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The Burr XII modified Weibull distribution: model, properties and applications

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A new distribution called Burr XII modified Weibull (BXIIMW or BMW) distribution is presented and its properties explored. This new distribution contains several new and well known sub-models, including Burr-Weibull, Burr-exponential, Burr-Rayleigh, Burr XII, Lomax modified Weibull, Lomax-Weibull, Lomax-exponential, Lomax-Rayleigh, Lomax, Weibull, Rayleigh, and exponential distributions. Some structural properties of the proposed distribution including the shapes of the hazard rate function, moments, conditional moments, moment generating function, skewness and kurtosis are presented. Mean deviations, Lorenz and Bonferroni curves, Rényi entropy and the distribution of the order statistics are given. Maximum likelihood estimation technique is used to estimate the model parameters and finally applications of the model to real datasets are presented to illustrate the usefulness of the proposed distribution.

Keywords: Weibull distribution, Structural properties, Failure-time, Maximum likelihood estimation.

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1 Introduction

Burr XII and Weibull distributions are two of the most important and most widely used distributions for lifetime as well as the wealth data analysis and modeling. Burr Type XII is a member of a system of continuous distributions introduced by Burr (1942). The flexibilities of Burr XII distribution were studied by Hatke (1949), Burr and Cislak (1968), Rodrigues (1977) and Tadikamalla (1980). Burr XII distribution has been used in many applications such as actuarial science, quantal bio-assay, economics, forestry, toxicology, life testing and reliability. According to Soliman (2005), this distribution covers shapes that is characteristic of a large number of distributions. The versatility and flexibility of the Burr XII distribution makes it quite attractive as a tentative model for data whose underlying distribution is unknown.

Weibull distribution by Weibull (1951) on the other hand is one of the most important and widely used lifetime distributions in reliability engineering and lifetime analysis. This is because of its flexible ability to fit a variety of data. However, it does not exhibit bathtub shape for its hazard rate function and as a result many researchers have proposed modifications and generalizations of the Weibull distribution to accommodate bathtub shaped hazard rates and non-monotone hazard rates in general. Among the various extension and modified forms of the Weibull distribution are the exponentiated Weibull family presented by Mudholkar and Srivastava (1993), generalized Weibull by Pham and Lai (2007), Gurvich et al. (1997), modified Weibull distribution by Lai et al. (2003) and the gamma generalized modified Weibull distribution given by Oluyede et al. (2015). The modified Weibull distribution of Lai et al. (2003) is one of the most important modification of the Weibull distribution. This distribution has increasing or bathtub-shaped failure rate functions.

The primary motivation for developing this distribution is the versatility and flexibility obtained from the product of the reliability or survival functions of the Burr XII and modified Weibull distribution via the use of competing risks to obtain a new model called the Burr XII modified Weibull distribution with desirable properties including hazard function that exhibits increasing, decreasing, bathtub and upside down bathtub shapes.

The results in this paper are organized in the following manner. The new model called Burr XII modified Weibull (BXIIMW or BMW) distribution, its sub-models, quantile function, hazard and reverse hazard functions are given in section 2. In section 3, moments, moment generating function and conditional moments are presented. Mean deviations, Lorenz and Bonferroni curves are given in section 4. Section 5 contain results on Rényi entropy, density of the order statistics and L-moments. Maximum likelihood estimates of the model parameters are given in section 6. A Monte Carlo simulation study to examine the bias and mean square error of the maximum likelihood estimates is presented in section 7. Section 8 contains applications of the new model to real data sets. A short concluding remark is given in section 9.

2 Burr XII Modified Weibull Distribution

In this section, the model, hazard and reverse hazard functions and quantile function are presented. First, the modified Weibull and Burr XII distributions are presented. The cumulative distribution function (cdf) of the modified Weibull (MW) distribution of Lai et al. (2003) is given by

$$F_{MW}(y;\alpha,\beta,\lambda) = 1-\exp(-\alpha y^{\beta}e^{\lambda y}), \quad y\geq 0,\ \alpha>0,\ \beta>0,\ \lambda\geq 0. \tag{1}$$

Note that α controls the scale of the distribution, β controls the shape, whereas λ can be considered to be an accelerating factor in the imperfection time and a factor of fragility in the survival of the individual as time increases. The cdf of Burr XII distribution is given by

$$G_{B}(y;c,k) = 1-(1+y^{c})^{-k}, \quad y\geq 0,\ k>0,\ c>0, \tag{2}$$

where k and c are shape parameters.

The reliability function of the new distribution called Burr XII Modified Weibull (BMW) distribution (obtained via the use of competing risk model) is constructed by combining the reliability functions for both the Burr XII and modified Weibull distributions. The resulting reliability or survival function is given by

$$\overline{F}_{BMW}(y;c,k,\alpha,\beta,\lambda) = (1+y^{c})^{-k}e^{-\alpha y^{\beta}e^{\lambda y}}, \quad c,k,\alpha,\beta>0 \text{ and } \lambda\geq 0. \tag{3}$$

The cdf of the new BMW distribution is given by

$$F_{BMW}(y;c,k,\alpha,\beta,\lambda) = 1-(1+y^{c})^{-k}e^{-\alpha y^{\beta}e^{\lambda y}}, \tag{4}$$

for $c,k,\alpha,\beta>0$ and $\lambda\geq 0$. The corresponding probability density function (pdf) is given by

$$f_{BMW}(y;c,k,\alpha,\beta,\lambda) = (1+y^{c})^{-k-1}e^{-\alpha y^{\beta}e^{\lambda y}}\left\{kcy^{c-1}+(1+y^{c})\alpha e^{\lambda y}y^{\beta-1}(\beta+\lambda y)\right\} \tag{5}$$

for $c,k,\alpha,\beta>0$ and $\lambda\geq 0$. Plots of the pdf for selected values of the model parameters are given in Figure 1. The plots suggests that the pdf can be right skewed or decreasing among many other shapes for the selected values of the model parameters.

2.1 Quantile Function

In this sub-section, the quantile function of the BMW distribution is presented. The quantile function can be obtained by inverting $\overline{F}_{BMW}(y) = 1-u$, $0\leq u\leq 1$, where

$$\overline{F}_{BMW}(y) \quad = \quad (1+y^{c})^{-k}e^{-\alpha y^{\beta}e^{\lambda y}}. \tag{6}$$

The quantile function is obtained by solving the non-linear equation

$$k\log(1+y^{c})+\alpha y^{\beta}e^{\lambda y}+\log(1-u)=0, \tag{7}$$

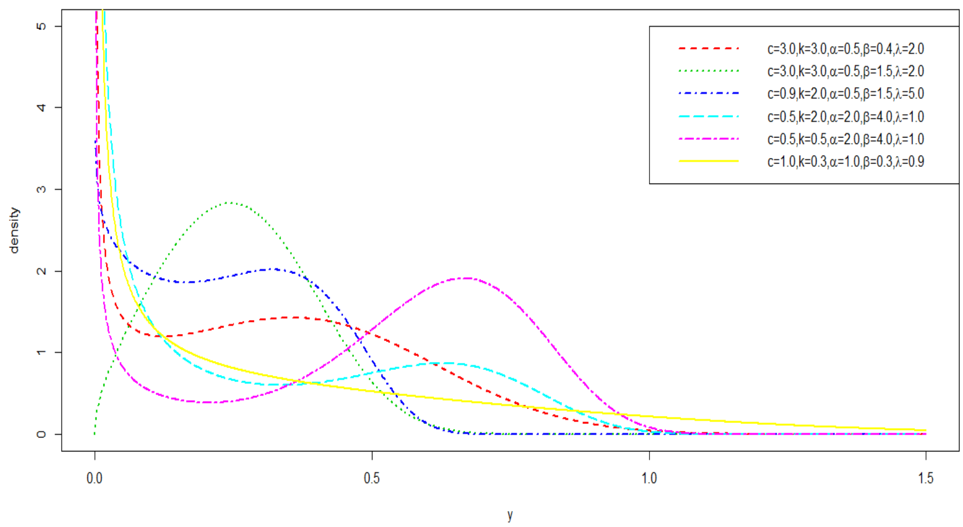

Figure 1: Plots of BMW PDF

using numerical methods. Consequently, random number can be generated based on equation (7). Table 1 lists the quantile for selected values of the parameters of the BMW distribution.

Table 1: BMW Quantile for Selected Values

	$(c, k, \alpha, \beta, \lambda)$					
u	(3.0,3.0,0.5,0.4,2.0)	(3.0,3.0,0.5,1.5,2.0)	(0.9,2.0,0.5,1.5,5.0)	(0.5,2.0,2.0,4.0,1.0)	(0.5,0.5,2.0,4.0,1.0)	(1.0,0.3,0.1,0.3,0.9)
0.1	0.01853977	0.2141278	0.03726958	0.002951125	0.05499965	0.1469918
0.2	0.08512607	0.2964359	0.08408788	0.01393109	0.2699589	0.4338763
0.3	0.1678449	0.3591675	0.1359876	0.03811902	0.4295427	0.7711961
0.4	0.245917	0.4141029	0.1896706	0.08463554	0.5171622	1.124044
0.5	0.3183341	0.4660199	0.2424022	0.1704162	0.5817035	1.470701
0.6	0.3885882	0.5181238	0.2931962	0.3138966	0.6370646	1.806173
0.7	0.4613197	0.5738216	0.3428492	0.4715588	0.6896507	2.138562
0.8	0.5436832	0.6385877	0.3944155	0.5976683	0.7450474	2.488656
0.9	0.654306	0.7270882	0.4556709	0.716097	0.813945	2.911513

2.2 Hazard and Reverse Hazard Functions

In this sub-section, the hazard and reverse hazard functions of the BMW distribution are presented. Graphs of the hazard function for selected values of the model parameters are also presented. The hazard and reverse hazard functions are given by

$$\begin{aligned} h_F(y) &= \frac{f_{BMW}(y)}{\overline{F}_{BMW}(y)} \\ &= \frac{(1+y^c)^{-k-1}e^{-\alpha y^\beta e^{\lambda y}}\left\{kcy^{c-1}+(1+y^c)\alpha e^{\lambda y}y^{\beta-1}(\beta+\lambda y)\right\}}{(1+y^c)^{-k}e^{-\alpha y^\beta e^{\lambda y}}} \\ &= kcy^{c-1}(1+y^c)^{-1}+\alpha e^{\lambda y}y^{\beta-1}(\beta+\lambda y), \end{aligned} \tag{8}$$

and

$$\begin{aligned} \tau_F(y) &= \frac{f_{BMW}(y)}{F_{BMW}(y)} \\ &= \frac{(1+y^c)^{-k-1}e^{-\alpha y^\beta e^{\lambda y}}\left\{kcy^{c-1}+(1+y^c)\alpha e^{\lambda y}y^{\beta-1}(\beta+\lambda y)\right\}}{1-(1+y^c)^{-k}e^{-\alpha y^\beta e^{\lambda y}}}, \end{aligned} \tag{9}$$

respectively.

The limiting behavior of the hazard function of the BMW distribution, which can be readily established is as follows:

- Note that,

$$\lim_{y\to 0} h_F(y) = \begin{cases} \infty & 0<c<1,\ \beta=1, \\ \alpha & c>1,\ \beta=1, \\ \alpha+k & c=1,\ \beta=1, \\ k & c=1,\ \beta>1, \\ 0 & c>1,\ \beta>1. \end{cases}$$

- For $\beta > 0$ and $c > 0$, $\lim_{y\to\infty} h_F(y) = \infty$.

Plots of the hazard function are given in Figure 2. The graphs exhibit decreasing, increasing, bathtub followed by upside down bathtub, and bathtub shapes for the selected values of the model parameters. This very attractive flexibility makes the BMW hazard function useful and suitable for non-monotonic empirical hazard behaviors which are more likely to be encountered in practice or real life situations.

2.3 Some Sub-models

There are several new as well as well known distributions that can be obtained from the BMW distribution. The sub-models include the following distributions:

- When $\alpha \to 0^+$, we obtain Burr XII or Burr (B) distribution.
- When $\lambda = 0$, we obtain Burr-Weibull (BW) distribution.
- If $\lambda = 0$ and $\beta = 1$, we obtain the Burr-Exponential (BE) distribution.
- If $\lambda = 0$ and $\beta = 2$, we have the Burr-Rayleigh (BR) distribution.

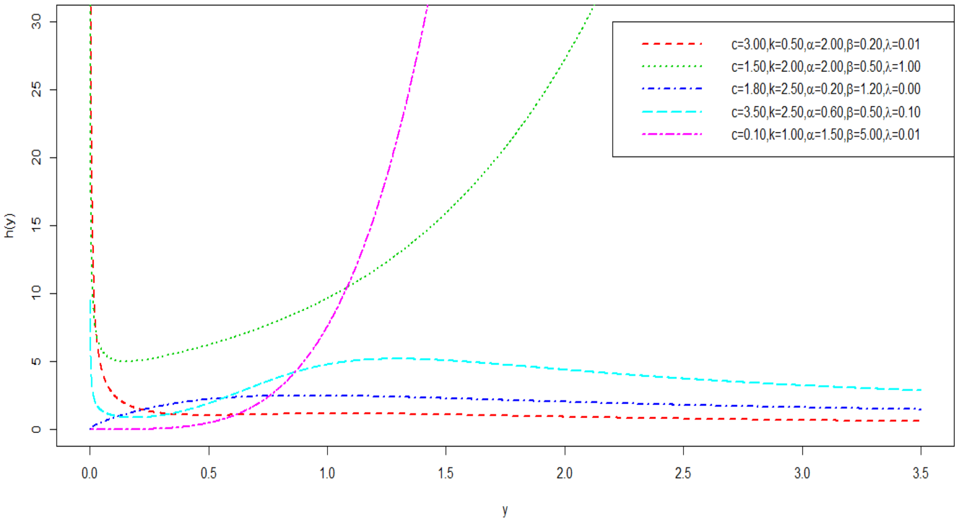

Figure 2: Plots of BMW Hazard Function

- If $c = 1$, we obtain the Lomax-modified Weibull (LMW) distribution.
- If $c = 1$, $\lambda = 0$ and $\beta = 2$, we obtain the Lomax-Rayleigh (LR) distribution.
- If $c = 1$, $\lambda = 0$ and $\beta = 1$, we obtain the Lomax-Exponential (LE) distribution.
- If $k \to 0^+$ and $\lambda = 0$, we obtain Weibull (W) distribution.
- If $k \to 0^+$, $\lambda = 0$ and $\beta = 2$, we obtain Rayleigh (R) distribution.
- If $k \to 0^+$, $\lambda = 0$ and $\beta = 1$, then we have the Exponential (E) distribution.
- If $k \to 0^+$, we obtain the modified Weibull (MW) distribution.
- If $\beta = 1$, then we obtain the Burr-modified Exponential (BME) distribution with cdf given by

  $$F(y) = 1 - (1 + y^c)^{-k} e^{-\alpha y e^{\lambda y}}, \tag{10}$$

  for c, α, $k > 0$, $\lambda \geq 0$ and $y \geq 0$.
- If $\beta = k = 1$ and $\lambda = 0$, then the BMW cdf reduces to the two parameter Log-logistic Exponential (LLoGE) distribution given by

  $$F(y) = 1 - (1 + y^c)^{-1} e^{-\alpha y}, \tag{11}$$

  for c, $\alpha > 0$, and $y \geq 0$.

- If $\lambda = 0, k = 1$ and $\beta = 2$, then the BMW cdf reduces to the two parameter Log-logistic Rayleigh (LLoGR) model with the cdf

$$F(y) = 1 - (1 + y^c)^{-1} e^{-\alpha y^2}, \tag{12}$$

for c, $\alpha > 0$ and $y \geq 0$.

- If $\lambda = 0$, and $k = c = \beta = 1$, then the BMW cdf reduces to the one parameter model with the cdf

$$F(y) = 1 - (1 + y)^{-1} e^{-\alpha y}, \tag{13}$$

for $\alpha > 0$ and $y \geq 0$.

3 Moments, Moment Generating Function and Conditional Moments

Moments are very important and necessary in any statistical analysis, especially in applications. Moments can be used to study the most important features and characteristics of a distribution (e.g. central tendency, dispersion, skewness and kurtosis). In this section, moments, moment generating function (mgf) and conditional moments are given for the BMW distribution. Measures of dispersion, skewness and kurtosis, as well as tables of the first six moments for selected values of the model parameters are also presented.

3.1 Moments

The r^{th} moment of the BMW distribution is given by

$$\begin{aligned}
E(Y^r) &= \int_0^\infty y^r f_{BMW}(y) dy \\
&= \int_0^\infty y^r (1 + y^c)^{-k-1} e^{-\alpha y^\beta e^{\lambda y}} \left\{ kcy^{c-1} + (1 + y^c)\alpha e^{\lambda y} y^{\beta-1}(\beta + \lambda y) \right\} dy \\
&= \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^m (m\lambda)^p kc}{m!p!} \int_0^\infty y^{r+c+m\beta+p-1} (1 + y^c)^{-k-1} dy \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} ((m+1)\lambda)^p \beta}{m!p!} \int_0^\infty y^{r+m\beta+\beta+p-1} (1 + y^c)^{-k} dy \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} (m+1)^p \lambda^{p+1}}{m!p!} \int_0^\infty y^{r+m\beta+\beta+p} (1 + y^c)^{-k} dy.
\end{aligned}$$

Let $x = (1 + y^c)^{-1}$, then

$$
\begin{aligned}
E(Y^r) &= \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^m (m\lambda)^p k}{m!p!} \int_0^1 x^{k+3-\frac{1}{c}(r+m\beta+p)} (1-x)^{\frac{1}{c}(r+m\beta+p)} dx \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} \beta((m+1)\lambda)^p}{m!p!c} \int_0^1 x^{k+3-\frac{1}{c}(r+m\beta+p)} (1-x)^{\frac{1}{c}(r+m\beta+\beta+p)-1} dx \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} (m+1)^p \lambda)^{p+1}}{m!p!c} \int_0^1 x^{k+3-\frac{1}{c}(r+m\beta+\beta+p+1)} (1-x)^{\frac{1}{c}(r+m\beta+\beta+p+1)-1} dx \\
&= \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^m (m\lambda)^p k}{m!p!} B\left(k - \frac{1}{c}(r+m\beta+p-4c), \frac{1}{c}(r+m\beta+p+c)\right) \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} \beta((m+1)\lambda)^p}{m!p!c} B\left(k - \frac{1}{c}\left(r+m\beta+p-4c\right), \frac{1}{c}(r+m\beta+\beta+p)\right) \\
&+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} (m+1)^p \lambda^{p+1}}{m!p!c} \\
&\times \; B\left(k - \frac{1}{c}\left(r+m\beta+\beta+p-4c+1\right), \frac{1}{c}(r+m\beta+\beta+p+1)\right), \qquad (14)
\end{aligned}
$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the beta function. The moment generating function of the BMW distribution is given by $E(e^{tY}) = \sum_{i=0}^{\infty} \frac{t^i}{i!} E(Y^i)$, where $E(Y^i)$ is given above.

The coefficients of variation (CV), Skewness (CS) and Kurtosis (CK) can be readily obtained. The variance (σ^2), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) are given by

$$
\sigma^2 = \mu_2' - \mu^2, \quad CV = \frac{\sigma}{\mu} = \frac{\sqrt{\mu_2' - \mu^2}}{\mu} = \sqrt{\frac{\mu_2'}{\mu^2} - 1}, \qquad (15)
$$

$$
CS = \frac{E\left[(X-\mu)^3\right]}{\left[E(X-\mu)^2\right]^{3/2}} = \frac{\mu_3' - 3\mu\mu_2' + 2\mu^3}{(\mu_2' - \mu^2)^{3/2}}, \qquad (16)
$$

and

$$
CK = \frac{E\left[(X-\mu)^4\right]}{\left[E(X-\mu)^2\right]^2} = \frac{\mu_4' - 4\mu\mu_3' + 6\mu^2\mu_2' - 3\mu^4}{(\mu_2' - \mu^2)^2}, \qquad (17)
$$

respectively. Table 2 lists the first six moments of the BMW distribution for selected values of the parameters, by fixing $\alpha = \beta = 1.5$ and $\lambda = 0.5$. Table 3 lists the first six moments of the BMW distribution for selected values of the parameters, by fixing $c = 2.0$ and $k = 1.0$. These values can be determined numerically using R and MATLAB.

Table 2: Moments of the BMW distribution for some parameter values; $\alpha = \beta = 1.5$ and $\lambda = 0.5$.

μ'_s	$c = 0.5, k = 0.5$	$c = 0.5, k = 1.5$	$c = 1.5, k = 0.5$	$c = 1.5, k = 1.5$
μ'_1	0.43766	0.29290	0.48848	0.40979
μ'_2	0.29569	0.17537	0.32477	0.23753
μ'_3	0.24297	0.13506	0.25999	0.17015
μ'_4	0.22732	0.12105	0.23710	0.14145
μ'_5	0.23433	0.12088	0.23875	0.13165
μ'_6	0.26090	0.13127	0.26028	0.13408
SD	0.32271	0.29929	0.29352	0.26382
CV	0.73735	1.02180	0.60089	0.64378
CS	0.66650	1.16473	0.67919	0.85869
CK	2.92547	3.86304	3.13235	3.56673

Table 3: Moments of the BMW distribution for some parameter values; $c = 2.0$ and $k = 1.0$.

μ'_s	$\alpha = 3.5, \beta = 1.5, \lambda = 0.5$	$\alpha = 3.5, \beta = 0.8, \lambda = 0.0$	$\alpha = 1.0, \beta = 1.5, \lambda = 0.2$	$\alpha = 0.8, \beta = 3.5, \lambda = 0.5$
μ'_1	0.31249	0.21032	0.50889	0.68308
μ'_2	0.13308	0.10248	0.36010	0.54319
μ'_3	0.06870	0.07905	0.31582	0.47550
μ'_4	0.04074	0.08507	0.32470	0.44606
μ'_5	0.02689	0.12021	0.37847	0.44154
μ'_6	0.01938	0.21511	0.48899	0.45660
SD	0.18823	0.24133	0.31801	0.27675
CV	0.60233	1.14746	0.62491	0.40514
CS	0.74599	2.34797	0.92163	-0.00847
CK	3.36566	11.76067	3.92708	2.45747

3.2 Conditional Moments

For lifetime models, it may be useful to know the conditional moments. The r^{th} conditional moment is defined as $E(Y^r \mid Y > t)$. The r^{th} conditional moment of the BMW distribution is given by

$$
\begin{aligned}
E(Y^r|Y>t) &= \frac{1}{\overline{F}_{BMW}(t)}\int_t^{\infty} y^r f_{BMW}(y)dy \\
&= \frac{1}{\overline{F}_{BMW}(t)}\int_t^{\infty} y^r\,[1+y^c]^{-k-1}\sum_{m=0}^{\infty}\frac{(-1)^m\alpha^m y^{m\beta}e^{m\lambda y}}{m!} \\
&\times \left[kcy^{c-1}+(1+y^c)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)\right]dy.
\end{aligned}
\tag{18}
$$

Let $x=(1+y^c)^{-1}$, then the r^{th} conditional moment of the BMW distribution is given by

$$
\begin{aligned}
E(Y^r|Y>t) &= \frac{1}{\overline{F}_{BMW}(t)}\Bigg[\sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^m(m\lambda)^p k}{m!p!} \\
&\times\ B_{(1+t^c)^{-1}}\left(k-\frac{1}{c}\left(r+m\beta+p-4c\right),\frac{1}{c}(r+m\beta+p+c)\right) \\
&+\ \sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^{m+1}\beta((m+1)\lambda)^p}{m!p!c} \\
&\times\ B_{(1+t^c)^{-1}}\left(k-\frac{1}{c}\left(r+m\beta+\beta+p-4c\right),\ (r+m\beta+\beta+p)\right) \\
&+\ \sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^{m+1}(m+1)^p\lambda^{p+1}}{m!p!c} \\
&\times\ B_{(1+t^c)^{-1}}\left(k-\frac{1}{c}\left(r+m\beta+\beta+p-4c+1\right),\frac{1}{c}(r+m\beta+\beta+p+1)\right)\Bigg],
\end{aligned}
$$

where $B_y(a,b)=\int_0^y x^{a-1}(1-x)^{b-1}dx$ is the incomplete beta function, and $c>r+m\beta+\beta+p+1$. The mean residual lifetime function is given by $E(Y|Y>t)-t=V_F(t)-t$, where $V_F(t)$ is called the vitality function of the distribution function F.

4 Mean Deviations, Lorenz and Bonferroni Curves

Mean deviation about the mean and mean deviation about the median as well as Lorenz and Bonferroni curves for the BMW distribution are presented in this section. Bonferroni and Lorenz curves are widely used tool for analyzing and visualizing income inequality. Lorenz curve, $L(p)$ can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume y, and Bonferroni curve, $B(p)$ is the scaled conditional mean curve, that is, ratio of group mean income of the population.

4.1 Mean Deviations

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median and are defined by

$$\delta_1(y) = \int_0^\infty |y-\mu| f_{BMW}(y)dy \quad \text{and} \quad \delta_2(y) = \int_0^\infty |y-M| f_{BMW}(y)dy, \tag{19}$$

respectively, where $\mu = E(Y)$ and M =Median(Y) denotes the median. The measures $\delta_1(y)$ and $\delta_2(y)$ can be calculated using the relationships

$$\delta_1(y) = 2\mu F_{BMW}(\mu) - 2\mu + 2\int_\mu^\infty y f_{BMW}(y)dy, \tag{20}$$

and

$$\delta_2(y) = -\mu + 2\int_M^\infty y f_{BMW}(y)dy, \tag{21}$$

respectively. When $r = 1$, we get the mean $\mu = E(Y)$. Note that $T(\mu) = \int_\mu^\infty y f_{BMW}(y)dy$ and $T(M) = \int_M^\infty y f_{BMW}(y)dy$, where

$$\begin{aligned} T(\mu) = \int_\mu^\infty y f_{BMW}(y)dy &= \Bigg[\sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^m (m\lambda)^p k}{m!p!} \\ &\times B_{(1+\mu^c)^{-1}} \left(k - \frac{1}{c}(1+m\beta+p-4c), \frac{1}{c}(1+m\beta+p+c) \right) \\ &+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} \beta((m+1)\lambda)^p}{m!p!c} \\ &\times B_{(1+\mu^c)^{-1}} \left(k - \frac{1}{c}(1+m\beta+\beta+p-4c),\ (1+m\beta+\beta+p) \right) \\ &+ \sum_{m,p=0}^{\infty} \frac{(-1)^m \alpha^{m+1} (m+1)^p \lambda^{p+1}}{m!p!c} \\ &\times B_{(1+\mu^c)^{-1}} \left(k - \frac{1}{c}(m\beta+\beta+p-4c+2), \frac{1}{c}(m\beta+\beta+p+2) \right) \Bigg]. \end{aligned}$$

Consequently, the mean deviation about the mean and the mean deviation about the median reduces to

$$\delta_1(y) = 2\mu F_{BMW}(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(y) = -\mu + 2T(M),$$

respectively.

4.2 Bonferroni and Lorenz Curves

In this sub-section, we present Bonferroni and Lorenz Curves. Bonferroni and Lorenz curves have applications not only in economics for the study income and poverty, but also in other fields such as reliability, demography, insurance and medicine. Bonferroni and Lorenz curves for the BMW distribution are given by

$$B(p)=\frac{1}{p\mu}\int_0^q yf_{BMW}(y)dy=\frac{1}{p\mu}[\mu-T(q)],$$

and

$$L(p)=\frac{1}{\mu}\int_0^q yf_{BMW}(y)dy=\frac{1}{\mu}[\mu-T(q)],$$

respectively, where

$$\begin{aligned}
T(q)=\int_q^\infty yf_{BMW}(y)dy &= \Bigg[\sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^m(m\lambda)^p k}{m!p!}\\
&\times B_{(1+q^c)^{-1}}\left(k-\frac{1}{c}\left(1+m\beta+p-4c\right),\frac{1}{c}(1+m\beta+p+c)\right)\\
&+\sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^{m+1}\beta((m+1)\lambda)^p}{m!p!c}\\
&\times B_{(1+q^c)^{-1}}\left(k-\frac{1}{c}\left(1+m\beta+\beta+p-4c\right),\ (1+m\beta+\beta+p)\right)\\
&+\sum_{m,p=0}^{\infty}\frac{(-1)^m\alpha^{m+1}(m+1)^p\lambda^{p+1}}{m!p!c}\\
&\times B_{(1+q^c)^{-1}}\left(k-\frac{1}{c}\left(m\beta+\beta+p-4c+2\right),\frac{1}{c}(m\beta+\beta+p+2)\right)\Bigg],
\end{aligned}$$

and $q=F^{-1}(p)$, $0\le p\le 1$.

5 Order Statistics, L-Moments and Rényi Entropy

The concept of entropy plays a vital role in information theory. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty. In this section, we present the distribution of the order statistic, L-moments and Rényi entropy for the BMW distribution.

5.1 Order Statistics

Order statistics play an important role in probability and statistics. In this sub-section, we present the distribution of the i^{th} order statistic from the BMW distribution. The pdf of the i^{th} order statistic from the BMW pdf $f(y)$ is given by

$$
\begin{aligned}
g_{i:n}(y) &= \frac{n! f_{_{BMW}}(y)}{(i-1)!(n-i)!}[F_{_{BMW}}(y)]^{i-1}[1-F_{_{BMW}}(y)]^{n-i} \\
&= \frac{n! f_{_{BMW}}(x)}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}(-1)^j\binom{n-i}{j}[F_{_{BMW}}(y)]^{j+i-1},
\end{aligned}
$$

and is obtained by using the binomial expansion

$$
[1-F_{_{BMW}}(y)]^{n-i} = \sum_{m=0}^{n-i}\binom{n-i}{m}(-1)^m[F_{_{BMW}}(y)]^m,
$$

so that

$$
\begin{aligned}
g_{i:n}(y) &= \frac{1}{B(i,n-i+1)}\sum_{m=0}^{n-i}\binom{n-i}{m}\frac{(-1)^m}{m+i}(m+i)[F_{_{BMW}}(y)]^{m+i-1}f(y) \\
&= \sum_{m=0}^{n-i} w_{i,m} f_{_{m+i}}(y),
\end{aligned}
$$

where $f_{_{m+i}}(y)$ is the pdf of the exponentiated BMW distribution with parameters $c, k, \alpha, \beta, \lambda$, and $m+i$.

The t^{th} moment of the i^{th} order statistic from the BMW distribution can be derived via a result of Barakat and Abdelkader (2004) as follows:

$$
E(Y_{i:n}^t) = t\sum_{p=n-i+1}^{n}(-1)^{p-n+i-1}\binom{p-1}{n-i}\binom{n}{p}\int_0^\infty x^{t-1}[1-F_{_{BMW}}(y)]^p dy. \tag{22}
$$

Note that by setting $x=(1+y^c)^{-1}$, we have

$$
\begin{aligned}
\int_0^\infty x^{t-1}[1-F_{_{BMW}}(y)]^p dy &= \sum_{m,s=0}^{\infty}\frac{(-1)^m(\alpha p)^m(m\lambda)^s}{m!s!}\int_0^\infty y^{t+m\beta+s-1}(1+y^c)^{-pk}dy \\
&= \sum_{m,s=0}^{\infty}\frac{(-1)^m(\alpha p)^m(m\lambda)^s}{m!s!}\frac{1}{c} \\
&\times \int_0^1 x^{pk-\frac{1}{c}(t+m\beta+s)-1}(1-x)^{\frac{t+m\beta+s}{c}-1}dx \\
&= \sum_{m,s=0}^{\infty}\frac{(-1)^m(\alpha p)^m(m\lambda)^s}{m!s!}\frac{1}{c} \\
&\times B\left(pk-\frac{1}{c}(t+m\beta+s), \frac{t+m\beta+s}{c}\right).
\end{aligned} \tag{23}
$$

Consequently, the t^{th} moment of the i^{th} order statistic from the BMW distribution is given by

$$\begin{aligned} E(Y_{i:n}^{t}) &= t\sum_{p=n-i+1}^{n}\sum_{m,s=0}^{\infty}(-1)^{p-n+i-1}\binom{p-1}{n-i}\binom{n}{p}\frac{(-1)^{m}(\alpha p)^{m}(m\lambda)^{s}}{m!s!}\frac{1}{c} \\ &\times\ B\bigg(pk-\frac{1}{c}(t+m\beta+s),\frac{t+m\beta+s}{c}\bigg), \end{aligned} \tag{24}$$

where $B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the complete beta function.

5.2 L-Moments

The $L-$moments (Hoskings, 1990) are expectations of some linear combinations of order statistics and they exist whenever the mean of the distribution exits, even when some higher moments may not exist. They are relatively robust to the effects of outliers and are given by

$$\lambda_{k+1}=\frac{1}{k+1}\sum_{j=0}^{k}(-1)^{j}\binom{k}{j}E(Y_{k+1-j:k+1}),\quad k=0,1,2,....... \tag{25}$$

The $L-$moments of the BMW distribution can be readily obtained from equation (24). The first four $L-$moments are given by $\lambda_1 = E(Y_{1:1})$, $\lambda_2 = \frac{1}{2}E(Y_{2:2} - Y_{1:2})$, $\lambda_3 = \frac{1}{3}E(Y_{3:3} - 2Y_{2:3} + Y_{1:3})$ and $\lambda_4 = \frac{1}{4}E(Y_{4:4} - 3Y_{3:4} + 3Y_{2:4} - Y_{1:4})$, respectively.

5.3 Rényi Entropy

In this sub-section, Rényi entropy (Rényi, 1960) of the BMW distribution is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy is an extension of Shannon entropy. Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^{\infty}[f_{BMW}(y;c,k,\alpha,\beta,\lambda)]^{v}dy\right),v\neq 1,v>0.$$

Rényi entropy tends to Shannon entropy as $v \to 1$. Note that $[f(y;c,k,\alpha,\beta,\lambda)]^v = f^v_{BMW}(y)$ can be written as

$$\begin{aligned}
f^v_{BMW}(y) &= \left[kcy^{c-1} + (1+y^c)\,\alpha y^{\beta-1} e^{\lambda y}(\beta+\lambda y)\right]^v \\
&\times\ (1+y^c)^{-kv-v}\, e^{-v\alpha y^\beta e^{\lambda y}} \\
&= \sum_{m,r=0}^{\infty}\sum_{p=0}^{v}\binom{v}{p}\frac{(-1)^m (v\alpha)^m \alpha^p[\lambda(m+p)]^r}{r!m!}\,(kc)^{v-p} \\
&\times\ y^{(r+m\beta+p\beta+cv-cp-v)}(1+y^c)^{-kv-v+p}(\beta+\lambda y)^p \\
&= \sum_{m,r=0}^{\infty}\sum_{p=0}^{v}\sum_{w=0}^{p}\binom{v}{p}\binom{p}{w}\frac{(-1)^m (v\alpha)^m \alpha^p[\lambda(m+p)]^r \lambda^w \beta^{p-w}}{r!m!}\,(kc)^{v-p} \\
&\times\ y^{(r+m\beta+p\beta+cv-cp-v+w)}(1+y^c)^{-kv-v+p}.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^\infty f^v_{BMW}(y)dy &= \sum_{m,r=0}^{\infty}\sum_{p=0}^{v}\sum_{w=0}^{p}\binom{v}{p}\binom{p}{w}\frac{(-1)^m (v\alpha)^m \alpha^p[\lambda(m+p)]^r \lambda^w \beta^{p-w}}{r!m!}\,(kc)^{v-p} \\
&\times\ \int_0^\infty y^{(r+m\beta+p\beta+cv-cp-v+w)}(1+y^c)^{-kv-v+p}dy \\
&= \sum_{m,r=0}^{\infty}\sum_{p=0}^{v}\sum_{w=0}^{p}\binom{v}{p}\binom{p}{w}\frac{(-1)^m (v\alpha)^m \alpha^p[\lambda(m+p)]^r \lambda^w \beta^{p-w}}{r!m!c} \\
&\times\ (kc)^{v-p}B\left(\frac{1}{c}a^*, kv+v-p-\frac{1}{c}a^*+4\right),
\end{aligned}$$

where we have used the substitutions $x = (1+y^c)^{-1}$ and $a^* = (r+m\beta+p\beta+cv-cp-v+w+1)$.

Consequently, Rényi entropy is given by

$$\begin{aligned}
I_R(v) &= \left(\frac{1}{1-v}\right)\log\Bigg[\sum_{m,r=0}^{\infty}\sum_{p=0}^{v}\sum_{w=0}^{p}\binom{v}{p}\binom{p}{w}\frac{(-1)^m (v\alpha)^m \alpha^p[\lambda(m+p)]^r \lambda^w \beta^{p-w}}{r!m!c} \\
&\times\ (kc)^{v-p}B\left(\frac{1}{c}a^*, kv+v-p-\frac{1}{c}a^*+4\right)\Bigg],
\end{aligned}$$

for $v \neq 1$ and $v > 0$.

6 Maximum Likelihood Estimation

Let $Y \sim BMW(c,k,\alpha,\beta,\lambda)$ and $\mathbf{\Delta} = (c,k,\alpha,\beta,\lambda)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\mathbf{\Delta})$ for a single observation of y of Y is given by

$$\begin{aligned}
\ell(\mathbf{\Delta}) &= -\alpha y^\beta e^{\lambda y} - (k+1)\log(1+y^c) \\
&+\ \log\left(kcy^{c-1} + (1+y^c)\,\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)\right). \qquad (26)
\end{aligned}$$

The first derivative of the log-likelihood function with respect to the components of the parameter vector $\mathbf{\Delta} = (c, k, \alpha, \beta, \lambda)^T$ are given by

$$\frac{\partial \ell}{\partial c} = -\frac{c(k+1)y^{c-1}}{1+y^c} + \frac{ky^{c-1}\left(1+c\log(y)\right) + \alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda x)y^c \log(y)}{kcy^{c-1} + \left(1+y^c\right)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)},$$

$$\frac{\partial \ell}{\partial k} = -\log\left(1+y^c\right) + \frac{cy^{c-1}}{kcy^{c-1} + \left(1+y^c\right)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)},$$

$$\frac{\partial \ell}{\partial \alpha} = -y^{\beta}e^{\lambda y} + \frac{\left(1+y^c\right)y^{\beta-1}e^{\lambda y}(\beta+\lambda y)}{kcy^{c-1} + \left(1+y^c\right)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)},$$

$$\frac{\partial \ell}{\partial \beta} = -\alpha e^{\lambda y}y^{\beta}\log(y) + \frac{\alpha e^{\lambda y}\left(1+y^c\right)\left(y^{\beta-1} + (\beta+\lambda y)y^{\beta-1}\log(y)\right)}{kcy^{c-1} + \left(1+y^c\right)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)},$$

and

$$\frac{\partial \ell}{\partial \lambda} = -\alpha y^{\beta+1}e^{\lambda y} + \frac{\left(1+y^c\right)\alpha y^{\beta-1}(ye^{\lambda y}(\beta+\lambda y+1))}{kcy^{c-1} + \left(1+y^c\right)\alpha y^{\beta-1}e^{\lambda y}(\beta+\lambda y)},$$

respectively.

The total log-likelihood function based on a random sample of n observations: $x_1, x_2, ..., x_n$ drawn from the BMW distribution is given by $\ell_n^* = \sum_{i=1}^n \ell_i(\mathbf{\Delta})$, where $\ell_i(\mathbf{\Delta})$, $i = 1, 2, ..., n$ is given by equation (26). The equations obtained by setting the above partial derivatives to zero are not in closed form and the values of the parameters $c, k, \alpha, \beta, \lambda$ must be found by using iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\mathbf{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial \ell_n^*}{\partial c}, \frac{\partial \ell_n^*}{\partial k}, \frac{\partial \ell_n^*}{\partial \alpha}, \frac{\partial \ell_n^*}{\partial \beta}, \frac{\partial \ell_n^*}{\partial \lambda})^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix is given by $\mathbf{I}(\mathbf{\Delta}) = [\mathbf{I}_{\theta_i,\theta_j}]_{5X5} = E(-\frac{\partial^2 \ell}{\partial \theta_i \partial \theta_j})$, $i, j = 1, 2, 3, 4, 5$, can be numerically obtained by MATLAB or R software. The total Fisher information matrix $n\mathbf{I}(\mathbf{\Delta})$ can be approximated by

$$\mathbf{J}(\hat{\mathbf{\Delta}}) \approx \left[-\left.\frac{\partial^2 \ell_n^*}{\partial \theta_i \partial \theta_j}\right|_{\mathbf{\Delta}=\hat{\mathbf{\Delta}}} \right]_{5X5}, \quad i, j = 1, 2, 3, 4, 5. \tag{27}$$

For a given set of observations, the matrix given in equation (27) is obtained after the convergence of the Newton-Raphson procedure in MATLAB or R software.

7 Simulation Study

In this section, we examine the performance of the BMW distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800) via the R package. We simulate 1000 samples for the true parameters values $I : c = 2, k = 5, \alpha = 3, \beta = 4, \lambda = 2$ and $II : c = 0.5, k = 0.6, \alpha = 0.4, \beta = 2, \lambda = 1$. Table 4 lists the means MLEs of the five model parameters along with the respective root mean squared errors (RMSE). The bias and RMSEs are given by:

$$Bias(\hat{\theta}) = \frac{\sum_{i=1}^{n} \hat{\theta}_i}{n} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n} (\hat{\theta}_i - \theta)^2}{n}},$$

respectively. From the results, we can readily verify that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, since RMSEs decay toward zero.

Table 4: Monte Carlo Simulation Results

		(2,5,3,4,2)			(0.5,0.6,0.4,2,1)		
Parameter	Sample Size	Mean	RMSE	Bias	Mean	RMSE	Bias
c	25	2.064011	0.6574089	0.064011344	0.6198985	0.33424985	0.11989847
	50	1.987646	0.5394971	-0.012353789	0.5666287	0.20913064	0.06662871
	100	1.998268	0.4381318	-0.001731681	0.5554174	0.13150152	0.05541743
	200	1.994114	0.368368	-0.005886012	0.5383476	0.08491854	0.03834756
	400	1.964774	0.3077718	-0.035226137	0.5363581	0.06557844	0.0363581
	800	1.9954	0.192545	-0.004600092	0.5297441	0.04887756	0.02974413
k	25	6.686042	5.5355356	1.686041854	0.7540626	0.43450896	0.15406256
	50	5.88308	4.0400779	0.883079865	0.6676759	0.29135272	0.06767593
	100	5.669251	3.1769968	0.669250619	0.640382	0.20754712	0.04038197
	200	5.431808	2.6291927	0.431807777	0.6250959	0.14756323	0.02509593
	400	4.91291	2.0769845	-0.087089516	0.631652	0.11393034	0.03165203
	800	4.934186	1.5458762	-0.065814208	0.6203353	0.0868708	0.02033529
α	25	24.426851	50.3145461	21.426851188	0.5962957	0.5742553	0.19629573
	50	16.567506	31.5421246	13.567505535	0.5896418	0.50291512	0.18964181
	100	14.692613	26.4758735	11.692612555	0.5334652	0.42369271	0.13346519
	200	11.893248	18.7538517	8.89324764	0.5684291	0.4086208	0.16842914
	400	10.302287	14.2708566	7.302287044	0.5484378	0.36112655	0.14843781
	800	8.966029	11.8409376	5.966029444	0.5153346	0.28969162	0.11533459
β	25	26.791509	73.8957413	22.7915085	2.6309001	1.65853707	0.6309001
	50	13.359338	32.6575501	9.359337906	2.2839632	1.32742815	0.28396315
	100	8.404221	15.3638763	4.404220766	2.0982607	1.13824399	0.09826073
	200	5.416987	4.968963	1.416986899	2.1830113	0.97925435	0.18301127
	400	4.32704	2.253199	0.327039519	2.217929	0.8385417	0.21792904
	800	3.981956	1.7651961	-0.018043559	2.1953171	0.703408	0.19531713
λ	25	19.063334	55.1118464	17.063333807	1.2979795	1.48080935	0.2979795
	50	8.997401	25.5950541	6.997401121	1.1350924	1.24977805	0.13509237
	100	5.011334	10.7838647	3.011333885	1.1181211	1.05886226	0.11812109
	200	3.095257	4.2421011	1.09525717	0.9362896	0.86937335	-0.06371041
	400	2.39227	2.5980651	0.392270229	0.8759433	0.72212842	-0.12405666
	800	2.404533	2.3832869	0.404533042	0.8725937	0.57230195	-0.12740626

8 Applications

In this section, we present examples to illustrate the applicability and flexibility of the BMW distribution and its sub-models for data modeling. Estimates of the parameters of BMW distribution (standard error in parentheses), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Sum of Squares (SS, described in this section), Cramer Von Mises and Anderson-Darling statistics (W^* and A^*) are presented for each dataset. We compared the BMW distribution with the gamma-Dagum (GD) (Oluyede et al., 2014) and beta modified Weibull (BetaMW) (Nadarajah et al., 2011) distributions. The GD and BetaMW pdfs are given by

$$\begin{aligned} g_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\theta^{\alpha}\Gamma(\alpha)}(1+\lambda x^{-\delta})^{-\beta-1}(-\log[1-(1+\lambda x^{-\delta})^{-\beta}])^{\alpha-1}, \\ &\times \quad [1-(1+\lambda x^{-\delta})^{-\beta}]^{\frac{1}{\theta}-1}, \quad x>0, \end{aligned}$$

and

$$g_{BetaMW}(x) = \frac{\alpha x^{\gamma-1}(\gamma+\lambda x)\exp(\lambda x)}{B(a,b)} e^{-b\alpha x^{\gamma}\exp(\lambda x)}(1-e^{-\alpha x^{\gamma}\exp(\lambda x)})^{a-1}, \quad x>0,$$

respectively. We also compared the BMW distribution with the exponentiated Kumaraswamy Dagum (EKD) by (Huang and Oluyede, 2014), beta Weibull-geometric (BWG) (Bidram et al., 2013) and beta Weibull-Poisson (BWP) (Percontini et al., 2013) distributions. The pdf of EKD distribution is given by

$$\begin{aligned} g_{EKD}(x) &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi-1} \\ &\times \quad \{1-[1-(1+\lambda x^{-\delta})^{-\alpha}]^{\phi}\}^{\theta-1}, \end{aligned} \tag{28}$$

for $\alpha, \lambda, \delta, \phi, \theta > 0$, and $x > 0$. The BWG and BWP pdfs are given by

$$g_{BWG}(x) = \frac{(1-p)^b \alpha\beta^{\alpha} x^{\alpha-1} e^{-b(\beta x)^{\alpha}}(1-e^{-(\beta x)^{\alpha}})^{a-1}}{B(a,b)(1-pe^{-(\beta x)^{\alpha}})^{a+b}} \tag{29}$$

for $a, b, \alpha, \beta > 0, p \in (0,1)$, and $x > 0$, and

$$g_{BWP}(x) = \frac{\alpha\beta\lambda x^{\alpha-1} e^{\lambda e^{-\beta x^{\alpha}}-\lambda-\beta x^{\alpha}}(e^{\lambda}-1)^{2-a-b}(e^{\lambda}-e^{\lambda e^{-\beta x^{\alpha}}})^{a-1}(e^{\lambda e^{-\beta x^{\alpha}}}-1)^{b-1}}{B(a,b)(1-e^{-\lambda})} \tag{30}$$

for $a, b, \alpha, \beta, \lambda > 0$, and $x > 0$, respectively.

The maximum likelihood estimates (MLEs) of the BMW model parameters $\Delta = (c, k, \alpha, \beta, \lambda)$ are computed by maximizing the objective function via the subroutine mle2 in R. The estimated values of the model parameters (standard error in parenthesis), -2log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2\ln(L)$, Bayesian Information Criterion, $BIC = p\ln(n) - 2\ln(L)$, and Consistent Akaike Information Criterion, $AICC = AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Tables 5 and 6. The goodness-of-fit statistics W^* and A^*, described by Chen and Balakrishnan (1995) are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. The BMW distribution is fitted to the datasets and these fits are compared to the fits using the Burr XII Weibull (BW), Burr XII exponential (BE), Burr-XII Rayleigh (BR), Lomax modified Weibull (LMW), Lomax Rayleigh (LR), Lomax exponential (LE), Log-logistic Rayleigh (LLoGR), Log-logistic exponential (LLoGE), modified Weibull (MW), Weibull (W) and exponential (E) distributions.

We also maximized the likelihood function using NLmixed in Statistical Analysis System (SAS) as well as the function nlm in R (Team (2011)). These functions were applied and executed for a wide range of initial values. This process often results or lead to more than one maximum, however, in these cases, we take the MLEs corresponding to the largest value of the maxima. In a few cases, no maximum was identified for the selected initial values. In these cases, a new initial value was tried in order to obtain a maximum.

The issues of existence and uniqueness of the MLEs are theoretical interest and has been studied by several authors for different distributions including Seregin (2010), Santos Silva and Tenreyro (2010), Zhou (2009), and Xia et al. (2009).

We can use the likelihood ratio test (LRT) to compare the fit of the BMW distribution with its sub-models for a given data set. For example, to test $\beta = 1$, the LRT statistic is $\omega = 2[\ln(L(\hat{c}, \hat{k}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})) - \ln(L(\tilde{c}, \tilde{k}, \tilde{\alpha}, 1, \tilde{\lambda}))]$, where $\hat{c}$, $\hat{k}$, $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ are the unrestricted estimates, and $\tilde{c}$, $\tilde{k}$, $\tilde{\alpha}$ and $\tilde{\lambda}$ are the restricted estimates. The LRT rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

Plots of the fitted densities, the histogram of the data and probability plots by Chambers et al. (1983) are given in Figure 3, Figure 4, Figure 5 and Figure 6. For the probability plot, we plotted $F(y_{(j)}; \hat{c}, \hat{k}, \hat{\alpha}, \hat{\beta}, \hat{\lambda})$ against $\dfrac{j - 0.375}{n + 0.25}$, $j = 1, 2, \cdots, n$, where $y_{(j)}$ are the ordered values of the observed data. The measures of closeness are given by the sum of squares

$$SS = \sum_{j=1}^{n} \left[F(y_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2 .$$

8.1 Aarset Data

The data contains the times to failure of 50 devices put on life test at time 0, Aarset (1987). The parameter estimates, goodness-of-fit statistics and results for this data are given in Table 5. The estimated variance-covariance matrix for the BMW distribution is given by:

Table 5: Estimates of Models for Aarset Data

	Estimates					Statistics						
Model	c	k	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
BMW	0.7890	0.3732	0.0028	0.4695	0.6326	205.57	215.57	216.93	225.13	0.0963	0.8497	0.1114
	(0.1716)	(0.0994)	(0.0063)	(1.6129)	(0.2601)							
BW	0.7320	0.3510	0.0030	2.9162	0	219.98	227.98	229.34	235.63	0.1916	1.3603	0.2230
	(0.1883)	(0.1085)	(0.0113)	(1.7461)	-							
BE	0.4578	0.1442	0.1885	1	0	248.13	254.13	255.50	259.87	0.4515	2.7775	0.5082
	(0.1910)	(0.0974)	(0.0341)	-	-							
BR	0.5431	0.3177	0.0214	2	0	228.54	234.54	235.91	240.28	0.2725	1.7761	0.2807
	(0.1387)	(0.0949)	(0.0042)	-	-							
LMW	1	0.4244	0.0026	0.1648	0.7108	209.75	217.75	219.11	225.40	0.1368	1.0863	0.2854
	-	(0.1065)	(0.0067)	(0.5077)	(0.2995)							
LR	1	0.2679	0.0194	2	0	235.99	239.99	241.36	243.82	0.3337	2.1380	0.3735
	-	(0.0755)	(0.0041)	-	-							
LE	1	0.0316	0.2088	1	0	251.84	255.84	257.20	259.66	0.4931	2.9907	0.5235
	-	(0.1098)	(0.0458)	-	-							
W	0	0	0.2404	0.9490	0	251.75	255.75	257.11	259.57	0.4964	3.0079	0.5289
	-	-	(0.0626)	(0.1196)	-							
E	0	0	0.2189	1	0	251.92	253.92	255.28	255.83	0.4877	2.9622	0.5190
	-	-	(0.0310)	-	-							
MW	0	0	0.1413	0.3548	0.2332	224.05	230.05	231.42	235.79	0.2344	1.6043	0.2660
	-	-	(0.0458)	(0.1127)	(0.0484)							
LLoGE	0.4869	1	0.1117	1	0	277.53	281.53	282.89	285.35	0.5104	3.0740	2.8117
	(0.1252)	-	(0.0344)	-	-							
LLoGR	0.4250	1	0.0180	2	0	251.87	255.87	257.24	259.70	0.2638	1.7458	2.3218
	(0.0952)	-	(0.0040)	-	-							
	λ	β	δ	α	θ							
GD	52.6990	0.0046	2.1508	17.9287	0.2807	245.14	255.14	256.50	264.70	0.4114	2.5517	0.4281
	(0.0055)	(0.0073)	(0.5231)	(0.0311)	(0.1120)							
	a	b	α	γ	λ							
BetaMW	0.2315	0.3073	0.0030	1.2938	0.5534	211.51	221.51	222.87	231.07	0.1580	1.2042	0.1662
	(0.1504)	(0.4498)	(0.0045)	(0.8827)	(0.1588)							
	α	λ	δ	ϕ	θ							
EKD	2.8805	57.1359	1.4989	19.7414	0.1684	238.97	248.97	250.33	258.53	0.3713	2.3385	0.5964
	(3.7807)	(47.1848)	(1.0292)	(10.6947)	(0.3297)							
	a	b	α	β	p							
BWG	0.0861	0.0692	5.3428	0.2150	0.1571	207.43	217.43	218.79	226.99	0.1044	0.9059	0.0985
	(0.0227)	(0.0240)	(0.7859)	(0.0174)	(0.0723)							

$$\begin{pmatrix} 0.0295 & -0.0060 & 0.0000 & -0.0277 & 0.0081 \\ -0.0060 & 0.0099 & -0.0002 & 0.0303 & 0.0003 \\ 0.0000 & -0.0002 & 0.0000 & -0.0080 & 0.0003 \\ -0.0277 & 0.0303 & -0.0080 & 2.6014 & -0.3224 \\ 0.0081 & 0.0003 & 0.0003 & -0.3224 & 0.0677 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by $c \in (0.7890 \pm 1.96 \times 0.1716), k \in (0.3732 \pm 1.96 \times 0.0994), \alpha \in (0.0028 \pm 1.96 \times 0.0063), \beta \in (0.4695 \pm 1.96 \times 1.6129)$ and $\lambda \in (0.6326 \pm 1.96 \times 0.2601)$, respectively.

The LRT statistic for testing H_0: BW against H_a: BMW and H_0: BR against H_a: BMW are 14.41 (p-value = 0.00015) and 22.9736 (p-value $<$ 0.0001), respectively. We

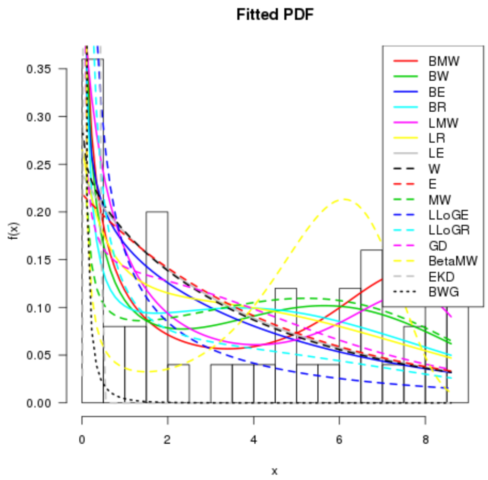

Figure 3: Fitted Densities for Aarset Data

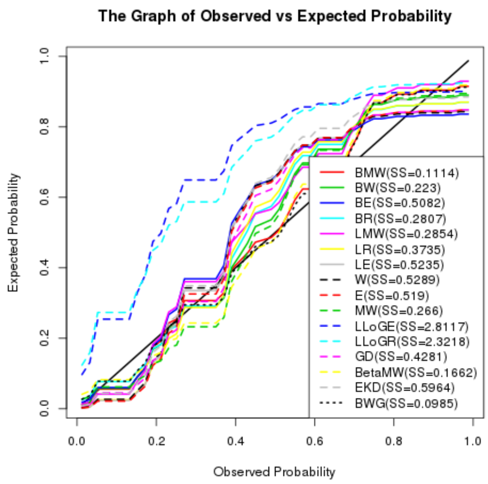

Figure 4: Probability Plots for Aarset Data

conclude that there is a significant difference between the BW and the BMW distributions, and also between the BR and the BMW distributions. The LRT statistic for testing H_0: LMW against H_a: BMW is 4.18 (p-value=0.0409 $<$ 0.05). We conclude that there is a significant difference between the LMW and the BMW distribution at the 5% level. There is indeed clear and convincing evidence based on the goodness-of-fit statistics W^* and A^* that the BMW distribution is far better than the sub-models, and the non-nested models. Also, the values of AIC and BIC shows that the BMW distribution is better than the non-nested GD, EKD, BetaMW and BWG distributions.

Table 6: Estimates of Models for Kevlar Failure Data

	Estimates					Statistics							
Model	c	k	α	β	λ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	SS
BMW	66.8117	0.0016	0.0067	1.4615	0.1204	277.53	287.53	288.93	296.99	0.0318	0.2215	0.0314
	0.0005	0.0009	0.0106	1.0006	0.0937							
BW	0.9402	0.0013	0.0103	1.9953	0	284.74	292.74	294.14	300.31	0.0736	0.4808	0.0806
	0.0309	0.0023	0.0062	0.2358	-							
BR	36.4457	0.0016	0.0091	2	0	282.01	288.01	289.40	293.68	0.0665	0.4228	0.1077
	5.1E-08	0.0012	0.0015	-	-							
LR	1	0.0048	0.0101	2	0	284.72	288.72	290.11	292.50	0.0711	0.4644	0.0795
	-	0.0344	0.0016	-	-							
W	0	0	0.0104	1.9940	0	284.74	288.74	290.14	292.53	0.0746	0.4875	0.0810
	-	-	0.0064	0.2412	-							
E	0	0	0.1136	1	0	311.19	313.19	314.58	315.08	0.1410	0.9161	0.8446
	-	-	0.0162	-	-							
MW	0	0	0.0225	1.1986	0.0952	281.85	287.85	289.25	293.53	0.0330	0.2371	0.0334
	-	-	0.0154	0.4783	0.0539							
LLoGR	0.1049	1	0.0091	2	0	349.79	353.79	355.19	357.58	0.0720	0.5349	4.4770
	0.0774	-	0.0016	-	-							
	λ	β	δ	α	θ							
GD	3.3401	3.2101	0.4143	3.7627	0.0167	294.62	304.62	306.01	314.07	0.2143	1.3772	0.2321
	1.9539	2.4007	0.1627	6.3883	0.0185							
	α	λ	δ	ϕ	θ							
EKD	3.5492	955.03	1.8521	1175.93	0.2163	281.08	291.08	292.48	300.54	0.0306	0.2308	0.0290
	3.0301	0.0040	0.5944	0.0008	0.2758							
	a	b	α	β	λ							
BetaWP	1.6464	1.5997	1.4400	0.0048	7.9336	289.09	299.09	300.48	308.55	0.1325	0.8604	0.1277
	3.4322	1.3842	1.7706	0.0245	0.4675							

8.2 Kevlar 49/Epoxy Strands Failure Data

The 49 data points represent the stress-rupture life of kevlar 49/epoxy strands which were subjected to constant sustained pressure at the 70% stress level until all had failed, as in the previous example, we have complete data with exact times of failure, Andrews and Herzberg (1985). The estimate results for this data is given in Table 6. The estimated variance-covariance matrix for the BMW distribution is given by:

$$\begin{pmatrix} 2.3E-07 & 6.9E-08 & -4.7E-06 & 4.8E-04 & -4.0E-05 \\ 6.9E-08 & 7.8E-07 & -3.5E-06 & 1.4E-04 & 5.9E-06 \\ -4.7E-06 & -3.5E-06 & 1.1E-04 & -9.7E-03 & 6.4E-04 \\ 4.8E-04 & 1.4E-04 & -9.7E-03 & 1.0E+00 & -8.4E-02 \\ -4.0E-05 & 5.9E-06 & 6.4E-04 & -8.4E-02 & 8.8E-03 \end{pmatrix},$$

and the 95% confidence intervals for the model parameters are given by $c \in (66.8117 \pm 1.96 \times 0.0005), k \in (0.0016 \pm 1.96 \times 0.0009), \alpha \in (0.0067 \pm 1.96 \times 0.0106), \beta \in (1.4615 \pm 1.96 \times 1.0006)$ and $\lambda \in (0.1204 \pm 1.96 \times 0.0937)$, respectively.

The LRT statistic for testing H_0: LR against H_a: BMW and H_0: W against H_a: BMW are 7.1878 (p-value = 0.06615) and 7.2124 (p-value = 0.06543), respectively. We conclude

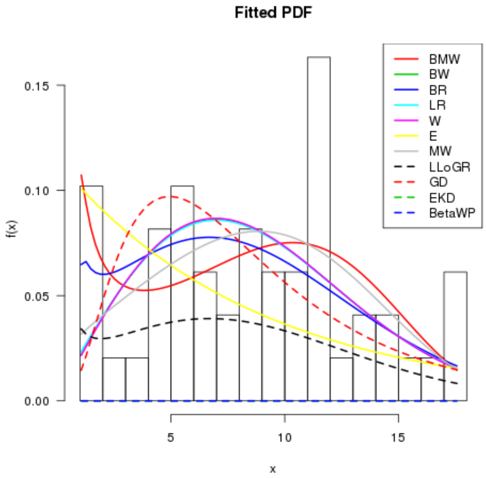

Figure 5: Fitted Densities for Kevlar Failure Data

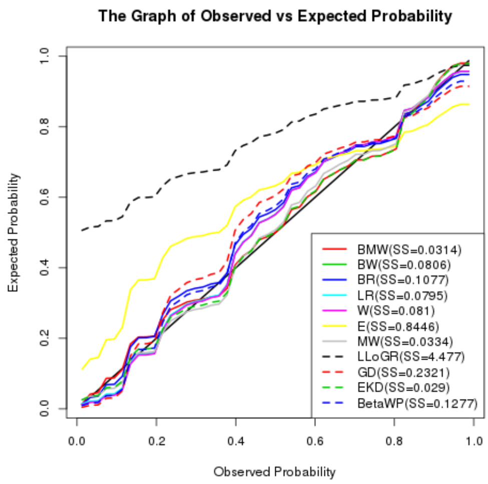

Figure 6: Probability Plots for Kevlar Failure Data

that there is a significant difference between the LR and the BMW distributions, and also between the W and the BMW distributions at the 10% level. We conclude that there is a significant difference between the BW and the BMW distributions (LRT statistic $\omega = 7.21$, p-value-0.00725). There is also a significant difference between the LLoGR and the BMW distributions based on the LRT. There is no significant difference between the MW and the BMW distributions based on the LRT, however, there is indeed clear and convincing evidence based on the goodness-of-fit statistics W^* and A^* that the BMW distribution is far better than the sub-models, and the non-nested models. Also, the values of AIC and BIC shows that the BMW distribution is better than the sub-models and the non-nested GD, EKD, and BetaWP distributions.

9 Concluding Remarks

In this paper, a new distribution called Burr XII modified Weibull (BMW) distribution was introduced. The statistical properties of the BMW distribution including the hazard and reverse hazard functions, quantile function, moments, incomplete moments, generating functions, mean deviations, Bonferroni and Lorenz curves, order statistics, Rényi entropy and maximum likelihood estimation for the model parameters are given. Simulation studies was conducted to examine the performance of the new BMW distribution. We also presented applications of this new model to real life datasets in order to illustrate the usefulness of the distribution.

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References

Aarset, M. V. (1987). How to identify bathtub hazard rate. *IEEE Transactions on Reliability*, 36(1):106–108.

Andrews, D. F. and Herzberg, A. M. (1985). *A Collection of Problems from Many Fields for the Students and Research Workers*. Springer Series in Statistics, New York: Springer.

Barakat, H. M. and Abdelkader, Y. H. (2004). Computing the moments of order statistics from nonidentical random variables. *Statistical Methodology and Applications*, 13.

Bidram, H., Behboodian, J., and Towhidi, M. (2013). The beta weibull geometric distribution. *Journal of Statistical Computation and Simulation*, 83(1):52–67.

Burr, I. W. (1942). Cumulative frequency functions. *The Annals of Mathematical Statistics*, 13(2):215232.

Burr, I. W. and Cislak, P. J. (1968). On a general system of distributions: I. its curve shaped characteristics, ii . the sample median. *The Annals of Mathematical Statistics*, 63:627–635.

Chambers, J., Cleveland, W., Kleiner, B., and Tukey, P. (1983). *Graphical Methods of Data Analysis*. Chapman and Hall.

Chen, G. and Balakrishnan, N. (1995). A general purpose approximate goodness-of-fit test. *Journal of Quality Technology*, 27:154–161.

Gurvich, M. R., DiBenedetto, A. T., and Ranade, S. V. (1997). A new statistical distribution for characterizing the random strength of brittle materials. *Journal of Materials Science*, 32:2559–2564.

Hatke, M. A. (1949). A certain cumulative probability function. *The Annals of Mathematical Statistics*, 20:461–463.

Hoskings, J. R. M. (1990). L-moments: Analysis and estimation of distributions using linear combinations of order statistics. *Journal of the Royal Statistical Society*, B 52:105–124.

Huang, S. and Oluyede, B. O. (2014). Exponentiated kumaraswamy-dagum distribution with applications to income and lifetime data. *Journal of Statistical Distributions and Applications*, 1(8).

Lai, C. D., Xie, M., and Murthy, D. N. P. (2003). A modified weibull distribution. *IEEE Transactions on Reliability*, 52(1):33–37.

Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated weibull family for analyzing bathtub failure-rate data. *IEEE Transactions on Reliability*, 42:299–302.

Nadarajah, S., Cordeiro, G. M., and Ortega, E. M. M. (2011). General results for the beta-modified weibull distribution. *Journal of Statistical Computation and Simulations*, 81(10):1211–1232.

Oluyede, B. O., Huang, S., and Pararai, M. (2014). A new class of generalized dagum distribution with applications to income and lifetime data. *Journal of Statistical and Econometric Methods*, 3(2):125–151.

Oluyede, B. O., Huang, S., and Yang, T. (2015). A new class of generalized modified weibull distribution with applications. *Austrian Journal of Statistics*, 44:45–68.

Percontini, A., Blas, B., and Cordeiro, G. M. (2013). The beta weibull poisson distribution. *Chilean Journal of Statistics*, 4(2):3–26.

Pham, H. and Lai, C. D. (2007). On recent generalizations of the weibull distribution. *IEEE Transaction on Reliability*, 56:454–458.

Rényi, A. (1960). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, volume 1, pages 547 – 561.

Rodrigues, R. N. (1977). A guide to the burr type xii distributions. *Biometrika*, 64:129–134.

Santos Silva, J. M. and Tenreyro, S. (2010). On the existence of maximum likelihood estimates in poisson regression. *Economics Letters*, 107:310–312.

Seregin, A. (2010). Uniqueness of the maximum likelihood estimator for k-monotone densities. *Proceedings of the American Mathematical Society*, 138:4511–4515.

Soliman, A. A. (2005). Estimation of parameters of life from progressively censored data using burr-xii model. *IEEE Transactions on Reliability*, 54:34–42.

Tadikamalla, P. R. (1980). A look at the burr and related distributions. *International Statistical Reviews*, 48:337–344.

Team, R. D. C. (2011). *A Language and Environment for Statistical Computing.* R

Foundation for Statistical Computing, Vienna, Austria.

Weibull, W. A. (1951). Statistical distribution function of wide applicability. *Journal of Applied Mechanics*, 18:293–296.

Xia, J., Mi, J., and Zhou, Y. Y. (2009). On the existence and uniqueness of the maximum likelihood estimators of normal and log-normal population parameters with grouped data. *Journal of Probability and Statistics*, Article id 310575:16 pages.

Zhou, C. (2009). Existence and consistency of the maximum likelihood estimator for the extreme index. *Journal Multivariate Analysis*, 100:794–815.

Appendix

R Algorithms

```
#### define BMW pdf
BMW_pdf=function(c,k,alpha,beta,lambda,x){
(1+x^c)^(-k-1)*(exp(-alpha*x^beta*exp(lambda*x)))*
(k*c*x^(c-1)+(1+x^c)*alpha*exp(lambda*x)*x^(beta-1)*(beta+lambda*x))
}

#### define BMW cdf
BMW_cdf=function(c,k,alpha,beta,lambda,x){
  1-(1+x^c)^(-k)*(exp(-alpha*(x^beta)*exp(lambda*x)))
}

#### define BMW quantile
BMW_quantile=function(c,k,alpha,beta,lambda,u){
f=function(x){
k*log(1+x^c)+alpha*x^beta*exp(lambda*x)+log(1-u)
}
x=uniroot(f,c(0,100))$root
return(x)
}

#### define BMW hazard
BMW_hazard=function(y,c,k,alpha,beta,lambda){
BMW_pdf(y,c,k,alpha,beta,lambda)/(1-BMW_cdf(y,c,k,alpha,beta,lambda))
}

#### define moments of BMW
BMW_moments=function(c,k,alpha,beta,lambda,r){
f=function(y,c,k,alpha,beta,lambda,r){
(y^r)*(BMW_pdf(y,c,k,alpha,beta,lambda))
```

```
}
y=integrate(f,lower=0,upper=Inf,subdivisions=1000000,
c=c,k=k,alpha=alpha,beta=beta,lambda=lambda,r=r)
return(y$value)
}

#### define simulation process of BMW
BMW_simulation=function(size=c(25,50,100,200,400,800),samp,par1){
Mean=vector()
RMSE=vector()
Bias=vector()
for(iter_size in 1:length(size)){
coef1=matrix(NA,samp,5)
colnames(coef1)=c('c','k','alpha','beta','lambda')
for(nsamp in 1:samp){
tryCatch(
{
x1_BMW=NULL
q=runif(size[iter_size],0,1)
x1=sapply(q,BMW_quantile,
c=par1[1],k=par1[2],alpha=par1[3],beta=par1[4],lambda=par1[5])
### BMW for x1
x1_BMW<-mle2(BMW_neglogl,
start=list(c=par1[1],k=par1[2],alpha=par1[3],beta=par1[4],lambda=par1[5]),
method="L-BFGS-B",data=list(x=x1),
lower=c(c=0,k=0,alpha=0,beta=0,lambda=0),
upper=c(c=Inf,k=Inf,alpha=Inf,beta=Inf,lambda=Inf),use.ginv=TRUE)
coef1[nsamp,]=coef(x1_BMW)
},error=function(e){}
)
}
Mean[length(size)*(0:4)+iter_size]=apply(coef1,2,mean,na.rm=TRUE)
RMSE[length(size)*(0:4)+iter_size]=apply((coef1-matrix(rep(par1,nsamp),
ncol=5,byrow=T))^2,2,function(x){sqrt(mean(x,na.rm=TRUE))})
  }
Bias=as.vector(sapply(1:5,function(x){Bias[(length(size)*(x-1)+1):
(length(size)*x)]=Mean[(length(size)*(x-1)+1):(length(size)*x)]-par1[x]}))
samplesize=as.vector(t(mapply(rep,size,5)))
return(cbind(samplesize,Mean,RMSE,Bias))
}

# funciton to calculate Cramer-von and Anderson-Darling statistics
WABMW=function(fit,x,sign){
if(!is.null(get0(paste0(sign,'_cdf')))){
```

```
u=get(paste0(sign,'_cdf'))
(coef(fit)[1],coef(fit)[2],coef(fit)[3],coef(fit)[4],coef(fit)[5],x)
  }
else{
u=BMW_cdf(coef(fit)[1],coef(fit)[2],coef(fit)[3],coef(fit)[4],coef(fit)[5],x)
}
y=qnorm(u)
v=pnorm((y-mean(y))/(as.numeric(sqrt(var(y)))))
W2=sum((v-((2*seq(1,length(x))-1)/(2*length(x))))^2)+1/(12*length(x))
A2=-length(x)-sum(((2*seq(1,length(x))-1)*(log(v))+
((2*length(x)+1-2*seq(1,length(x)))*(log(1-v))))/length(x)
W=W2*(1+0.5/length(x))
A=A2*(1+0.75/length(x)+2.25/length(x)/length(x))
result=list(W=W,A=A)
return(result)
}

# define the function to calculate SS value
SS_BMW=function(fit,x,sign){
pp=(seq(1,length(x),1)-0.375)/(length(x)+0.25)
if(!is.null(get0(paste0(sign,'_cdf')))){
op=get(paste0(sign,'_cdf'))(coef(fit)[1],coef(fit)[2],coef(fit)[3],
coef(fit)[4],coef(fit)[5],sort(x))
}
else{
op=BMW_cdf(coef(fit)[1],coef(fit)[2],coef(fit)[3],coef(fit)[4]
,coef(fit)[5],sort(x))
}
 SS=0
  for(i in 1:length(x)){
    SS=SS+(op[i]-pp[i])^2
  }
  result=list(pp=pp,op=op,SS=SS)
  return(result)
}

# BMW Fit
mysample_BMW<-mle2(BMW_neglogl,
start=list(c=1,k=1,alpha=1,beta=1,lambda=1),
method="L-BFGS-B",data=list(x=mysample),
lower=c(c=0,k=0,alpha=0,beta=0,lambda=0),
upper=c(c=Inf,k=Inf,alpha=Inf,beta=Inf,lambda=Inf),use.ginv=TRUE)
```