =900.6			DIC=898.0	
BE	$\alpha 1$	0.1917	0.0189	(0.1546,0.2287)	0.1924	(0.1580, 0.2315)
	$\alpha 2$	0.0846	0.0114	(0.0622,0.1069)	0.0858	(0.0653, 0.1101)
	ϕ	0.6654	0.2521	(0.1712, 1.1595)	0.8045	(0.3996, 0.9901)
		AIC= 904.8			DIC=902.9	
BRay	$\alpha 1$	0.0493	0.0048	(0.0398,0.0587)	0.0492	(0.0403, 0.0595)
	$\alpha 2$	0.0203	0.0027	(0.0150,0.0256)	0.0200	(0.0152, 0.0257)
	ϕ	0.5469	0.1924	(0.1687, 0.9240)	0.6940	(0.3331, 0.9638)
		AIC=1119.4			DIC=1119.0	

BWM: Bivariate modified Weibull; BW: bivariate standard Weibull; BE: bivariate exponential. RRay: bivariate Rayleigh; Std. error: standard error; 95%CI: 95% confidence interval; 95%CrI: 95% credible interval; AIC: Akaike information criterion; DIC: Deviance information criterion.

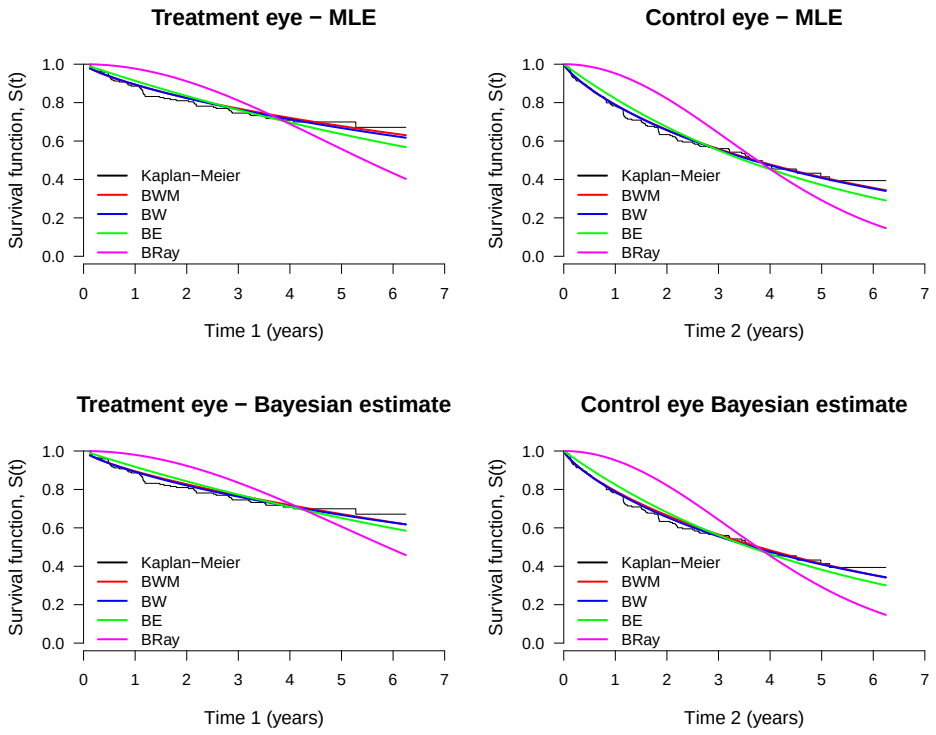

Figure 7: Plots of survival function estimated by the Kaplan-Meier and assuming the model based on the BMW with $\phi = -0.8$ under maximum likelihood estimates and Bayesian estimates.

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Appendices

The following R function bellow can be used to generate m random samples of size n from a BWM with parameters α_1, β_1, λ_1, α_2, β_2 λ_2 and ϕ.

```
time1 <-matrix(NA,nrow = n,ncol=m)
cens1 <-matrix(NA,nrow = n,ncol=m)

unif1 <-matrix(NA,nrow = n,ncol=m)

for(i in 1:m){
u1        <- runif(n,0,1)
unif1[,i] <-u1
t01       <-c()
for(j in 1:n)
    t01[j]    <-rwm(u1[j],alpha1,beta1,lambda1)
c01       <- rexp(n,rate = theta)
time1[,i] <- pmin(t01,c01)
cens1[,i] <- as.numeric(c01>=t01)
}

time2   <-matrix(NA,nrow = n,ncol=m)
cens2   <-matrix(NA,nrow = n,ncol=m)

for(i in 1:m){
c02       <- rexp(n,theta)
u2        <- runif(n,0.001,0.999)
t02<-c()
for(j in 1:n){
    w      <- wunif(u2[j],unif1[j,i],phi)
    t02[j] <- rwm(w,alpha2,beta2,lambda2)
}
time2[,i]  <- pmin(t02,c02)
cens2[,i]  <- as.numeric(c02>=t02)
}
```

Note that this R-code uses the package 'LambertW' introduced by Goerg (2016) and the functions *rwm* and *wunif* presented below.

```
library(LambertW)

rwm <-function(u,alpha,beta,lambda){
v1 <- 1 / beta;
```

```
v2 <- log(1 - u);
v3 <- log(-v2 / alpha);
v4 <- exp(v3 * v1);
v5 <- W(lambda * v1 * v4);
v6 <- exp((-beta * v5 + v3) * v1);
return(v6)
}
```

and

```
wunif<-function(u2,u1,phi){
a1 <- u2-phi*u1^2+phi*u1-u1
a2 <- 2*phi*u1*(1-u1)
a3 <- a1/a2
return(a3)
}
```

This is the OpenBUGS code for the Bayesian model based on the Bivariate modified Weibull distribution:

```
model
{
for (i in 1:N)
{
St1[i] <- exp(-alpha1*pow(t1[i],beta1)* exp(lambda1*t1[i]))
ft1[i] <- alpha1*pow(t1[i],(beta1-1))*(beta1+lambda1*t1[i])*
     exp(lambda1*t1[i]-alpha1*pow(t1[i],beta1)*
     exp(lambda1*t1[i]))

St2[i] <- exp(-alpha2*pow(t2[i],beta2) * exp(lambda2*t2[i]))
ft2[i] <- alpha2*pow(t2[i],(beta2-1))*(beta2+lambda2*t2[i])*
    exp(lambda2*t2[i]-alpha2*pow(t2[i],beta2)*
    exp(lambda2*t2[i]))

F1[i]  <- 1-St1[i]
F2[i]  <- 1-St2[i]

S1[i]  <- ft1[i]  *St2[i]  *(1+phi*(F2[i])*(1-2*St1[i]))
S2[i]  <- ft2[i] *St1[i]  *(1+phi*(F1[i])*(1-2*St2[i]))
S12[i] <- ft1[i]  *ft2[i]  *(1+phi*(1-2*St1[i])*(1-2*St2[i]))
S[i]   <- St1[i]  *St2[i]  *(1+phi*F1[i]*F2[i])

L[i] <- pow(S12[i],d1[i]*d2[i])*pow(S1[i],d1[i]*(1-d2[i]))*
     pow(S2[i],(1-d1[i])*d2[i])*pow(S[i],(1-d1[i])*(1-d2[i]))
```

```
logL[i] <- log(L[i])
zeros[i] <- 0
zeros[i] ~ dloglik(logL[i])
}

alpha1 ~dgamma(1,1)
beta1  ~dgamma(1,1)
lambda1~dgamma(1,1)
alpha2 ~dgamma(1,1)
beta2  ~dgamma(1,1)
lambda2~dgamma(1,1)
k ~ dbeta(1,1)
phi<-1-2*k
}
```

In this code: N is the sample size, St1[i] is the survival function given in equation (9), St2[i] is the survival function given in equation (10), ft1[i] is the densities function given in equation (7), ft2[i] is the densities function given in equation (8), S1[i] is a expression given in equation (15), S2[i] is a expression given in equation (16), S12[i] is a expression given in equation (17), S[i] is a joint survival function given in equation (5) and L[i] is the likelihood function given in equation (14).