BAYESIAN ANALYSIS OF CHANGE POINT PROBLEM IN AUTOREGRESSIVE MODEL: A MIXTURE MODEL APPROACH

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Abstract
In this paper, we discuss the problem of gradual changes in the parameters of an autoregressive (AR) time series model of p^{th} order, through Bayesian mixture approach. This model incorporates the beginning and end points of the interval of switch. Further, the marginal posterior densities of the parameters are obtained by employing the ordinary numerical integration technique.

Keywords: **Autoregressive model; Bayesian estimation; Structural change; Mixture model, Numerical integration.**

1. INTRODUCTION

In recent years there has been many evidence for the parameter of economic models undergone the structural changes. When a parametric model changes parameter value it is important to know the time when the change occurred and pre and post change value of the parameters.
The literature on structural change problems is by now enormous. Here we consider some of the work related with linear and time series models. Bacon and watts (1971), Ferreira (1975), Holbert and Broemeling (1977), Chin Choy and Broemeling (1980), Smith and Cook (1990) and Moen *et al* (1985) look at change points in linear models. West and Harrison (1986), Salazar (1982), Broemeling (1985) and Venkatesan and Arumugam (2007) have studied the structural change problems in time series model through the parameter change, while Baufays and Rasson (1985) have studied a variance change in autoregressive model. Most of the work in the literature is based on the parameter change in the time series model. In this paper, a Bayesian analysis of structural changes in autoregressive model of higher order is studied through the mixture model approach by introducing the distribution function of the beta random variable to model the nature of change in a finite interval of time.
In this paper the structural changes are incorporated in a model by the mixture approach. Consider, for example the case of permanent change in a firite internal (t_1, t_2). It is now assumed that one model operates before time t_1, another model operates after time t_2 and in the interval the second model gradually replaces the first model. That is, at time t $(t_1<t<t_2)$ the first model operates with probability $(1-P_t)$ and the second model operates with probability P_t and P_t goes from zero to one as t goes from t_1 to t_2. Thus, in this formulation, the likelihood function of the data will be based upon mixture distributions. One special advantage of this approach in the construction of switching models is that the number of parameter describing the nature of switch will always be fixed.
An out line of this paper is as follows. The p^{th} order autoregressive model and likelihood function are described in Section 2. Section 3 describes the posterior analysis of the model under the mixture model approach.

2. THE MODEL AND LIKELIHOOD FUNCTION

The autoregressive model of order p (AR(p)) time series model $\{X_t\}$ is defined by

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \ldots + \alpha_p X_{t-p} + e_t \qquad \ldots (1)$$

and suppose that there is a shift in $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ starts at some time point t_1 and ends at some time point t_2, the model is given by

$$X_t = (1-P_t)\sum_{i=1}^{p}\alpha_i X_{t-i} + P_t\sum_{i=1}^{p}\beta_i X_{t-i} + e_t \qquad \ldots (2)$$

where $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $(\beta_1, \beta_2, \ldots, \beta_p)$ are real unknown autoregressive parameters of before and after change respectively, e_t's are iid normal with zero mean and common variance σ^2 and denote $\theta' = (\alpha_1, \alpha_2, \ldots, \alpha_p, ; \beta_1, \beta_2, \ldots, \beta_p)$ (say)

Let $X_1, X_2 \ldots\ldots X_n$ be a sequence of n observations. Then the conditional density of X_t / X_{t-1} has the following probability density function.

$$f(X_t / X_{t-1}) = \begin{cases} f_{1t} & : \ t \le t_1 \\ (1-P_t)f_{1t} + P_t f_{2t} & : \ t_1 < t < t_2 \\ f_{2t} & : \ t \ge t_2 \end{cases} \qquad \ldots (3)$$

where f_{1t} and f_{2t} are the probability density function of a Normal random variable with means $\sum_1^n \alpha_i X_{t-i}$ and $\sum_1^n \beta_i X_{t-i}$ respectively and common variance σ^2.

$$P_t = \begin{cases} 0 & : \ t \le t_1 \\ F(t') & : \ t_1 < t < t_2 \\ 1 & : \ t \ge t_2 \end{cases}$$

$$F(t') = \frac{1}{B(\alpha,\beta)}\int_0^{t'} u^{\alpha-1}(1-u)^{\beta-1}\,du \qquad t' = \frac{t-t_1}{t_1-t_2} \quad ; \quad 1 < t_1 < t_2 < n$$

B (α,β) denotes the complete beta function with arguments α and, β; t_1 and t_2 are positive integers with $1<t_1<t_2<n$ then, α and β determine the nature of increase P_t from 0 to 1 as t goes from t_1 to t_2. Here, (t_1,t_2) in the switching interval the problem is to estimate $U = (t_1,t_2,\alpha,\beta,\theta',\sigma^2)$ but attention is mainly focused on the estimation of t_1,t_2, α and β using the observation $X=(X_1,X_2\ldots\ldots X_n)$.

The prior distributions for the parameters are assigned as follows

i. σ^2 is non-informative

ii. Given σ^2, θ' follows the multivariate normal distribution with mean zero and precision τ_i/σ^2; $i = 1, 2$.

iii. α and β follows the exponential distribution with parameters 'a' and 'b' respectively

iv. (t_1, t_2) is uniformly distributed over all possible values.

v. The parameters (θ', σ^2), α, β and (t_1,t_2) are apriori independent.

Therefore the joint prior distribution is

$$P(U) \propto \frac{ab}{\sigma} e^{-(\alpha a+\beta b)}; \qquad \sigma, a, b, \alpha, \beta > 0 \qquad \ldots (4)$$

Let $X_1, X_2, \ldots, X_n$ are observations and $X_0, X_{-1}, \ldots, X_{1-p}$ are assumed to be known values.

The likelihood function of the observations $X = (X_1, X_2, \ldots, X_n)$ given the parameter $U = (t_1, t_2, \theta', \alpha, \beta, \sigma^2)$ is given by

$$P(X/U) \propto \sum_{r=0}^{P}\Sigma_r\left(\prod_{t\in C_r^*}(1-P_t)\right)\left(\prod_{t\in C_r}P_t\right)\sigma^{-n}$$

$$\bullet \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{1}^{t_1}(X_t-B_1)^2+\sum_{t\in C_r^*}(X_t-B_1)^2 + \sum_{t\in C_r}(X_t-B_2)^2+\sum_{t_2+1}^{n}(X_t-B_2)^2\right\}\right] \qquad \ldots(5)$$

which, on simplification, becomes

$$P(X/U) \propto \sum_{r=0}^{P} A_r\ \sigma^{-n} \exp\left(\frac{-Q}{2\sigma^2}\right) \qquad \ldots (6)$$

Where $A_r \quad = \quad \Sigma_r\left(\prod_{t\in C_r^*}(1-P_t)\right)\left(\prod_{t\in C_r}P_t\right)\sigma^{-n}$

$$\begin{aligned} Q \quad = C(X) &+ [\theta_1' A_1(P,X)\theta_1 - 2\theta_1' B_1(P,X)] \\ &+ [\theta_1' A_2(P,X)\theta_1 - 2\theta_1' B_2(P,X)] \\ &+ [\theta_2' A_3(P,X)\theta_2 - 2\theta_2' B_3(P,X)] \\ &+ [\theta_2' A_4(P,X)\theta_2 - 2\theta_2' B_4(P,X)] \end{aligned}$$

where $C(X) \quad = \quad \sum_{1}^{n} X_t^2$

$B_1(P, X)$ is $p \times 1$ vector with i^{th} element $\sum_{i=1}^{t_1} X_t X_{t-i}$; $B_2(P, X)$ is $p \times 1$ vector with i^{th} element $\sum_{t\in C_r^*} X_t X_{t-i}$;

$B_3(P, X)$ is $p \times 1$ vector with i^{th} element $\sum_{t\in C_r} X_t X_{t-i}$;$B_4(P, X)$ is $p \times 1$ vector with element $\sum_{t_2+1}^{n} X_t X_{t-i}$; θ_1' $= (\alpha_1, \alpha_2, \ldots \alpha_p)$; $\theta_2' = (\beta_1, \beta_2, \ldots, \beta_p)$

$A_1(P, X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t=1}^{t_1} X_{t-i}^2$ and ij^{th} off-diagonal element is $\sum_{t=1}^{t_1} X_{t-i}X_{t-j}$.;

$A_2(P, X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t\in C_r^*} X_{t-i}^2$ and ij^{th} off-diagonal element is $\sum_{t\in C_r^*} X_{t-i}X_{t-j}$;$A_3(P, X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t\in C_r} X_{t-i}^2$ and ij^{th} off-diagonal element is $\sum_{t\in C_r} X_{t-i}X_{t-j}$;$A_4(P, X)$ is $p \times p$ matrix with i^{th} diagonal element is $\sum_{t=t_{2+1}}^{n} X_{t-i}^2$ and ij^{th} off-diagonal element is $\sum_{t=t_2+1}^{n} X_{t-i}X_{t-j}$; $P=t_2-t_1$

$C= \{t_1+1, t_1+2\ldots\ldots\ldots,t_2\}$; C_r is any subset of C with 'r' elements;C_r^* is the compliment of C_r
Σ r stands for the summation taken over all the (p_{Cr}) combination of ($t_1+1,\ldots.t_2$) selecting 'r' at a time of the second term and remaining (p-r) of the first term.

3. THE POSTERIOR ANALYSIS

Using (4), (6) and Bayes theorem, the joint posterior distribution of the parameter is after simplification given by

$$P(U/X) \quad \propto \quad \sum_{r=0}^{P} A_r\, e^{-(\alpha a+\beta b)}\sigma^{-(n+1)} \exp\left(\frac{-Q^*}{2\sigma^2}\right) \qquad \ldots (7)$$

where

$Q^* = C(X) + [\theta_1' M(P,X)\theta_1 - 2\theta_1' D(P,X)] + [\theta_2' M_1(P,X)\theta_2 - 2\theta_2' D_1(P,X)]$

$M(P, X) \quad = \quad A_1(P, X) + A_2(P, X)$

$M_1(P, X) = A_3(P, X) + A_4(P, X)$
$D(P, X) = B_1(P, X) + B_2(P, X)$
$D_1(P, X) = B_3(P, X) + B_4(P, X)$

After simplification, one can get,

$$P(U/X) \propto \sum_{r=0}^{P} A_r \sigma^{-(n+1)} e^{-(\alpha a+\beta b)} \exp\left(\frac{-Q^{*'}}{2\sigma^2}\right) \quad \ldots (8)$$

where $Q^{*'} = [\theta_1 - M^{-1}(P, X) D(P, X)]' M(P, X) [\theta_1 - M^{-1}(P, X) D(P, X)]$
$+ [\theta_2 - M_1^{-1}(P, X) D_1(P, X)]' M_1(P, X)$
$. [\theta_2 - M_1^{-1}(P, X) D_1(P, X)] + C^*(X)$

and $C^*(X) = [C(X) - D'(P, X)M^{-1}(P,X) D(P, X) - D_1'(P, X) M_1^{-1}(P, X) D_1(P, X)]$

Eliminating θ_1 and θ_2 and σ^2 from the above expression (8), one gets

$$P(t_1, t_2, \alpha, \beta / X) \propto \sum_{r=0}^{P} A_r (e^{-(\alpha a+\beta b)}) \,.\, |M(P,X)|^{-1/2} |M_1(P,X)|^{-1/2} [C^*(X)]^{\frac{-(n-2p+3)}{2}} \quad \ldots (9)$$

The elimination of the parameters from (9) is analytically not possible since the joint posterior distribution of t_1, t_2, α and β is a complicated function of t_1, t_2, α and β. Therefore, one may have to resort to numerical integration technique to determine the marginal posterior distribution of the parameter.

The first order autoregressive model is obtained by taking p=1 in (1) and in the joint posterior density (9) is reduced to the following, after simplification

$$P(t_1, t_2, \alpha, \beta / X) \propto \sum_{r=0}^{P} A_r e^{-(\alpha a+\beta b)} |V|^{-1/2}; \ \alpha, \beta > 0, \ \ 1 \le t_1 < t_2 \le n-1 \quad \ldots (10)$$

where

$$V = \begin{bmatrix} \frac{A_2(n+2)}{C} & 0 \\ 0 & \frac{A_3(n+2)}{C} \end{bmatrix} \text{ and } C = A_1 - \frac{B_1^2}{A_2} - \frac{B_2^2}{A_3}.$$

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