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Reliability estimation for the generalised inverted Rayleigh distribution

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In this article, we consider a generalized inverted Rayleigh distribution. The maximum likelihood estimators (MLEs) of scale and shape parameters are obtained. Also, we compute MLE of reliability function. Asymptotic confidence intervals for the parameters and reliability function are constructed. Simulation study is conducted to investigate performance of MLEs and confidence intervals. An illustration with real data is provided.

keywords: Generalized inverted Rayleigh distribution, MLE, Reliability, Confidence interval.

1 Introduction

A generalised inverted scale family of distributions is important to analyse lifetime data. Potdar and Shirke (2013) introduced generalized inverted scale family of distributions. Generalised inverted exponential distribution, generalised inverted Rayleigh distribution, generalised inverted half-logistic distribution etc. are some of the members of this family of distributions. Lin et al. (1989) and Dey (2007) studied inverted exponential distribution (IED) to analyse lifetime data. Singh et al. (2013) discussed Bayes estimators of the parameters and reliability function of IED using Type-I and Type-II censored samples.

Generalised exponential distribution was introduced by Gupta and Kundu (1999), Gupta and Kundu (2001a) and Gupta and Kundu (2001b). Abouammoh and Alshingiti (2009) generalised inverted exponential distribution (GIED) by introducing a shape parameter. They discussed statistical and reliability properties of GIED. They also studied

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estimation of both scale and shape parameters. Krishna and Kumar (2013) used type-II censored data to estimate reliability characteristics of GIED. They proposed maximum likelihood estimation and least square estimation procedures. Potdar and Shirke (2014) discussed inference for the scale family of lifetime distributions based on progressively censored data. Dey and Dey (2014) and Dube et al. (2015) studied GIED for progressively censored data. Dey and Pradhan (2014) discussed GIED for hybrid censored data. Kumar and Garg (2014) estimated parameters of generalized inverted Rayleigh distribution (GIRD) based on randomly censored samples. Bakoban and Abubakar (2015) proved applicability of generalized inverted Rayleigh distribution by considering two real data sets. Bakoban and Abubakar (2015) have studied inference of generalized inverted Rayleigh distribution with real data applications, whereas, we have studied reliability estimation for the GIRD. We have also studied the performance of the methods by simulation study.

In this article, the generalised inverted Rayleigh distribution is proposed by introducing a shape parameter to the inverted Rayleigh distribution. Inferential procedures are considered for both the parameters and reliability function. In Section 2, we introduce the model and obtain maximum likelihood estimators (MLEs) for scale and shape parameters. We also computed MLE of reliability function in same Section. Expression for elements of Fisher information matrix are derived in Section 3. Asymptotic confidence intervals (CIs) for scale and shape parameters and reliability function are also discussed. In Section 4, the performance of the MLEs and CIs are investigated through simulation study. Real data application is discussed in Section 5. Conclusions are reported in Section 6.

2 Model and maximum likelihood estimation

Consider generalised inverted Rayleigh distribution with scale parameter λ and shape parameter α. Let X be a generalised inverted Rayleigh random variable. The cdf and pdf of X are respectively given as

$$F_X(x;\lambda,\alpha) = 1 - \left[1 - e^{-(1/(\lambda x))^2}\right]^{\alpha} \qquad x \geq 0,\ \alpha,\ \lambda > 0. \tag{1}$$

$$f_X(x;\lambda,\alpha) = \frac{2\alpha}{\lambda^2 x^3}\, e^{-(1/(\lambda x))^2}\left[1 - e^{-(1/(\lambda x))^2}\right]^{\alpha-1} \qquad x \geq 0,\ \alpha,\ \lambda > 0. \tag{2}$$

In the following, we discuss method of finding MLEs of α and λ as well as MLE of reliability function R(t).

- **Maximum likelihood estimation α and λ**

Suppose, we observe lifetimes of n units having lifetime distribution given by equation (2). The likelihood function for the observed data is

$$l(\underline{x}|\lambda,\alpha) = \prod_{i=1}^{n} f_X\left(x_i;\lambda,\alpha\right).$$

$$l(\underline{x}|\lambda,\alpha) = \prod_{i=1}^{n} \frac{2\alpha}{\lambda^2 x_i^3}\, e^{-(1/(\lambda x_i))^2} \left[1 - e^{-(1/(\lambda x_i))^2}\right]^{\alpha-1}.$$

Then log-likelihood function is,

$$L = nlog(2\alpha) - 2nlog(\lambda) - 3\sum_{i=1}^{n} log(x_i) - \frac{1}{\lambda^2}\sum_{i=1}^{n}\frac{1}{x_i^2} + (\alpha-1)\sum_{i=1}^{n} log\left[1 - e^{-(1/(\lambda x_i))^2}\right]. \quad (3)$$

When λ is known, the MLE of α is the solution of $\frac{dL}{d\alpha} = 0$. Thus MLE of α is solution of the equation

$$\frac{n}{\alpha} + \sum_{i=1}^{n} log\left[1 - e^{-(1/(\lambda x_i))^2}\right] = 0. \quad (4)$$

Therefore, when λ is known the MLE of α is

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} log\left[1 - e^{-(1/(\lambda x_i))^2}\right]} \quad (5)$$

Similarly, when α is known, the MLE of λ is the solution of $\frac{dL}{d\lambda} = 0$. Thus MLE of λ is solution of the equation

$$-n\lambda^2 - \sum_{i=1}^{n}\frac{1}{x_i^2} + (\alpha-1)\sum_{i=1}^{n}\frac{1}{x_i^2\left[e^{(1/(\lambda x_i))^2} - 1\right]} = 0. \quad (6)$$

When both parameters α and λ are unknown, the MLEs of α and λ are the solutions of the two simultaneous equations (4) and (6). We substitute $\hat{\alpha}$ given in equation (5) into equation (6) so as to get a nonlinear equation in λ only, which does not have closed form solution. Therefore, we use Newton-Raphson method to compute $\hat{\lambda}$. In Newton-Raphson method, we have to choose initial value of λ. We use least square estimate as an initial value of λ. The estimate of the parameters can be obtained by least square fit of simple linear regression. Empirical distribution function is computed on the line of Escobar and Meeker (1998).

$$y_i = \beta x_i \quad \text{with} \quad \beta = \lambda.$$

$$y_i = \frac{1}{\sqrt{-log\left(1 - \left[\frac{\left(1-\hat{F}(x_{i-1})\right)^{1/\alpha} + \left(1-\hat{F}(x_i)\right)^{1/\alpha}}{2}\right]\right)}} \quad for\ i = 1, 2, ..., n.$$

$$\hat{F}(x_0) = 0.$$

The least square estimates of λ is given by

$$\hat{\lambda}_0 = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}. \quad (7)$$

We use $\hat{\lambda}_0$ as an initial value of λ to obtain the MLE $\hat{\lambda}$ using Newton-Raphson method. Then we obtain $\hat{\alpha}$ using equation (5). We use these MLEs $\hat{\alpha}$ and $\hat{\lambda}$ to obtain MLE of reliability function $\hat{R}(t)$.

- **Maximum likelihood function of reliability function R(t)**

The probability that a system survives until time t is called the reliability of the system at time t and it is denoted by $R(t)$. Thus, reliability function at time t is

$$R(t) = P(X > t) = 1 - F(t). \quad (8)$$

For generalised inverted Rayleigh distribution, reliability function is

$$R(t) = \left[1 - e^{-(1/(\lambda t))^2}\right]^{\alpha} \qquad t \geq 0,\ \alpha > 0,\ \lambda > 0.$$

Let $\hat{\alpha}$ and $\hat{\lambda}$ are MLEs of α and λ respectively. Using invariance property of MLE, the MLE of $R(t)$ is

$$\hat{R}(t) = \left[1 - e^{-(1/(\hat{\lambda} t))^2}\right]^{\hat{\alpha}} \qquad t \geq 0. \quad (9)$$

Equation (9) describes MLE of R(t).

3 Interval estimation

Now, we consider Fisher information in the following.

- **Fisher information matrix**

Log-likelihood function L is described by equation (3). Now, Fisher information matrix of $\underline{\theta} = (\alpha, \lambda)'$ is

$$I(\underline{\theta}) = -E \begin{bmatrix} \frac{d^2L}{d\alpha^2} & \frac{d^2L}{d\alpha d\lambda} \\ \\ \frac{d^2L}{d\lambda d\alpha} & \frac{d^2L}{d\lambda^2} \end{bmatrix}, \quad (10)$$

$$\frac{d^2L}{d\alpha^2} = -\frac{n}{\alpha^2},$$

$$\frac{d^2L}{d\alpha d\lambda} = \frac{d^2L}{d\lambda d\alpha} = -\frac{2}{\lambda^3} \sum_{i=1}^{n} \frac{e^{-(1/(\lambda x_i))^2}}{x_i^2 \left(1 - e^{-(1/(\lambda x_i))^2}\right)},$$

$$\frac{d^2L}{d\lambda^2} = \frac{2n}{\lambda^2} - \frac{6}{\lambda^4} \sum_{i=1}^{n} \frac{1}{x_i^2} - \frac{4(\alpha - 1)}{\lambda^6} \sum_{i=1}^{n} \frac{e^{(1/(\lambda x_i))^2}}{x_i^4 \left[e^{(1/(\lambda x_i))^2} - 1\right]^2}$$

$$+\frac{6(\alpha-1)}{\lambda^4}\sum_{i=1}^{n}\frac{1}{x_i^2\left[e^{(1/(\lambda x_i))^2}-1\right]^2}. \tag{11}$$

To obtain expectation of the above expression is a tedious job. Therefore, we use the observed Fisher information matrix, which is given as,

$$I(\hat{\underline{\theta}}) = \begin{bmatrix} -\frac{d^2L}{d\alpha^2} & -\frac{d^2L}{d\alpha d\lambda} \\ \\ -\frac{d^2L}{d\lambda d\alpha} & -\frac{d^2L}{d\lambda^2} \end{bmatrix}_{\alpha=\hat{\alpha},\lambda=\hat{\lambda}.} \tag{12}$$

The asymptotic variance-covariance matrix of the MLEs is the inverse of $I(\hat{\underline{\theta}})$. After obtaining inverse matrix, we get variance of $\hat{\alpha}$, variance of $\hat{\lambda}$ and covariance between $\hat{\alpha}$ and $\hat{\lambda}$. We use these variances to obtain confidence intervals for α and λ respectively.

- **Confidence interval for α and λ**

Assuming asymptotic normal distribution of the MLEs, CIs for α and λ are constructed. Let $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of α and λ respectively. Let $\hat{\sigma}^2(\hat{\alpha})$ and $\hat{\sigma}^2(\hat{\lambda})$ is the estimated variances of $\hat{\alpha}$ and $\hat{\lambda}$ respectively. Therefore, $100(1-\xi)\%$ asymptotic CIs for α and λ are respectively given by,

$$\left(\hat{\alpha}-\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}, \quad \hat{\alpha}+\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}\right)$$

$$\text{and} \quad \left(\hat{\lambda}-\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}, \quad \hat{\lambda}+\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}\right), \tag{13}$$

where $\tau_{\xi/2}$ is the upper $100(1-\xi/2)^{th}$ percentile of standard normal distribution. Escobar and Meeker (1998) reported that the asymptotic CI based on log(MLE) has better coverage probability. An approximate $100(1-\xi)\%$ CI for $\log(\alpha)$ and $\log(\lambda)$ are

$$\left(log(\hat{\alpha})-\tau_{\xi/2}\sqrt{\hat{\sigma}^2(log(\hat{\alpha}))}, \quad log(\hat{\alpha})+\tau_{\xi/2}\sqrt{\hat{\sigma}^2(log(\hat{\alpha}))}\right)$$

$$\text{and} \quad \left(log(\hat{\lambda})-\tau_{\xi/2}\sqrt{\hat{\sigma}^2(log(\hat{\lambda}))}, \quad log(\hat{\lambda})+\tau_{\xi/2}\sqrt{\hat{\sigma}^2(log(\hat{\lambda}))}\right), \tag{14}$$

where $\hat{\sigma}^2(log(\hat{\alpha}))$ is the estimated variance of $log(\hat{\alpha})$ which is approximately obtained by $\hat{\sigma}^2(log(\hat{\alpha})) \approx \frac{\hat{\sigma}^2(\hat{\alpha})}{\hat{\alpha}^2}$. $\hat{\sigma}^2(log(\hat{\lambda}))$ is the estimated variance of $log(\hat{\lambda})$ which is approximately obtained by $\hat{\sigma}^2(log(\hat{\lambda})) \approx \frac{\hat{\sigma}^2(\hat{\lambda})}{\hat{\lambda}^2}$. Hence, an approximate $100(1-\xi)\%$ CIs for α and λ are respectively given by,

$$\left(\hat{\alpha}\, e^{\left(-\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}}{\hat{\alpha}}\right)}, \quad \hat{\alpha}\, e^{\left(\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\alpha})}}{\hat{\alpha}}\right)}\right) \text{ and } \left(\hat{\lambda}\, e^{\left(-\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}}{\hat{\lambda}}\right)}, \quad \hat{\lambda}\, e^{\left(\frac{\tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{\lambda})}}{\hat{\lambda}}\right)}\right). \tag{15}$$

Equations (13) and (15) provides CIs for α and λ. Now, we discuss CIs for reliability function.

- **Confidence interval for R(t)**

Let $\hat{R}(t)$ is the MLE of reliability function $R(t)$ and $\sigma^2(\hat{R}(t))$ is the variance of $\hat{R}(t)$. To construct confidence interval for $R(t)$ based on $\hat{R}(t)$, we have to compute variance of $\hat{R}(t)$; It is obtained by using the following lemma.

Lemma (3.1) : Let $X_1,\ X_2,\ ....,\ X_k$ be the random variables with means $\theta_1,\ \theta_2,\ .....,\ \theta_k$, and define $X = (X_1,\ X_2,\ ......,\ X_k)$ and $\theta = (\boldsymbol{\theta_1},\ \boldsymbol{\theta_2},\ ......,\ \boldsymbol{\theta_k})$. Suppose $h(X)$ is differentiable function then variance of $h(X)$ is approximated as

$$V\left[h(X)\right] \approx \sum_{i=1}^{k} \left[h_i'(\theta)\right]^2 V(X_i)\ +\ 2 \sum_{i=1, i>j}^{k} \sum_{j=1}^{k} h_i'(\theta) h_j'(\theta) COV(X_i, X_j).$$

$$\text{where}\ \ h_i'(\theta) = \frac{d}{dx_i} h(x)|_{x_1=\theta_1,\ldots,x_k=\theta_k}.$$

Proof : Please refer to Casella and Berger (2002) pp.241-242.

□

Using the above lemma, we obtain $\sigma^2(\hat{R}(t))$. Hence, $100(1-\xi)\%$ asymptotic CI for $R(t)$ is

$$\left(\hat{R}(t) - \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{R}(t))},\qquad \hat{R}(t) + \tau_{\xi/2}\sqrt{\hat{\sigma}^2(\hat{R}(t))}\right) \tag{16}$$

where $\hat{\sigma}^2(\hat{R}(t))$ is an ML estimate of $\sigma^2(\hat{R}(t))$. Using Lemma (3.1) estimated variance of $\hat{R}(t)$ is given by

$$\begin{aligned}\hat{\sigma}^2(\hat{R}(t)) = \hat{V}\left(\hat{\alpha}\right) & \left[\left(1 - e^{-(1/(\hat{\lambda}t))^2}\right)^{\alpha} log\left(1 - e^{-(1/(\hat{\lambda}t))^2}\right)\right]^2 \\ & + \hat{V}\left(\hat{\lambda}\right)\left[\frac{2\hat{\alpha}}{\hat{\lambda}^3 t^2}\ e^{-(1/(\hat{\lambda}t))^2}\left(1 - e^{-(1/(\hat{\lambda}t))^2}\right)^{\hat{\alpha}-1}\right]^2 \\ & - \frac{4\ \hat{\alpha}\ C\hat{O}V\left(\hat{\alpha}, \hat{\lambda}\right)}{\hat{\lambda}^3 t^2}\ e^{(-1/(\hat{\lambda}t))^2}\ \left[1 - e^{-(1/(\hat{\lambda}t))^2}\right]^{\hat{\alpha}} \\ & log\left[1 - e^{-(1/(\hat{\lambda}t))^2}\right]\ \left[1 - e^{-(1/(\hat{\lambda}t))^2}\right]^{(\hat{\alpha}-1)}.\end{aligned} \tag{17}$$

Using ML estimates of $\sigma^2(\hat{R}(t))$ from equation (17), we obtain $100(1-\xi)\%$ asymptotic CI for $R(t)$ from equation (16).

4 Simulation Study

A simulation study is carried out to study the performance of MLEs of α and λ when the generalised inverted Rayleigh distribution is lifetime distribution. We consider bias and mean square error (MSE) to compare MLEs. Asymptotic CIs based on MLEs and log transformed MLEs are compared with their confidence levels.

Simulation is carried out for (α, λ)=(0.5, 0.5), (0.5, 1), (1, 0.5), (1, 1), (1.5, 0.5) with sample size n= 20, 30, ..., 100. Newton-Raphson method is used to compute MLE of λ. For each sample size, 10,000 sets of samples are generated. The bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and reliability estimator $\hat{R}(t)$ are displayed in the Tables 1 to 5 for various values of α and λ. Further, the coverage probability and lengths of 95% CIs based on MLE and log transformed MLE of α and λ are given in Tables 6 to 10. Also, the coverage probability and lengths of the CIs of R(t) based on MLE are given in Tables 6 to 10.

Table 1: Bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{R}(t)$ when α=0.5, λ=0.5 and t=2.

	MLE of α		MLE of λ		MLE of $R(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.0655	0.0340	-0.0183	0.0081	0.0088	0.0051
30	0.0411	0.0183	-0.0122	0.0055	0.0058	0.0035
40	0.0298	0.0124	-0.0089	0.0041	0.0049	0.0025
50	0.0218	0.0092	-0.0058	0.0033	0.0034	0.0020
60	0.0202	0.0074	-0.0059	0.0027	0.0029	0.0017
80	0.0140	0.0051	-0.0048	0.0020	0.0022	0.0013
100	0.0120	0.0040	-0.0038	0.0017	0.0017	0.0010

Table 2: Bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{R}(t)$ when α=0.5, λ=1 and t=2.

	MLE of α		MLE of λ		MLE of $R(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.0651	0.0344	-0.0355	0.0328	-0.0035	0.0081
30	0.0425	0.0186	-0.0233	0.0220	-0.0004	0.0050
40	0.0309	0.0122	-0.0168	0.0160	-0.0002	0.0037
50	0.0230	0.0088	-0.0125	0.0132	0.0001	0.0029
60	0.0199	0.0071	-0.0118	0.0109	-0.0009	0.0024
80	0.0145	0.0052	-0.0074	0.0082	-0.0009	0.0018
100	0.0121	0.0041	-0.0078	0.0066	0	0.0014

Table 3: Bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{R}(t)$ when α=1, λ=0.5 and t=2.

	MLE of α		MLE of λ		MLE of $R(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.1567	0.2033	-0.0124	0.0055	0.0049	0.0079
30	0.0991	0.0979	-0.0087	0.0036	0.0032	0.0050
40	0.0760	0.0649	-0.0074	0.0026	0.0033	0.0037
50	0.0549	0.0486	-0.0052	0.0022	0.0025	0.0029
60	0.0474	0.0380	-0.0047	0.0018	0.0017	0.0024
80	0.0355	0.0259	-0.0032	0.0014	0.0014	0.0018
100	0.0274	0.0200	-0.0025	0.0011	0.0010	0.0014

Table 4: Bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{R}(t)$ when α=1, λ=1 and t=2.

	MLE of α		MLE of λ		MLE of $R(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.1560	0.2027	-0.0269	0.0215	-0.0082	0.0056
30	0.0983	0.0978	-0.0183	0.0142	-0.0056	0.0039
40	0.0780	0.0681	-0.0144	0.0109	-0.0032	0.0029
50	0.0522	0.0458	-0.0095	0.0086	-0.0032	0.0023
60	0.0467	0.0383	-0.0097	0.0071	-0.0026	0.0019
80	0.0373	0.0268	-0.0076	0.0054	-0.0024	0.0014
100	0.0293	0.0204	0.0062	0.0043	-0.0022	0.0011

Table 5: Bias, MSE of $\hat{\alpha}$, $\hat{\lambda}$ and $\hat{R}(t)$ when α=1.5, λ=0.5 and t=2.

	MLE of α		MLE of λ		MLE of $R(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
20	0.2899	0.6400	-0.0131	0.0046	0.0020	0.0085
30	0.1736	0.2870	-0.0089	0.0030	0.0009	0.0054
40	0.1246	0.1753	-0.0059	0.0022	0.0004	0.0040
50	0.0970	0.1270	-0.0052	0.0018	-0.0002	0.0031
60	0.0877	0.1068	-0.0049	0.0015	0.0004	0.0026
80	0.0612	0.0699	-0.0033	0.0011	0.0002	0.0019
100	0.0453	0.0529	-0.0021	0.0009	0.0001	0.0015

Table 6: Coverage probability(CP) and lengths of 95 % CIs for α , λ and $R(t)$ when α=0.5, λ=0.5 and t=2.

	α				λ				$R(t)$	
	CI based on MLE		CI based on Log-MLE		CI based on MLE		CI based on Log-MLE			
n	CP	Length	CP	Length	CP	Length	CP	Length	CP	Length
20	0.9648	0.5989	0.9344	0.6275	0.8863	0.3371	0.9161	0.3442	0.9049	0.2660
30	0.9591	0.4636	0.9386	0.4780	0.9052	0.2801	0.9245	0.2840	0.9165	0.2207
40	0.9552	0.3914	0.9420	0.4004	0.9169	0.2447	0.9341	0.2472	0.9231	0.1926
50	0.9526	0.3439	0.9422	0.3502	0.9266	0.2207	0.9388	0.2226	0.9360	0.1733
60	0.9530	0.3127	0.9452	0.3174	0.9268	0.2013	0.9375	0.2027	0.9360	0.1586
80	0.9514	0.2670	0.9467	0.2700	0.9353	0.1750	0.9435	0.1759	0.9358	0.1378
100	0.9529	0.2377	0.9472	0.2398	0.9323	0.1568	0.9410	0.1574	0.9427	0.1236

Table 7: Coverage probability(CP) and lengths of 95 % CIs for α , λ and $R(t)$ when α=0.5, λ=1 and t=2.

	α				λ				$R(t)$	
	CI based on MLE		CI based on Log-MLE		CI based on MLE		CI based on Log-MLE			
n	CP	Length	CP	Length	CP	Length	CP	Length	CP	Length
20	0.9640	0.5987	0.9369	0.6273	0.8868	0.6753	0.9127	0.6895	0.9202	0.3232
30	0.9592	0.4649	0.9368	0.4793	0.9044	0.5602	0.9228	0.5680	0.9347	0.2649
40	0.9576	0.3923	0.9421	0.4013	0.9164	0.4893	0.9335	0.4945	0.9370	0.2298
50	0.9597	0.3447	0.9480	0.3510	0.9221	0.4405	0.9369	0.4442	0.9395	0.2057
60	0.9573	0.3125	0.9450	0.3172	0.9255	0.4025	0.9390	0.4053	0.9385	0.1879
80	0.9517	0.2672	0.9456	0.2703	0.9330	0.3506	0.9399	0.3524	0.9408	0.1629
100	0.9506	0.2377	0.9441	0.2398	0.9320	0.3136	0.9404	0.3149	0.9441	0.1457

Table 8: Coverage probability(CP) and lengths of 95 % CIs for α , λ and $R(t)$ when α=1, λ=0.5 and t=2.

	α				λ				$R(t)$	
	CI based on MLE		CI based on Log-MLE		CI based on MLE		CI based on Log-MLE			
n	CP	Length	CP	Length	CP	Length	CP	Length	CP	Length
20	0.9654	1.3843	0.9373	1.4699	0.8988	0.2772	0.9194	0.2811	0.9170	0.3230
30	0.9636	1.0560	0.9410	1.0975	0.9168	0.2286	0.9316	0.2306	0.9326	0.2655
40	0.9592	0.8894	0.9473	0.9151	0.9269	0.1985	0.9360	0.1999	0.9360	0.2305
50	0.9552	0.7764	0.9397	0.7941	0.9274	0.1789	0.9363	0.1798	0.9354	0.2064
60	0.9552	0.7021	0.9440	0.7154	0.9347	0.1634	0.9405	0.1642	0.9446	0.1887
80	0.9565	0.5992	0.9494	0.6077	0.9364	0.1420	0.9402	0.1425	0.9433	0.1636
100	0.9528	0.5307	0.9481	0.5366	0.9387	0.1273	0.9441	0.1277	0.9463	0.1464

Table 9: Coverage probability(CP) and lengths of 95 % CIs for α , λ and $R(t)$ when α=1, λ=1 and t=2.

	α				λ				$R(t)$	
	CI based on MLE		CI based on Log-MLE		CI based on MLE		CI based on Log-MLE			
n	CP	Length	CP	Length	CP	Length	CP	Length	CP	Length
20	0.9703	1.3825	0.9422	1.4680	0.9006	0.5527	0.9234	0.5603	0.9022	0.2799
30	0.9613	1.0552	0.9418	1.0966	0.9197	0.4567	0.9307	0.4609	0.9175	0.2322
40	0.9597	0.8918	0.9406	0.9176	0.9211	0.3973	0.9330	0.4000	0.9280	0.2031
50	0.9576	0.7738	0.9437	0.7915	0.9308	0.3582	0.9372	0.3601	0.9272	0.1823
60	0.9519	0.7016	0.9435	0.7148	0.9336	0.3268	0.9380	0.3283	0.9341	0.1669
80	0.9540	0.6004	0.9451	0.6089	0.9363	0.2837	0.9433	0.2847	0.9371	0.1451
100	0.9553	0.5319	0.9483	0.5378	0.9364	0.2542	0.9431	0.2549	0.9403	0.1300

Table 10: Coverage probability(CP) and lengths of 95 % CIs for α , λ and $R(t)$ when α=1.5, λ=0.5 and t=2.

	α				λ				$R(t)$	
	CI based on MLE		CI based on Log-MLE		CI based on MLE		CI based on Log-MLE			
n	CP	Length	CP	Length	CP	Length	CP	Length	CP	Length
20	0.9657	2.3456	0.9371	2.5221	0.8996	0.2502	0.9177	0.2530	0.9239	0.3362
30	0.9645	1.7491	0.9424	1.8309	0.9138	0.2068	0.9279	0.2083	0.9328	0.2757
40	0.9596	1.4556	0.9442	1.5052	0.9296	0.1804	0.9386	0.1814	0.9383	0.2391
50	0.9562	1.2732	0.9479	1.3074	0.9294	0.1618	0.9382	0.1625	0.9403	0.2141
60	0.9556	1.1529	0.9444	1.1786	0.9335	0.1477	0.9395	0.1482	0.9457	0.1956
80	0.9578	0.9769	0.9523	0.9929	0.9362	0.1285	0.9412	0.1288	0.9442	0.1695
100	0.9573	0.8621	0.9516	0.8734	0.939	0.1153	0.9417	0.1155	0.9468	0.1517

Bias, MSE of MLEs of various values of α, λ and reliability function R(t) are reported in Tables 1 to 5. For small values of α and λ, MLEs show good performance. The bias and MSE of the estimates are relatively smaller for small value of parameter. The Bias and MSE of estimates of α are not affected due to different values of λ. Similarly, bias and MSE of estimates of λ are not affected for different values of α. The bias and MSE of the MLE of reliability function R(t) shows better performance for all values of α and λ. The bias and MSE of the MLEs decrease with increase in sample size n.

Coverage probability and lengths of 95% CIs of α, λ and reliability function R(t) for various values of α and λ are reported in Tables 6 to 10. When sample size is small, coverage probability and lengths of CIs of α based on MLE are moderately large. Increase in sample size reduces the lengths of CIs and coverage probability approach to nominal levels. Coverage probability of CIs of α based on log transformed MLE increase and lengths of CIs decrease as sample size increases. When sample size is small, CIs of λ based on MLE show poor coverage probability as compared to CIs based on log transformed MLE. When sample size is large coverage probability of CIs of λ based on log transformed MLE are close to nominal levels. In both MLE and log-transformed MLE case, increase in sample size considerably reduces the lengths of CIs of λ and increases the coverage probability.

Coverage probability of CIs of reliability function R(t) is small and lengths of CIs are moderately large for small sample size. Increase in sample size reduces the lengths of CIs of R(t) and coverage probability approaches to nominal levels. Length of CIs of R(t) are small for smaller values of α and λ, whereas coverage probability of CIs of R(t) does not depend on value of α and λ.

5 Real data application

Consider following real data which represents the number of million revolutions before failure for each of 23 ball bearings in a life test given by Lawless (2011).

17.88, 28.92, 33, 41.52, 42.12, 45.6, 48.4, 51.84, 51.96, 54.12, 55.56, 67.8, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.4.

According to Kolmogrov- Smirnov (K-S) test criterion generalised inverted Rayleigh distribution satisfactory fits to this data as compared to exponential distribution. For generalised inverted Rayleigh distribution K-S statistic is 0.09928 whereas for exponential distribution it is 0.26224. For this real data set, we obtain MLEs of α, λ and reliability function R(t) for time period t=40. We construct CIs of α and λ based on MLE and log transformed MLE. The MLE and CIs of α and λ and their lengths are displayed in Table 11 and 12 respectively. We also construct CI for reliability function. MLE and CIs of reliability function with their lengths are presented in Table 13.

Table 11: MLE, Confidence intervals for α and their lengths

MLE	Based on MLE		Based on log-MLE	
	90% CI	95% CI	90% CI	95% CI
1.0320	(0.5429, 1.5211) Length=0.9782	(0.4492, 1.6148) Length=1.1656	(0.6425, 1.6577) Length=1.0152	(0.5867, 1.8153) Length=1.2286

Table 12: MLE, Confidence intervals for λ and their lengths

MLE	Based on MLE		Based on log-MLE	
	90% CI	95% CI	90% CI	95% CI
0.0209	(0.0160, 0.0258) Length=0.0098	(0.0151, 0.0268) Length=0.0117	(0.0166, 0.0265) Length=0.0099	(0.0158, 0.0277) Length=0.0119

Table 13: MLE, Confidence intervals for R(t) and their lengths

MLE	90% CI	95% CI
0.7533	(0.3458, 1.1608)	(0.2677, 1.2389)
	Length=0.8150	Length=0.9712

Method of MLE and confidence interval based on MLE of reliability function gives best performance for real data.

6 Conclusion

This article considers generalised inverted Rayleigh distribution having scale and shape parameters. Point and interval estimation procedures for the parameters and reliability function are discussed. Through simulation study, performance of estimators are studied. In this study, both MLE and CI of parameters as well as reliability function give better performance. Expressions given in this article can also be used for generalised inverted scale family distributions.

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