Normalized priority vectors for fuzzy preference relations

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Abstract

We consider the problem of normalizing the priority vectors associated with fuzzy preference relations and we show that a widely used normalization procedure may lead to unsatisfactory results whenever additive consistency is involved. We give some examples from the literature and we propose an alternative normalization procedure which is compatible with additive consistency and leads to better results.
Keywords: fuzzy preference relation, priority vector, normalization.

Introduction

Several problems involving vectors calculation have an infinite number of solutions, eigenvectors of a square matrix being perhaps the most popular. Therefore, in order to obtain uniqueness, a normalization procedure is usually performed by dividing every component w_i of a vector w by a suitable value k. Frequently used values of k are $k = \|w\|$ and $k = \sum_{i=1}^{n} w_i$. In the first case a unit–norm vector is obtained, $\|w\| = 1$, while in the second case the components of the obtained vector sum up to one. In the Analytical Hierarchy Process (AHP) [3], as well as in other similar methods, this type of normalization is justified and it is usually applied.

In this paper we show that, on the contrary, as long as fuzzy preference relations (FPR in the following) are concerned, this kind of normalization is incompatible with additive consistency. To show this incompatibility we first prove some theoretical results and then examine some (not all) of the published papers performing this kind of normalization [2], [5], [6], [8], [9], [10]. To conclude, we propose an alternative normalization procedure which is compatible with additive consistency and leads to better results.

1 The priority vector of a fuzzy preference relations and its normalization

Fuzzy Preference Relations, *FPR*s in the following, are nonnegative relations $R : \Lambda \times \Lambda \rightarrow [0, 1]$ on a set of alternatives $\Lambda = \{A_1, A_2, ..., A_n\}$. Additive reciprocity is assumed, $r_{ij} + r_{ji} = 1, \quad i, j = 1, ..., n$, where $r_{ij} := R(A_i, A_j)$. The $n \times n$ matrix $R = [r_{ij}]$ is also called *additive pairwise comparison matrix* or *additive preference matrix*. Tanino [4] defines 'additive consistent' a *FPR* satisfying $(r_{ih} - 0.5) = (r_{ij} - 0.5) + (r_{jh} - 0.5) \quad \forall i, j, h$. He proves also a characterization of an additive consistent *FPR*, stating that a *FPR* $R = [r_{ij}]_{n \times n}$ is additive consistent if and only if a non negative vector $w = (w_1, ..., w_n)$ exists with $|w_i - w_j| \leq 1 \;\; \forall i, j$, such that the entries r_{ij} of R are given by

$$r_{ij} = 0.5 + 0.5(w_i - w_j) \quad i, j = 1, ..., n. \tag{1}$$

Moreover, components w_i are unique up to addition of a real constant.

Given an additive consistent *FPR* $R = [r_{ij}]$ and a nonnegative vector w, we say that w is 'associated' with R if and only if it satisfies (1) as well as the assumption $|w_i - w_j| \leq 1 \;\; \forall i, j$. vector w is said to 'represent' the associated *FPR*.

Tanino's characterization (1) is used as optimization criterion in [2], [5] and [9], but in these three papers also the normalization constraint

$$\sum_{i=1}^{n} w_i = 1 \tag{2}$$

is imposed. With the following proposition we prove the incompatibility of (2) with Tanino's characterization (1).

Proposition 1. *Given a positive integer $n \geq 3$, it is impossible to associate, by means of (1), every additive consistent FPR with a weight vector w satisfying $\sum_{i=1}^{n} w_i < n - 1$. This association is always possible if the inequality is not strict, i.e. if $\sum_{i=1}^{n} w_i \leq n - 1$.*

Proof

Let us consider the additive consistent *FPR* $\hat{R} = [\hat{r_{ij}}]$, where $\hat{r}_{ij} = 0.5$ for $i, j = 1, ..., n-1$, $\hat{r}_{in} = 1$ for $i = 1, ..., n-1$, $\hat{r}_{nj} = 0$ for $j = 1, ..., n-1$ and $\hat{r}_{nn} = 0.5$. It is easy to prove that no vector w associated with $\hat{R}$ exists satisfying $\sum_{i=1}^{n} w_i < n-1$. By substituting $\hat{r}_{in} = 1$ in (1) for $i = 1, ..., n - 1$, one obtains $w_i = w_n + 1, \quad i =$

$1, ..., n-1$, and therefore $\sum_{i=1}^{n} w_i = (n-1)(w_n+1) + w_n = nw_n + n - 1$. Since $w_n \geq 0$, inequality $\sum_{i=1}^{n} w_i < n-1$ is violated and the first part of the proposition is proved. To prove the second part, let us observe that Tanino's characterization of an additive consistent FPR guarantees the existence of a vector $w = (w_1, ..., w_n)$ satisfying (1). Let us assume, without loss of generality, $w_n \leq w_{n-1} \leq \cdots \leq w_1$. Since w_i's are unique up to addition of a real constant k, by choosing $k = -w_n$, it is always possible to associate to an additive consistent FPR a vector w with $w_n = 0$, $w = (w_1, \cdots, w_{n-1}, 0)$. From $w_n = 0$ and constraint $|w_i - w_j| \leq 1$, it follows $0 \leq w_i \leq 1$. Then it is $\sum_{i=1}^{n} w_i \leq n-1$. □

Tanino [4] also considers an alternative kind of consistency for FPR which is called *multiplicative*. A FPR is 'multiplicative consistent' if and only if $r_{ih}r_{hj}r_{ji} = r_{ij}r_{jh}r_{hi}$, $\forall i, j, h$. In this case, a positive vector $v = (v_1, \ldots, v_n)$ exists such that $r_{ij} = v_i/(v_i+v_j)$ $\forall i, j$. Components v_i are unique up to multiplication by a positive constant. Therefore, such a priority vector can be normalized using (2).

1.1 An alternative normalization

Uniqueness of priority vector satisfying (1) can be achieved simply by adding to each component w_i the constant $k = -\min\{w_1, ..., w_n\}$, thus obtaining a vector with the minimum component equal to zero. Assuming $w_n \leq w_{n-1} \leq \cdots \leq w_1$, it is $k = -w_n$ and the normalized vector becomes

$$w = (w_1, \cdots, w_{n-1}, 0). \tag{3}$$

Contrary to (2), this alternative normalization procedure is compatible with (1) and, as proved above, it guarantees that all the priority weights w_i are in the interval [0, 1]. This is a good standard result that also allows an easier and more familiar understanding of the obtained priorities. To summarize, the normalization constraint we propose is

$$\begin{array}{l} \min\{w_1, ..., w_n\} = 0 \\ 0 \leq w_i \leq 1 \quad i = 1, ..., n \end{array} \tag{4}$$

2 Consistency optimization and vector normalization

In the previous section we have considered the case of additive consistent FPRs. Let us now consider the case in which additive consistency is, in general, not achievable, but it is the goal of a proposed optimization model. In [5], incomplete FPRs are considered and some goal programming models for obtaining the priority vector are proposed. The proposal is based on the idea of moving as close as possible to (1) and it is therefore appropriate and effective but, as proved in the previous section, the normalization constraint (2) required in the proposed models conflicts with the goal and leads to unsatisfactory results. Several examples can be constructed showing that better results are obtained if in these models constraint (2) is substituted by (4). Analogous remarks can be made on the approach proposed in [2] and on the case of the interval fuzzy preference relations considered in [9]. Also in the definitions 3 and 4 given in [9] for an 'additive consistent interval *FPR*', the constraint (2) leads to incoherent results.

We conclude by stressing that, given the very frequent use of vector normalization, it is important that researchers are warned not to consider it as a risk–free routine when they are dealing with *FPR*. Otherwise, interesting proposals can become useless, due to an inadequate choice of the normalization constraint.

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