A topological approach to real networks.

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Abstract:
Topological structures of real networks are investigated using some relevant descriptive indicators suitable for specific applicative contexts.

Keywords: Graph, centrality, spectral radius, topological indices.

1. Introduction

Graph-theoretic techniques may be conveniently used for describing the topological structure of a real network in many different contexts, from social networks (scientific co-authorships, corporate board of directors,…) to the World WideWeb, Internet router levels, electrical grid, etc... Recently, studies on networks highlight their common properties through the use of topological descriptors, that capture important structural features of the net such spectral radius (Jamakovic *et* al., 2006), diameter, S-metric (Li *et al*., 2006; Grassi *et al*., 2008), Wiener index (Dobrynin *et al.*, 2001; Wang, 2007), energy of a graph (Brankov *et al*., 2004). Centrality, related to single nodes, (see for a recent survey Da Costa *et al*., 2008) is also an important issue.

2. The network topology: some useful indicators.

Here we mainly focus on the properties of a network through its relevant descriptive parameters, especially in sight of specific applicative contexts. Just to mention an example among the many, the smaller the spectral radius of the adjacency matrix is, the higher is the robustness of a network against the spread of viruses (Jamakovic et al., 2006). Similarly, the design of communication networks often requires to construct networks having a small diameter so that a message, originating at a specific node, runs through the network as efficiently as possible, where efficiency is related to communication times for receiving a message (Bermond *et al*., 1986).
Among the various topological indices that have been widely considered in graph theory (Devillers *et al*.; 1999, Dobrynin *et al.*, 2001), the most meaningful for our application are the Wiener index $\sigma(G)=\frac{1}{2}\sum_{(u,v)\in V(G)} d_G(u,v)$, with $d(u, v)$ being the geodetic between node u and v in the graph G, and the Randić index $w_a(G)=\sum_{uv\in E(G)}(d(u)d(v))^{\alpha}$ where E(G) is the edge set of G, $d(u)$ the degree of vertex u and $\alpha\neq 0$. For $\alpha=1$, when G is a tree, $w_1(T)$ is also called the *weight* of T and corresponds to the "metric" S(T). These indices are usual in chemical literature. In Wang (2007, 2008) trees with the maximum value of Wiener Index and of the Randić index for a given degree sequence are studied.

In Grassi *et al*. (2008) the "metric" $S(G) = \sum_{(i,j)\in E} d_i d_j$ is studied in the set of trees with a given degree sequence, where d_i is the degree of the i-th node. This metric characterizes the hub structure of the net.

Meanwhile, some more theoretical questions will be considered and, in particular, we will study the role of centrality (better yet, of the various definitions of a "central vertex" for a graph), identifying the most suitable measure that tightly depends on the empirical problem under analysis. High centrality scores of centrality indicate that a vertex is directly connected to many others (*degree-centrality*); or that it can reach others on relatively short paths (*closeness-centrality*), or that the vertex lies on considerable fractions of shortest paths connecting others (*betweenness-centrality*), or that a maximum flow is transmitted between two nodes (*flow-betweenness-centrality*), or that contributions from random walks are included (*random-walk-betweenness*), or that the vertex is connected to vertexes that themselves are highly central (*eigenvector-centrality*).

We observe that each measure captures different aspects of a vertex's importance in a graph and choosing an appropriate measure is a crucial problem.

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